Rational and Integral Points via Analytic and Geometric Methods

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1 Overview of the Field

The study of rational or integral solutions to systems of Diophantine equations is a topic that is almost as old as mathematics itself. The most natural point of view is geometric wherein one views any such system of polynomial equations as an algebraic variety. The most basic question is one of decidability: when one restricts to suitable classes of varieties is there an algorithm for deciding whether or not the variety has rational (or integral) points?

In case the variety is found to have such points the next most basic goal is to describe them in some way. The focus of this conference was on

— the Hasse principle and weak approximation for rational (or integral) points on varieties; and

— the density of rational (or integral) points on varieties.

For both of these questions it is expected that the behaviour of the set of rational points on a given variety is governed by its geometry.

Existence of rational or integral points on a variety

Necessary conditions for the existence of rational or integral points on varieties are the solubility of the defining equations over the reals and modulo arbitrary prime powers. If these conditions are also sufficient for a class of varieties, we say that it satisfies the *Hasse principle*. For example, Hasse and Minkowski showed that the Hasse principle holds for quadrics in projective space, but Swinnerton-Dyer found some cubic surfaces failing the Hasse principle. Going further, we may ask whether the rational (respectively, integral) points on a variety are dense in the adelic points; this is known as *weak approximation* (respectively, *strong approximation*).

A deep conjecture of Colliot-Thélène predicts that the Brauer–Manin obstruction provides an algorithm for deciding whether or not the Hasse principle or weak approximation hold for the class of smooth projective rationally connected varieties over any number field. This is a cohomological obstruction that was put forward by Manin in 1970 in his ICM address and undoubtedly lies very deep. As pointed out by Serre, for example, a positive answer to this conjecture would result in a positive solution to the inverse Galois problem!

The corresponding problem for integral points on non-proper varieties, namely decidability for the integral Hasse principle and strong approximation, seems to be much harder. We are now starting to get a sense of which of the tools developed for rational points can be transported successfully to the setting of integral points.

Density of rational and integral points on a variety

Given a higher-dimensional variety with rational points, the next natural question is to ask about the distribution of its rational points. For example, we may wonder whether all of its rational points lie on one algebraic curve, or more generally on a finite number of lower-dimensional subvarieties. More precisely, the question is whether the rational points are dense on the variety in the Zariski topology. By Colliot-Thélène's conjecture, this is expected to be true for smooth projective rationally connected varieties over all global fields, but Zariski density is still open even for some rational surfaces. At the other extreme, Lang's conjecture predicts that rational points are not Zariski dense on varieties of general type.

Techniques for proving that the set of rational points on a given variety is Zariski dense include, in the case of surfaces, finding families of elliptic curves on the surface. Bogomolov and Tschinkel have proved that surfaces defined over a number field that admit an elliptic fibration have potentially dense set of rational points; i.e. the rational points are Zariski dense after a finite extension of the ground field. This result covers a very large class of surfaces: all surfaces of Kodaira dimension one, K3 surfaces with large Picard number and Enriques surfaces. If one restricts attention to rational surfaces, it is enough to deal with minimal surfaces, since being Zariski dense is a birational property. Iskovskikh has classified minimal rational surfaces showing that they are divided essentially in two kinds: del Pezzo surfaces and conic bundles. Such surfaces can be subdivided further according to their degree (i.e. the self-intersection of the canonical divisor). Thanks to work of Manin and Kollár, we now know that rational points are Zariski dense in del Pezzo surfaces of degree at least three over perfect fields.

The Batyrev–Manin conjecture, which had its origins roughly 25 years ago, is concerned with a quantitative description of rational points on Fano varieties over global fields. Assuming that the set of rational points is Zariski dense, it gives a precise prediction for the number of rational points of bounded anticanonical height on the variety in terms of certain associated geometric invariants. By working over the function field $\mathbb{F}_q(C)$, for a smooth curve C, Ellenberg and Venkatesh have uncovered important connections with the geometry of moduli spaces of curves on varieties over finite fields. Over number fields, analytic number theory has had a profound impact on the status of the Batyrev–Manin conjecture. This conjecture is now known for several classes of equivariant compactifications of algebraic groups (e.g. toric varieties) and for many examples of del Pezzo surfaces. Much less is known about the distribution of integral points; here, Chambert-Loir and Tschinkel have developed a conjectural picture for log-Fano varieties, while Harpaz has initiated an investigation into the possible distribution of integral points on affine cubic surfaces.

2 Recent Developments and Open Problems

Hasse principle and weak approximation

Colliot-Thélène's conjecture is still wide open but there are several recent examples where tools from analytic number theory have been successfully paired with algebro-geometric methods to handle important special cases of the conjecture. One particularly striking example is based on the resolution of the inverse Gowers $U^{s+1}[N]$-norm conjecture by Green, Tao and Ziegler in additive combinatorics. This was used by Browning, Matthiesen and Skorobogatov to settle Colliot-Thélène's conjecture for some conic bundle surfaces with arbitrarily many singular fibres.

For general smooth projective varieties we are very far from even a conjectural understanding of obstructions to the existence of rational points. Thanks to highly innovative input from algebraic topology, Harpaz and Schlank have used étale homotopy theory to unify all known obstructions. In order to identify the limitations of these obstructions it is important to test them on explicit classes of varieties, such as remarkable work of Poonen on the insufficiency of étale–Brauer obstructions for certain threefolds fibred by Châtelet surfaces.

The arithmetic of K3 surfaces is a particularly active area of current research. There is a general expectation that the Brauer–Manin obstruction should control the existence of rational points on these varieties, but much remains to be done. For K3 surfaces it appears that the transcendental part of the Brauer group plays an important rôle, as shown by Hassett and Várilly-Alvarado using a powerful construction motivated by Hodge theory. A proper understanding of the transcendental Brauer–Manin obstruction throws up huge challenges, both computational and otherwise.

Density of rational points

Whilst the existence of a single rational point is enough to ensure the Zariski density of rational points on del Pezzo surfaces of degree at least three over number fields, the situation for del Pezzo surfaces of degree one or two is much more difficult to understand. Salgado and van Luijk have recently made inroads into the most difficult case of degree one del Pezzo surfaces, subject to an appropriate arithmetic hypothesis. There remains much to be done here, however, with conic bundles of low degree being next in line for scrutiny.

The Batyrev–Manin conjecture is relatively well understood over the rational numbers in dimension 2 (using universal torsors combined with various analytic techniques) and for equivariant compactifications of algebraic groups (using harmonic analysis on adelic points). Only recently has the situation over other global fields been systematically addressed in work of Derenthal, Frei and Pieropan (number fields) and work of Bourqui (function fields). As we look to Fano threefolds, it appears that the presence of thin sets can sometimes throw off the Batyrev–Manin conjecture. Recent work of Lehmann and Tanimoto proffers a bold geometric explanation of this phenomenon, while recent work of Ellenberg and Zurieck-Brown explores a version of Manin's conjecture for stacks as a bold new way to unify the Batyrev–Manin conjecture and Malle's conjecture.

The Brauer–Manin obstruction in families

Another exciting new development is the combination of the two main topics presented above: One can ask to quantify how often a point exists or weak approximation fails in a given family of varieties. Such questions have been recently considered for example by Bhargava, Poonen and Stoll in the case of families of curves, and by Loughran for families of Brauer–Severi varieties parameterized by toric varieties.

3 Presentation Highlights

The following results were among some of the topics discussed by workshop speakers.

Existence of rational points

Marta Pieropan: *Rational points over C_1-fields of characteristic 0.* In the 1950s Lang studied the properties of C_1-fields; that is, fields over which every hypersurface of degree at most n in a projective space of dimension n has a rational point. Later he conjectured that every smooth proper rationally connected variety over a C_1-field has a rational point. This talk addressed the search for rational points on rationally connected threefolds over C_1-fields of characteristic 0.

Zariski density

Ronald van Luijk: *Verifying Zariski density of rational points on del Pezzo surfaces of degree 1.* Let S be a del Pezzo surface of degree 1 over a number field k. The main goal of this talk was to give easily verifiable sufficient conditions under which its set $S(k)$ of rational points is Zariski dense. It is well known that almost all fibers of the anticanonical map $\phi : S \dashrightarrow \mathbb{P}^1$ are elliptic curves with the unique base point of ϕ as zero. Suppose that $P \in S(k)$ is a point of finite order $n > 1$ on its fiber. Then there is another elliptic fibration on the blow-up of S at P. This was used to define a proper closed subset $Z \subset S$ such that one can deduce that $S(k)$ is Zariski dense merely by checking whether or not $(S \setminus Z)(k)$ is non-empty. This was joint work with Jelle Bulthuis inspired by an example of Noam Elkies.

Hasse principle and weak approximation — algebraic techniques

Otto Overkamp: *Finite descent obstruction and non-Abelian reciprocity*. For a nice algebraic variety X over a number field F, one of the central problems of Diophantine Geometry is to locate precisely the set $X(F)$

inside $X(\mathbb{A}_F)$, where $\mathbb{A}_F$ denotes the ring of adèles of F. One approach to this problem is provided by the finite descent obstruction, which is defined to be the set of adelic points which can be lifted to twists of torsors for finite étale group schemes over F on X. More recently, Kim proposed an iterative construction of another subset of $X(\mathbb{A}_F)$ which contains the set of rational points. The main result of this talk was that the two approaches are equivalent.

Olivier Wittenberg: *Zero-cycles on homogeneous spaces of linear groups.* The Brauer-Manin obstruction is expected to control the existence and weak approximation properties of rational points on homogeneous spaces of linear algebraic groups over number fields. He discussed joint work with Yonatan Harpaz on the zero-cycle variant of this conjecture. The same method also leads to a new proof of Shafarevich's theorem that finite nilpotent groups are Galois groups over any number field.

Yang Cao: *Weak and strong approximation for a group scheme.* Weak and strong approximation for algebraic groups are established by using arithmetic duality. He talked about how to apply arithmetic duality to weak and strong approximation for some special group schemes over the projective line.

Jennifer Berg: *Odd order transcendental obstructions to the Hasse principle on general K3 surfaces.* After fixing numerical invariants such as dimension, it is natural to ask which birational classes of varieties fail the Hasse principle, and moreover whether the Brauer group (or certain distinguished subsets) explains this failure. This talk focused on K3 surfaces, which have been a testing ground for many conjectures on rational points. In 2014, Ieronymou and Skorobogatov asked whether any odd torsion in the Brauer group of a K3 surface could obstruct the Hasse principle. In joint work with Tony Varilly-Alvarado she answered this question in the affirmative for transcendental classes via a purely geometric approach.

Vladimir Mitankin: *Integral points on generalised affine Châtelet surfaces.* A classical result of Colliot-Thélène and Sansuc states that the only obstruction to the Hasse principle and weak approximation for generalised Châtelet surfaces is the Brauer-Manin one, conditionally on Schinzel's hypothesis. Inspired by their work, he studies the analogous questions concerning the existence and the density of integral points on the corresponding affine surfaces, again under Schinzel's hypothesis.

Rachel Newton: *Arithmetic of rational points and zero-cycles on Kummer varieties.* Yongqi Liang has shown that for rationally connected varieties over a number field K, sufficiency of the Brauer-Manin obstruction to the existence of rational points over all finite extensions of K implies sufficiency of the Brauer-Manin obstruction to the existence of zero-cycles of degree 1 over K. This talk discussed joint work with Francesca Balestrieri where Liang's result is extended to Kummer varieties.

Hasse principle and weak approximation — analytic techniques

Pankaj Vishe: *Quartic forms in 30 variables.* He discussed joint work with Oscar Marmon, showing that smooth quartic hypersurfaces satisfy the Hasse Principle as long as they are defined over at least 30 variables. The key tool here is employing Kloosterman's version of circle method.

Kevin Destagnol: *Prime (and squarefree) values of polynomials in moderately many variables.* The classical Schinzel hypothesis and its quantitative version, the Bateman-Horn conjecture, states that a system of polynomials in one variable takes infinitely many simultaneously prime values under some necessary assumptions. This talk presented a proof of a generalization of these conjectures to the case of an integer form in many variables. Moreover this result was applied to study the Hasse principle and weak approximation for some normic equations.

Manin's conjecture

Roger Heath-Brown: *Manin's conjecture for a bi-projective variety.* He talked about joint work with Tim Browning concerning the variety

$$X_1Y_1^2 + X_2Y_2^2 + X_3Y_3^2 + X_4Y_4^2 = 0$$

in $\mathbb{P}^3 \times \mathbb{P}^3$. The height of a point (x, y) is given by $H(x)^3H(y)^2$, and Manin's conjecture predicts asymptotically $cB \log B$ points of height at most B. To obtain this one must exclude points on the subvariety $x_1x_2x_3x_4 = 0$; but in order to achieve the Peyre constant one must exclude an infinite number of subvarieties in which $x_1x_2x_3x_4$ is a square. By combining the circle method with lattice point counting techniques, they were able to prove Manin's conjecture for this example, and the talk gave an overview of the various ingredients, and the way that they fit together.

Sho Tanimoto: *Log Manin's conjecture for klt Campana points*. The notion of Campana points has been introduced by Abramovich and Várilly-Alvarado and interpolates between rational points and integral points. In this talk a variant of this notion is discussed used to prove Manin's conjecture for Campana points on equivariant compactifications of vector groups using the height zeta functions method. This was joint work with Tony Varilly-Alvarado.

David Zureick-Brown: *Counting points, counting fields, and heights on stacks*. A folklore conjecture is that the number $N_d(K, X)$ of degree-d extensions of K with discriminant at most d is on order $c_d X$. In the case $K = \mathbb{Q}$, this is easy for $d = 2$, a theorem of Davenport and Heilbronn for $d = 3$, a much harder theorem of Bhargava for $d = 4$ and 5, and completely out of reach for $d > 5$. More generally, one can ask about extensions with a specified Galois group G; in this case, a conjecture of Malle holds that the asymptotic growth is on order $X^a(\log X)^b$ for specified constants a, b. The form of Malle's conjecture is reminiscent of the Batyrev–Manin conjecture, which says that the number of rational points of height at most X on a Batyrev-Manin variety also grows like $X^a(\log X)^b$ for specified constants a, b. What's more, an extension of $\mathbb{Q}$ with Galois group G is a rational point on a Deligne-Mumford stack called BG, the classifying stack of G. A natural reaction is to say "the two conjectures is the same; to count number fields is just to count points on the stack BG with bounded height?" The problem: there is no definition of the height of a rational point on a stack. In this joint work with Jordan Ellenberg and Matt Satriano the definition is introduced and shown to suggest a heuristic which has both the Malle conjecture and the Batyrev–Manin conjecture as special cases.

Further quantitative results

Christopher Frei: *Average bounds for l-torsion in class groups*. Let l be a positive integer. Improved average bounds for the l-torsion of the class groups for some families of number fields was discussed, including degree-d-fields for $2 \leq d \leq 5$. The improvements are based on refinements of a technique due to Ellenberg, Pierce and Wood. This was joint work with Martin Widmer.

Joseph Gunther: *Slicing the stars: counting algebraic numbers*. Masser and Vaaler gave an asymptotic formula for the number of algebraic numbers of given degree and increasing height. This problem was solved by counting lattice points in an expanding star body. In this talk it was explained how to estimate the volumes of slices of star bodies, which allows one to count algebraic integers, algebraic integers of given norm and/or trace, and more. This was joint work with Robert Grizzard.

Damaris Schindler: *Diophantine inequalities for ternary diagonal forms*. Small solutions to ternary diagonal inequalities of any degree are discussed, where all of the variables are assumed to be of similar size. This problem is studied on average over a one-parameter family of forms. It was shown how these Diophantine inequalities are related to counting rational points close to varieties.

Families of abelian varieties

Marc Hindry: *Variation of the Mordell-Weil rank in families of abelian varieties*. Consider a family of abelian varieties over a number field K, i.e. a variety X with a map to a curve B whose fibres are abelian varieties. The generic fibre is an abelian variety over the function field $K(B)$ and the group of $K(B)$-rational points has a rank r. For almost all points t in $B(K)$ the fibre is an abelian variety X_t over K and the group of K-rational point has rank $r(t)$. A specialisation theorem of Silverman says that for or almost all points t in $B(K)$ the rank $r(t)$ is greater or equal to r. One want to understand the distribution of $r(t)$, in particular we ask wether there are infinitely many t's 1) with $r(t) = r$, 2) with $r(t) > r$. Under specific geometric conditions, joint work with Cecília Salgado settles the second question, and provide interesting example where much more can be proven.

Julie Desjardins: *Variation of the root number in families of elliptic curves*. What can be said about the variation of the rank in a family of elliptic curves? One knows in particular that if infinitely many curves in the family have non-zero rank, then the set of rational points is Zariski dense in the associated elliptic surface. She uses a "conjectural substitute" for the geometric rank (or rather for its parity): the root number. For a non-isotrivial family, under two analytic number theory conjectures she showed that the root number is -1 (resp. +1) for infinitely many curves in the family. On isotrivial families however, the root number may be constant; she described its behaviour in this case.

Zhizhong Huang: *Density of rational points on certain elliptic K3 surfaces*. He discussed a 2-cover method to study rational points on elliptic surfaces that can be applied to several isotrivial Kummer-type families whose generic Mordell-Weil ranks are 0. He showed that for these families rational points are Zariski dense and even dense in real topology, giving evidence for a conjecture of Mazur.

Rational points in families

Efthymios Sofos: *The behaviour of rational points in families*. A topic of current interest regards 'how often' a variety has a rational point. This topic was initially studied by Serre who gave upper bounds in the case of families of conics parametrised by a projective space. In the last few years this topic has been significantly enriched by Loughran and others. This talk introduces the topic and finishes by discussing joint work with Erik Visse, where asymptotics are given for a family of conics parametrised by arbitrary smooth hypersurfaces of low degree.

Daniel Loughran: *An Erdős-Kac law for local solubility in families of varieties*. A famous theorem due to Erdős and Kac states that the number of prime divisors of an integer N behaves like a normal distribution. Talk considered analogues of this result in the setting of arithmetic geometry, with probability distributions for questions related to local solubility of algebraic varieties. This was joint work with Efthymios Sofos.

Brauer groups

Alexei Skorobogatov: *On uniformity conjectures for abelian varieties and K3 surfaces over number fields.* In joint work with Martin Orr and Yuri Zarhin, he showed that the uniform boundedness of the transcendental Brauer group of K3 surfaces and abelian varieties of bounded dimension defined over number fields of bounded degree is a consequence of a conjecture of Coleman about rings of endomorphisms of abelian varieties. They also showed that this conjecture of Coleman implies the conjecture of Shafarevich about the Néron-Severi lattices of K3 surfaces.

Martin Bright: *A uniform bound on the Brauer groups of certain log K3 surfaces*. There has been much interest recently in bounding the Brauer groups of K3 surfaces over number fields. On the other hand, the arithmetic of integral points on log K3 surfaces appears to share some features with that of rational points on K3 surfaces. Some of the simplest examples of log K3 surfaces are the open surfaces obtained by starting with a projective del Pezzo surface and removing a smooth anticanonical divisor. In this talk Merel's boundedness of torsion on elliptic curves is used to prove boundedness of the Brauer groups of such log K3 surfaces over a number field. This was joint work with Julian Lyczak.

Wei Ho: *Splitting Brauer classes with the universal Albanese.* In this talk it is proved that every Brauer class over a field splits over a torsor under an abelian variety. This was joint work with Max Lieblich.

Curves

Jackson Morrow: *Irrational points on random hyperelliptic curves.* Let d and g be positive integers with $1 < d < g$. If d is odd, he proves that there exists $B(d) > 0$ such that a positive proportion of odd genus g hyperelliptic curves over $\mathbb{Q}$ have at most $B(d)$ points of degree d. Similarly, if d is even, he bounds the degree d points not pulled back from degree $d/2$ points of the projective line. The proof proceeds by refining Park's recent application of tropical geometry to symmetric power Chabauty, and then applying results of Bhargava and Gross on average ranks of Jacobians of hyperelliptic curves. This was joint work with Joseph Gunther.

Anastassia Etropolski: *Chabauty-Coleman experiments for genus 3 hyperelliptic curves*. Given a curve of genus at least 2, it was proven in 1983 by Faltings that it has only finitely many rational points. Unfortunately, this result is ineffective, in that it gives no bound on the number of rational points. 40 years earlier, Chabauty proved the same result under the condition that the rank of the Jacobian of the curve is strictly smaller than the genus. While this is obviously a weaker result, the methods behind that proof could be made effective, and this was done by Coleman in 1985. Coleman's work led to a procedure known as the Chabauty-Coleman method, which has shown to be extremely effective at determining the set of rational points exactly, particularly in the case of hyperelliptic curves. This talk discussed how one can implement this method using Magma and Sage to provably determine the set of rational points on a large set of genus 3, rank 1 hyperelliptic curves, and how these calculations fit into the context of the state of the art conjectures in the field. This talk was joint work with Jennifer Balakrishnan, Francesca Bianchi, Victoria Cantoral-Farfan, and Mirela Ciperiani.

Jennifer Park: *Cycles in the supersingular l-isogeny graphs and corresponding endomorphisms*.

Jaap Top: *Two arithmetical aspects of Poncelet's closure theorem*. Already 190 years ago Jacobi in a paper in Crelle's journal described the celebrated closure theorem of Poncelet as a "bekanntes Problem der Elementargeometrie". Some natural number theoretical questions arising in this context, were discussed during this talk.

Adam Morgan: *Parity of 2-Selmer ranks of abelian varieties over quadratic extensions*. For an abelian variety A over a number field K, a consequence of the Birch and Swinnerton-Dyer conjecture is the 2-parity conjecture: the global root number agrees with the parity of the 2-infinity Selmer rank. It is a standard result that the root number may be expressed as a product of local terms and we show that, over any quadratic extension of K, the same holds true for the parity of the 2-infinity Selmer rank. Using this several new instances of the 2-parity conjecture are proved for general principally polarised abelian varieties by comparing the local contributions arising. Somewhat surprisingly, the local comparison relies heavily on results from the theory of quadratic forms in characteristic 2.

Felipe Voloch: *Obstructions to existence of rational points on curves from subgroups of the Brauer group*. It is widely expected that, if a curve over a global field has no rational points, that there is an obstruction to the existence of rational points coming from the Brauer group. One piece of evidence for this is an heuristic due to Poonen. In this talk it is shown that Poonen's argument also applies to p-primary subgroups of the Brauer group (for any prime p) but that there are examples of curves with no rational point but not having an obstruction coming from the p-primary subgroups of the Brauer group. This was joint work with B. Creutz and B. Viray.

4 Scientific Progress Made

The workshop brought together experts in analytic number theory and arithmetic geometry in order to drive forward the boundaries of knowledge regarding rational and integral points on higher-dimensional algebraic varieties, through a fusion of geometric, cohomological and analytic methods. The level of the talks was extremely high, and occasioned quite a lot of discussion and interaction. The time between and after talks provided the perfect opportunity for the participants to carry on discussions and collaborations. For example, D. Loughran and V. Mitankin were able to work on their project on Markov surfaces; C. Salgado, A. Várilly-Alvarado and F. Voloch made progress on their collaboration on error correcting codes; T. Browning and D. Schindler discussed work on generalised quadratic forms over number fields; and A. Várilly-Alvarado worked with S. Tanimoto and J. Berg on a number of different projects.