Special values of automorphic L-functions and associated p-adic L-functions

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1 Overview of the Field

The field of the subject area of this workshop was p-adic L-functions associated to various automorphic L-functions. The study of p-adic L-functions is one of the most important themes in p-adic number theory. This theme of the workshop was also related to various research areas, such as the theory of automorphic representations, p-adic Hodge theory, Shimura varieties etc. Finally, the p-adic L-function is also an important object for Iwasawa theory. According to the philosophy of Iwasawa Main Conjecture, the principal ideal of Iwasawa algebra generated by a p-adic L-function should be equal to the characteristic ideal of Selmer group. We covered some known cases of the construction of p-adic L-functions as well as such related topics.

The p-adic L-function is a p-adic counterpart of Hasse-Weil L-functions for various motives or various automorphic L-functions. The first p-adic L-function in the history appeared through the work by Kubota-Leopoldt in early 1960's, which was a p-adic analogue of Riemann zeta-function or Dirichlet L-function. Some time after this work, other constructions of Kubota-Leopoldt p-adic L-function were obtained by Mazur, Iwasawa, Coleman. These works brought us deeper understandings of the Kubota-Leopoldt p-adic L-function, as well as progress on Iwasawa Main Conjecture. After successful studies of Kubota-Leopoldt p-adic L-function, its totally real analogues by Barsky, Cassou-Nougès, Deligne-Ribet and CM analogues by Katz, people naturally thought of generalization to higher rank cases. Regarding the Kubota-Leopoldt case as a p-adic L-function for automorphic forms on GL(1), the next case was a p-adic L-function for elliptic modular cusp forms. Such constructions in elliptic modular situation were done by Manin, Amice-Vélu, Vishik, Mazur and a lot of others in early 1970's. Following pioneering works mentioned above, further generalizations to various higher rank situations were worked out by a lot of people through recent 40 years. The conjectural framework of the existence of p-adic L-functions which interpolate critical values was discussed by Coates, Perrin-Riou, Panchishkin, et al. Also, we had some progress on the construction of such cyclotomic one-variable p-adic L-functions, part of which is listed as follows:

(1) symmetric product of elliptic cusp forms (Coates, Hida, Schmidt, et al),

(2) Hilbert modular cusp forms and GL(2) over general number fields (Manin, Haran, et al),

(3) GL(n) and GL(n) × GL(n-1) under some conditions (Ash, Ginzburg, Schmidt, Januszewski, et al)

(4) standard representation of GSp(4) (Panchishkin et al)

(5) Generalization of Katz method in unitary case (Eishen, Harris, Skinner)

A lot of variants of were studied and constructed. Anticyclotomic variants of p-adic L-functions interpolating anticyclotomic twists in place of cyclotomic twists were studied by Bertolini, Darmon, Iovita, Spiess etc. Supersingular variants were also studied, first by Perrin-Riou, which was followed by outstanding plus-minus formulation of Pollack as well as Sprung et al. Extensions of p-adic L-functions to more general deformations were studied first by Greenberg, Stevens, Hida, Tilouine, which was also followed by more recent works by Bellaiche, Dimitrov, Ochiai et al.

In view of these situations, the first objective of this workshop is to look into the detail concerned with one-variable cyclotomic p-adic L-functions such as:

(i) Conjectural framework of the expected p-adic L-functions after Coates-Perrin-Riou, Panchichkine, etc

(ii) Some essential ingredients to construct p-adic L-functions (distribution properties, boundedness of denominators, congruences between special values, non-canonical choices of complex periods, tec)

(iii) Some known examples of automorphic L-functions for which the (partial) construction of the p-adic L-function has been done (including the cases (1) -(5) above).

2 Recent Developments and Open Problems

During the last decades, there are also some progress on automorphic representations and automorphic L-functions. Among such progress, it will quite useful to review:

(i) Actual state of calculating local integration for associated automorphic representations (Oda, Ishi, Sun et al).

(ii) Actual state of understanding for the periods for critical values of automorphic L-functions (Shimura, Deligne, Harris, et al).

(iii) Automorphic frame work of (relative) trace formula to understand special values of automorphic L-functions.

(iv) Delicate normalization of optimal complex periods (Stevens, Vassal et al)

Thanks to these recent advances on autormorphic side, it is quite timely to put together both experienced specialists on the basic of p-adic L-functions and specialists on automorphic L-functions. Putting knowledge and specialists of these two different background together, there were some active interactions between the participants. We expect that new developments on p-adic L-functions and automorphic L-functions will grow up from what we discussed in this workshop.

3 Overview of the workshop

Here is a brief overview of the workshop.

October 1 (Mon)

Speaker: Tadashi Ochiai (Osaka University)
Title: Overview Introduction (Lecture Talk)
In this talk, he formulated the conjectural framework of the conjecture on the existence of the cyclotomic p-adic L-function. First, he introduced Hasse-Weil L-function for motives. Then he discussed the critical condition of motives and the algebraicity of special values as well as some fundamental conditions which are essential to discuss p-adic properties of special values. After, he introduced some basic languages of Iwasawa algebra and measures, he finished by stating the conjecture on the existence of of the cyclotomic p-adic

L-function which was formulated thanks to Coates–Perrin-Riou and some others. This talk also contained a brief plan of the warkshop.

Speaker: Vinayak Vatsal (University of British Columbia)
Title: Cyclotomic p-adic L-function for $\mathrm{GL}(2)_{\mathbb{Q}}$ —Modular Symbol Construction (Lecture Talk)
He described the now-classical method of constructing p-adic L-functions for $GL(2, \mathbb{Q})$ based on modular symbols. The ideas of this method go back to Manin, Mazur, and Swinnerton-Dyer, as well as many others. He also pointed out some subtle points on the periods when modular forms in question is congruent to an Eisenstein series modulo p.

Speaker: Jacques Tilouine (University of Paris 13)
Title: Integral period relations and the Bloch–Kato formula for quadratic twists of the adjoint L-function
He talked about his joint work with E. Urban (related references are [13], [14]). They prove under mild assumptions Integral Period Relations for the quadratic base change of a modular form and we compute the relative congruence number in terms of the value at $s = 1$ of a quadratic twist of the adjoint L-function. This proves a conjecture of Hida (1999). It also implies, under standard Taylor-Wiles type assumptions, in the real quadratic case, resp. in the imaginary quadratic case, the Bloch-Kato formula, resp. an analogue of it. This analogue would be the exact conjectural Bloch-Kato formula if a Bianchi period defined by E. Urban in his thesis could be related to a Bloch-Kato-Beilinson regulator.

Speaker: Shin-ichi Kobayashi (Kyusyu University)
Title: Cyclotomic p-adic L-function for $\mathrm{GL}(2)_{\mathbb{Q}}$—Rankin-Selberg Construction (Lecture Talk)
He explained the classical construction of the cyclotomic p-adic L-function of elliptic modular forms by the Rankin-Selberg method. This gives another construction to the p-adic L-function which appeared in Vatsal's talk.

Speaker: Masaaki Furusawa (Osaka City University)
Title: Refined global Gross-Prasad conjecture on special Bessel periods and Böecherer's conjecture
He talked about a joint work with Kazuki Morimoto. First he gave an outline of the proof of the refined global Gross-Parasd conjecture for special Bessel periods on SO(2n+1). Then he discussed about its consequence to Böecherer's conjecture concerning the Fourier coefficients of the Siegel cusp forms of degree two which are Hecke eigenforms.

October 2 (Tue)

Speaker: Ken-ichi Namikawa (Tokyo Denki University)
Title: p-adic L-function for $\mathrm{GL}(n+1) \times \mathrm{GL}(n)$ I (Lecture series)
Abstract: A construction of p-adic L-functions for $\mathrm{GL}(2)$ via the modular symbol method is reviewed in this talk. I will summarize some technical points of the construction comparing with the works of F. Januszewski on p-adic L-functions for $\mathrm{GL}(n+1) \times \mathrm{GL}(n)$. In particular, the behavior under the Tate twists is emphasized in the talk, since it is the most important new ingredient in Januszewski's recent preprint.

Speaker: Fabian Januszewski (Karlsruher Institut für Technologie)
Title: p-adic L-function for $\mathrm{GL}(n+1) \times \mathrm{GL}(n)$ II (Lecture series)
(subtitle: Rankin-Selberg L-functions and their special values)
In this talk, he gave a quick overview of the analytic theory of Rankin-Selberg L-functions for GL(n+1) x GL(n) due to Jacquet, Piatetski-Shapiro and Shalika, which are the L-functions of interest in this lecture series. Subsequently he introduced the modular symbol of Schmidt, Kazhdan-Mazur-Schmidt, Kasten-Schmidt, Raghuram and himself and explained its relation to special values of Rankin-Selberg L-functions and the archimedean non-vanishing hypothesis established by Sun.

Speaker: Haruzo Hida (University of California, Los Angeles)
Title: Galois Deformation Ring and Base Change to a Quadratic Field
Let us consider the universal minimal p-ordinary deformation r_T into $GL(2, T)$ (for a prime $p \geq 5$) of a

modulo p induced representation from a quadratic field F. For almost all primes p split in F, he described how to determine T as an algebra over the weight Iwasawa algebra Λ as an extension of degree $1, 2, 3$. This implies that the Pontryagin dual of the adjoint Selmer group of r_T is isomorphic to $\Lambda/(L_p)$ as Λ-modules for an explicit power series L_p.

Speaker: Alexei Pantchichkine (Université Grenoble Alpes)
Title: Constructions of p-adic L-functions and admissible measures for Hermitian modular forms
For a prime p and a positive integer n, the standard zeta function $L_F(s)$ is considered, attached to an Hermitian modular form $F = \sum_H A(H)q^H$ on the Hermitian upper half plane $\mathcal{H}_m$ of degree n, where H runs through semi-integral positive definite Hermitian matrices of degree n, i.e. $H \in \Lambda_m(\mathcal{O})$ over the integers $\mathcal{O}$ of an imaginary quadratic field K, where $q^H = \exp(2\pi i \mathrm{Tr}(HZ))$. Analytic p-adic continuation of their zeta functions constructed by A.Bouganis in the ordinary case, is extended to the admissible case via growing p-adic measures. Previously this problem was solved for the Siegel modular forms. Main result is stated in terms of the Hodge polygon $P_H(t) : [0, d] \to \mathbb{R}$ and the Newton polygon $P_N(t) = P_{N,p}(t) : [0, d] \to \mathbb{R}$ of the zeta function $L_F(s)$ of degree $d = 4n$. Main theorem gives a p-adic analytic interpolation of the L values in the form of certain integrals with respect to Mazur-type measures.

Speaker: Siegfried Böcherer (University of Mannheim)
Title: Doubling method and exterior twists (a survey)
Abstract: We give a survey on our work with C.Schmidt on p-adic interpolation for standard L-functions for Siegel modular forms. We emphasize that there is a strong analogy between exterior twists of Eisenstein series and appliction of certain differential operators. The case of triple product L-functions for GL(2) can be handled in a similar way.

October 3 (Wed)

Speaker: Ellen Eischen (University of Oregon)
Title: p-adic L-functions obtained by Eisenstein measure for unitary group I (Lecture series)
This lecture covered the doubling method for unitary groups and the resulting integral representation of the standard L-functions for cuspidal representations of unitary groups.

Speaker: Zheng Liu (McGill University)
Title: p-adic L-functions obtained by Eisenstein measure for unitary group II (Lecture series)
This lecture focused on the case of the unitary group U(1) (so U(1)xU(1) inside U(1,1)) and will explain how Katz's p-adic L-function for a CM fields is a special case of the doubling method constructions.

October 4 (Thu)

Fabian Januszewski (Karlsruher Institut für Technologie)
Title: p-adic L-function for $\mathrm{GL}(n+1) \times \mathrm{GL}(n)$ III (Lecture series)
(subtitle: Rankin-Selberg L-functions: p-adic L-functions attached to nearly ordinary cohomology classes)
In the third talk of this lecture series, he defined p-adic L-functions attached to nearly ordinary cohomology classes for GL(n+1) x GL(n) and sketched a proof of the Manin congruences and the functional equation.

Speaker: Fabian Januszewski (Karlsruher Institut für Technologie)
Title: p-adic L-function for $\mathrm{GL}(n+1) \times \mathrm{GL}(n)$ IV (Lecture series)
(subtitle: Rankin-Selberg L-functions: Interpolation formulae)
In the last talk of this lecture series, he examined the relation between the p-adic L-functions constructed in the previous lecture and the complex analytic p-adic L-functions of Jacquet, Piatetski-Shapiro and Shalika. This involves p-stabilization in principal series representations, a careful study of local zeta integrals and non-vanishing properties of certain Whittaker vectors.

Speaker: Henri Darmon (McGill University)
Title: p-adic periods arising from Garrett-Rankin triple products, their tame refinements,

and Venkatesh's derived Hecke operators
He strived to make a reasonably self-contained lecture based on an earlier paper by Harris and Venkatesh that just appeared in Exp. Math.

Speaker: Lennart Gehrmann (Universität Duisburg–Essen)
Title: p-adic L-function of $\mathrm{GL}(2n)$ via method of p-adic representation
The p-adic L-functions for cohomological cuspidal automorphic representations of GL(2n) were first constructed by Ash and Ginzburg in the case of trivial coefficients. He discussed a new construction, which works for arbitrary coefficient systems. The construction relies on the representation theory of p-adic groups as well as properties of the cohomology of p-arithmetic groups. This is a generalization of Spiess' work on the GL(2)-case.

Speaker: Romyar Sharifi (University of California, Los Angeles)
Title: Modular symbols in Iwasawa theory
He gave an overview of in-part conjectural relationships between modular symbols in Eisenstein quotients of homology groups of locally symmetric spaces and cohomological operations on special units such as cyclotomic units. Some focus was given to aspects related to p-adic L-functions and special values. The talk was based in part on joint work with Takako Fukaya and Kazuya Kato. Especially he explained Beilinson-Kato Euler systems and two-variable Coleman map associated to nearly ordinary Hida deformations.

October 5 (Fri)

Speaker: Christopher Skinner (Princeton University) or Zheng Liu (McGill University)
Title: p-adic L-functions obtained by Eisenstein measure for unitary group III (Lecture series)
This lecture reviewed some general facts about p-adic L-functions and the Eisenstein measure and explained the strategy for constructing one from the other.

Speaker: Christopher Skinner (Princeton University),
Title: p-adic L-functions obtained by Eisenstein measure for unitary group IV (Lecture series)
This lecture covered some of the details of the constructions and especially the choice of data and calculations at the p-adic places.

4 Scientific Progress Made and Outcome of the Meeting

For those who work on the construction and the study of p-adic L-functions, this workshop provides a technical and conceptual art of generalizing the subject. For those who are not necessarily specialists of p-adic L-functions but who devote themselves on basic of automorphic representations, this workshop hopefully provides a strong new motivation to generalize some of their very technical technical works in view of applications to p-adic L-functions.

Here are some comments sent from participants.

1. The conference was very very beneficial for me. I am in the automorphic side and the first introductory talk and both the series lectures helped me to have better understanding on p-adic L-functions. On the other hand, the geometric talks in Thursday afternoon were rather interesting. The talks were difficult for beginners, but l got to know the hot topics in this area. All the speakers had high standards and the place was wonderful. It was a great pleasure to visit Mexico in my life. All in all, the conference was the most beneficial, enjoyable and memorable. Once again, I should like to thank the organizers for the invitation to such the great event this year.

2. The main feature of this conference, in my opinion, is that the two main series of four lectures were remarkably constructed, building from general to particular, therefore giving a clear overview of the two main methods to construct p-adic L-functions. Moreover, other talks presented variant of these methods, putting in perspective the approaches according to the tools available in various situations. I

must say that it gave me a clearer vision of the subject, which I knew well, say, around 2000, but which has considerably evolved in the last two decades. The general level of the talks (and posters) was very high, but nevertheless, a distinct effort for clarity was present in most talks.

3. I was always interested in the construction of p-adic L-functions for unitary groups and in the construction of p-adic L-functions for $GL(n) \times GL(n+1)$. So it was a very fruitful that I could attend the lecture series on these two subjects this time. I had a good occasion to think about the development of this area in near future. I also had a reflection on what I should study from now on. Hence I believe that it was an interesting workshop to participants.

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