Symmetry Breaking in Discrete Structures

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September 16 – 21, 2018

1 Overview of the Field

The *distinguishing number* or *asymmetric coloring number* of a group G on the set Ω, denoted $D(G, \Omega)$ or $ACN(G, \Omega)$ is the smallest k such that there is a coloring of Ω by k colors such that the only color-preserving elements of G fix all elements of Ω. In most instances, the action of G is faithful, so G is assumed to be a permutation group and we simply write $D(G)$ instead of $D(G, \Omega)$, and, for convenience, use $D(G)$ rather than $ACN(G)$. Albertson and Collins [1] introduced the concept in the context of the automorphism group of a graph Γ acting on the vertex set; in the case of graphs one says that a graph Γ has distinguishing number k, written $D(\Gamma) = k$, if $D(Aut(\Gamma)) = k$. The Albertson–Collins paper has spawned hundreds of publications in the last 15 years, nearly all in the context of graphs.

On the other hand, the case $D(G) = 2$ has a longer history. In 1977, Babai proved [2] that all infinite leafless trees have distinguishing number two. In 1981, Gluck [9] showed that if $|G|$ is odd, then $D(G) = 2$. In permutation group terms, $D(G) = 2$ if and only if G has a regular orbit on the power set of Ω. In 1984, Cameron, Neumann, and Saxl [7] showed that all but finitely many primitive permutation groups other than A_n, S_n have distinguishing number two; in 1997, Seress [20] classified those that do not.

The distinguishing concept has been applied to finite group actions arising as automorphism groups of various combinatorial or algebraic structures other than graphs: vector spaces over finite fields, automorphism groups of groups, maps, partitions (but not yet partially ordered sets in general or polytopes in particular). In most cases, the generic situation is that all but finitely many structures, perhaps with some restriction, have distinguishing number two. The reason is often related to the following observation of Russell and Sundaram [19], usually called the Motion Lemma: if every non-identity element of G moves at least m elements (has support of size at least m) and $m > 2\log_2(|G|)$, then $D(G) = 2$. For permutation groups, motion is called minimal degree and goes back to the 19th century. Again, a version of the Motion Lemma appeared 12 years earlier in [7].

There has also been interest in symmetry breaking for infinite permutation groups. Motion/minimal degree plays an important role here, just as in the finite case. In analogy to the Motion Lemma, we have the Infinite Motion Conjecture: for infinite, locally finite graphs (all vertices have finite valence), infinite motion of the automorphism group implies distinguishing number two. Certain classes of graphs automatically have infinite motion, such as leafless trees and certain products, and in these cases the conjecture is true. As with the Motion Lemma, the size of $Aut(\Gamma)$ also matters: the conjecture holds when $|Aut(\Gamma)|$ is countably infinite. This implies $D(G) = 2$ when G is the automorphism group of a map or a finitely generated group. It also hold for graphs with subexponential growth. The local finiteness condition (for permutation groups, one needs all point stabilizers to have finite orbits) is necessary: a modification of Cantor's back-and-forth proof that the rationals are the only linearly ordered infinite set without endpoints, gives a counterexample when valences are allowed to be countably infinite [15].

As with all work in infinite permutation groups, logic and topology play an important role. Here too, many results were found independently. For example some of the essentially topological results of Halin [10] from 1973 on the base size of an automorphism group were found previously by the logicians Kueker [14] (1968) and Reyes [18] (1970). In these cases the groups always act on countable sets. But, there are also numerous results on distinguishing numbers of groups acting on larger sets, in particular for graphs on uncountable vertex-sets, but these results still are more-or-less isolated, and the open problems deeply entrenched in set theory and logic.

As we have indicated, a larger issue here is that different communities of mathematicians studying the same idea, but with different language and notation and with no communication between the communities. Bailey and Cameron [6] cite distinguishing number as one of many cases where researchers in graph theory and permutation groups have not been talking to each other. For infinite permutation groups, the situation is exacerbated by topological and logical issues.

2 Aim of the workshop

The aim of the workshop was to bring together researchers in symmetry breaking, from both the graph theory and the permutation group communities. There has not been any other conference focussing on this topic, although a few other conferences have included talks or special sessions on the subject.

The workshop had three themes. One was applications to graph theory. Another is to apply symmetry breaking to other areas of mathematics besides graph theory: other discrete structures, group theory, logic, computer science. The third theme was to provide a coherent vision of symmetry breaking, developing tools and general structure theorems that can be used in a variety of contexts.

3 Recent Developments and Open Problems

One of the fundamental early results for symmetry breaking in graphs is due to Collins and Trenk [8]: if the finite graph Γ has maximum valence $\Delta(\Gamma) = d$, then $D(\Gamma) \leq d + 1$ with equality only for the complete graph K_{d+1}, the complete bipartite graph $K_{d,d}$ and the cycle C_5. For $d = 3$, this means if Γ is a "subcubic" connected graph other than K_4 or $K_{3,3}$, then either Γ is asymmetric (no nontrivial automorphism) or $D(\Gamma)$ is 2 or 3. As there are various reasons to think $D = 2$ is the generic situation for all graphs that are not asymmetric, one expects the subcubic graphs with $D(\Gamma) = 3$ to be classifiable in some way. A very recent paper [11] gives that classification. A subcubic graph Γ has $D(\Gamma) = 3$ if and only if it is the cube, the Petersen graph, or a balanced binary tree T_n to which "gadgets" have been added between sibling pairs of valence 1 vertices; in addition to the possibility of no gadget at all or a single edge, the three other gadgets are $K_{2,2}$, K_4 with an edge removed, and the cycle C_6 with two diagonals.

One problem with distinguishing is that a small modification in the local structure of a graph can lead to a big change in D: taking a single edge of an asymmetric graph with $\Delta(\Gamma) = d$ and replacing the middle third by $K_{2,d-1}$ increases $D(\Gamma)$ from 1 to $d-1$. There are various ways to restrict such structure:

1. require Γ to have large motion;
2. require Γ to be vertex or edge transitive;
3. require Γ to have large girth (no short cycles).

An immediate corollary of the above classification for subcubic graphs Γ is that $D(\Gamma) \leq 2$ whenever Γ has motion at least 3, or has girth at least 5, or is vertex transitive, with the exception of the cube and Petersen graph.

Question. What about $\Delta(\Gamma) > 3$?

Lehner and Verret [16] have obtained a classification for vertex transitive graphs with $\Delta(\Gamma) = 4$. Lehner, Pilśniak, and Stawiski have confirmed the Infinite Motion Conjecture for graphs of maximum valence 5. On the other hand, they warn that their results do not generalize to $\Delta(\Gamma) > 5$.

It is restriction (1) on motion that may have the most promise. Babai asked at the conference:

Question. Is there a function $f(d)$ such that if $\Delta(\Gamma) = d$, Γ is connected, and Γ has motion at least $f(d)$, then $D(\Gamma) = 2$ (possibly with known exceptions)?

Obviously, one might ask whether $f(d)$ is a polynomial, say quadratic. Note that this question subsumes the Infinite Motion Conjecture for infinite graphs with bounded valence[1].

There are various refinements and extensions of distinguishing number. When $D(G) = 2$, one can consider the *cost*, that is, the least number of times one color is used. For graphs, there is the *distinguishing chromatic number*, where the coloring is required to be proper, that is, is adjacent vertices get different colors. Again for graphs, there is the *edge-distinguishing number*, also called the *distinguishing index*, where the coloring is on edges instead of vertices. One interesting aspect of this is that if a graph has a hamiltonian path of length at least 7, then its edge-distinguishing number is two. Thus for vertex transitive graphs, there is a connection to Lovász's conjecture.

[1]In the meantime Imrich and Tucker showed that $f(d) = 2\lceil \log_2 d \rceil + 1$ for finite or infinite trees [12].

The other important recent development is László Babai's work on the Graph Isomorphism Problem using symmetry breaking ideas, like individualization of vertices. Although he was not on the original list of projected participants, the workshop was very fortunate that he agreed to attend and give a series of talks. These talks were the highlight of the workshop.

In this context it turned out that the cost, called *Boutin-Imrich cost* $BI(G)$ by Babai, was a rather useful in his investigations. He presented a cost-aware version of the Motion Lemma and asked, for example, whether $BI(G) = O(b(G))$ if $m(G) = \Omega(n)$, where $b(G)$ is the minimum size of a base of G and n the order of G.

4 Presentation Highlights

Each day of the workshop began with a morning of introductory lectures, presented by experts in the area. Since the audience included researchers from different fields, these talks were as general as possible, employing language, notation, and examples that are consistent and widely used. The remaining sessions were organized by topic: graphs, other combinatorial structures, permutation groups, refinements or extensions of distinguishing number.

The first day, Thomas Tucker began with an overview of symmetry breaking, with particular attention to motion/minimal degree as an overarching concept. He was followed by Florian Lehner, who talked about symmetry breaking for infinite graphs with bounded degree, and Gabriel Verret, who talk about the valence four case for vertex transitive graphs. The afternoon saw a series of shorter talks by Rafal Kalinowski on bounds for the distinguishing index, Mariusz Woźniak on distinguishing vertices with palettes, Mohammad Hadi Shekarriz on counting the number of distinguishing colorings, and Saeid Alikhani on symmetry breaking in various families of graphs.

The second day, László Babai gave a series of three talks on symmetry breaking as it relates to Graph Isomorphism [5, 3]. These lectures gave a vision that raised symmetry breaking from a small niche in graph theory to an overarching approach to the larger problems of graph theory and permutation groups. That afternoon, his student Bohdan Kivva gave a series of two lectures on coherent configurations, which can be viewed as highly regular colorings of the arcs of the directed complete graph; these arise naturally from the orbits of permutation groups on ordered pairs. The first [13] extended Babai's lower bound [4] on minimal degree for strongly regular graphs to primitive coherent configurations of rank 4. His second talk presented work by Xiaorui Sun and John Wilmes [21], also students of Babai, on bounds on base size for primitive coherent configurations. Both talks were algorithmic and both gave striking examples of computations that had been achieved before for permutation groups in a weaker form and only by using the Classification of Finite Simple Groups.

For the half-day of talks on the third day, Claude Laflamme applied symmetry breaking to homogeneous structures, Thomas Lachmann gave bounds of the cost of symmetry breaking for cubic vertex transitive graphs, Svenja Hüning addressed locally finite trees, and Sara Sabrina Zemljič discussed distinguishing Sierpinski products of graphs.

For the fourth day, Joy Morris viewed symmetry breaking in terms of edge-colorings of an oriented Cayley graph invariant under automorphisms of the graph. Luke Morgan and Scott Harper gave talks on their work semiprimitive permutation groups. In the afternoon, Marston Conder presented work by Verret, Lehner, Pablo Spiga and Primoz Potočnik on lower bounds on minimal motion for vertex transitive cubic graphs. Mark Ellingham presented a variation on symmetry breaking where one is allowed to interchange colors, in effect breaking symmetry with a partition, rather than a coloring. Ann Trenk and Karen Collins presented their work on distinguishing partially ordered sets. That evening participants presented various open problems.

Friday, Conder looked at restricting symmetry, rather than breaking all symmetry, with emphasis on examples from maps, such as chirality. Wilfried Imrich concluded the workshop with a talk on symmetry breaking for uncountable graphs.

5 Outcome of the Meeting

The purpose of the meeting was to raise symmetry breaking from a niche area of graph theory to a larger vision of symmetry breaking in permutation groups and combinatorics. It is clear the workshop succeeded in that purpose: indeed, fewer than one quarter of the talks could be considered graph-distinguishing.

A second purpose, was to bring together the different mathematical communities doing symmetry breaking, who have not, to this point, been communicating with each other. The list of participants alone guaranteed the success of the workshop. The organizers are grateful that so many were willing to attend, including the senior mathematicians who did not make

presentations: Chris Godsil, Richard Hammack, Alexander Hulpke, and Andrew Vince.

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