Refined global Gross-Prasad conjecture on special Bessel periods and Böcherer's conjecture

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Plan of today's talk

1. **Introduction**
2. **Gross-Prasad conjecture**
3. **Refined Gross-Prasad conjecture**
4. **Böcherer's conjecture**

Today's talk is a report on two of our papers:

- *On special Bessel periods and the Gross-Prasad conjecture for* $\mathrm{SO}(2n+1) \times \mathrm{SO}(2)$. Math. Ann. **368** (2017), 561–586.

- *Refined global Gross-Prasad conjecture on special Bessel periods and Böcherer's conjecture.* arXiv:1611.05567v4 (November 27, 2017), to be revised.

Notation

- F: a number field.
- E: a quadratic extension of F.
- χ_E: quadratic character of $\mathbb{A}^\times / F^\times$ corresponding to E.

- All global L-functions are complete L-functions.
- $\xi_F = \prod_{v:\ all} \zeta_{F_v}(s)$: complete Dedekind zeta of F.

- $(V, \langle\ ,\ \rangle)$: a quadratic space such that $\dim V = 2n+1 \quad (n \geq 2)$,

$$V = \mathbb{H}^{n-1} \oplus L \quad \text{(orthogonal sum) with } \mathbb{H}\text{: hyperbolic plane}$$

and

$$\dim L = 3, \quad L \supset \left(E, \mathrm{N}_{E/F}\right) \text{ as quadratic spaces.}$$

- $\mathcal{G}_n :=$ F-isomorphism classes of $\mathrm{SO}(V)$ for such V.
- We identify $\mathrm{SO}(V)$ with its F-isomorphism class in $\mathcal{G}_n$.
- We specify $\mathbb{G} = \mathbb{G}_n = \mathrm{SO}(\mathbb{V}_n) \in \mathcal{G}_n$ to denote the *split* one.

Bessel subgroup

For $G = \mathrm{SO}(V) \in \mathcal{G}_n$, we have $\mathrm{SO}(E) \subset G$.
But $\mathrm{SO}(E)$ is "too small."

Definition (Bessel subgroup)

Taking a certain unipotent subgroup S, a Bessel subgroup R_E is defined by

$$R_E := D_E \ltimes S \quad \textit{with } D_E := \mathrm{SO}(E),$$

which is contained in a maximal parabolic subgroup of G whose Levi component is $\mathrm{GL}(n-1) \times \mathrm{SO}(L)$.

For a non-trivial character $\psi : \mathbb{A}/F \to \mathbb{C}^\times$, we have a character on $S(\mathbb{A})$ also denoted by ψ, by abuse of notation, which is stable under the conjugate action of $D_E(\mathbb{A})$.

Bessel period & Special Bessel period

Definition (Bessel period)

Let τ be a character of $D_E(\mathbb{A})/D_E(F)$. Note: $D_E \simeq E^\times/F^\times$.
Then for an automorphic form f on $\mathrm{SO}(V,\mathbb{A})$, $B_{E,\tau,\psi}(f)$, a Bessel period of type (E,τ,ψ) is defined by

$$B_{E,\tau,\psi}(f) := \int_{D_E(F)\backslash D_E(\mathbb{A})} \int_{S(F)\backslash S(\mathbb{A})} f(ts)\,\tau^{-1}(t)\,\psi^{-1}(s)\,dt\,ds.$$

Definition (Special Bessel period)

When τ is trivial, the Bessel period of type $(E,1,\psi)$ is called the special Bessel period of type E and denoted by $B_E(f)$, i.e.

$$B_E(f) := \int_{D_E(F)\backslash D_E(\mathbb{A})} \int_{S(F)\backslash S(\mathbb{A})} f(ts)\,\psi^{-1}(s)\,dt\,ds.$$

Gross-Prasad conjecture for special Bessel periods

Theorem 1 (F & Morimoto, Math. Ann.)

- $\pi = \otimes_v \pi_v$: *an irreducible cuspidal automorphic representation of* $G(\mathbb{A})$ *for* $G \in \mathcal{G}_n$*. Let* V_π *be its space of automorphic forms.*
- *Assume that a local component* π_w *at some finite place* w *is generic.*

Suppose that $B_E \not\equiv 0$ *on* V_π*. Then* $L(1/2, \pi)\, L(1/2, \pi \times \chi_E) \neq 0$*.*
Moreover:

- $\exists\, \pi^\circ$: *globally generic irreducible cuspidal automorphic representation of* $\mathbb{G}(\mathbb{A})$ *which is nearly equivalent to* π*, i.e.* $\pi_v^\circ \simeq \pi_v$ *for almost all* v*.*

Ginzburg, Jiang & Rallis: more general theorem assuming the global genericity of π.
Jiang & Zhang: recently proved a more general theorem assuming the extension of Arthur's result to the non quasi-split case.

Theorem 1 follows from the following theorem.

Theorem 2 (F & Morimoto, Math. Ann.)

- π: *an irreducible cuspidal automorphic representation of* $G(\mathbb{A})$ *with* $G \in \mathcal{G}_n$. *Suppose that* $B_E \not\equiv 0$ *on* V_π.

Suppose moreover that:

- $\sigma := \Theta_n(\pi, \psi)$: *theta lift of* π *from* G *to* $\widetilde{\mathrm{Sp}}_n(\mathbb{A})$ *with respect to* ψ,
- $\mathit{\Pi} := \Theta_{\mathbb{V}_n}(\sigma, \psi^{-\lambda})$: *theta lift of* σ *to* $\mathbb{G}_n(\mathbb{A})$ *with respect to* $\psi^{-\lambda}$

are both non-zero and cuspidal. Note $E = F\left(\sqrt{-\lambda}\right)$ *and* $\psi^a(x) = \psi(ax)$. *Then we have:*

$$L(1/2, \pi)\, L(1/2, \pi \times \chi_E) \neq 0$$

and $\exists\, \pi^\circ$: *globally generic irreducible cuspidal automorphic representation of* $\mathbb{G}_n(\mathbb{A})$ *nearly equivalent to* π.

Remark

This line of thought concerning special Bessel periods goes back to Waldspurger $(n = 1)$ *and Piatetski-Shapiro & Soudry* $(n = 2)$.

Proof of Theorem 2

Remark (Continued)

- *Theorem 2 seems to have been known to experts for a long time.*
- *Now it is possible to give a rigorous proof thanks to essential contributions made towards theta correspondence over the years.*
- *Among them, the most notable ones are Adams & Barbasch (arch.), Gan & Savin (non-arch.), Gan & Takeda (Howe duality), Jiang & Soudry ($\mathrm{SO}_{2n+1} \leftrightarrow \widetilde{\mathrm{Sp}}_n$), Yamana (L-functions, L-values and theta).*

Proof of Theorem 2

Recall that ($n = 2$ by Piatetski-Shapiro & Soudry, $n \geq 2$ by F.):

$$B_E \not\equiv 0 \text{ on } V_\pi \iff \sigma = \Theta_n(\pi, \psi), \text{ theta lift to } \widetilde{\mathrm{Sp}}_n(\mathbb{A}), \text{ is } \psi_\lambda\text{-generic.}$$

$$\sigma \text{ is } \psi_\lambda\text{-generic} \iff \Pi = \Theta_{\mathbb{V}_n}\left(\sigma, \psi^{-\lambda}\right), \text{ theta lift to } \mathbb{G}_n(\mathbb{A}), \text{ is generic.}$$

Proof of Theorem 2 (continued)

Here the generic character ψ_λ *for* $\widetilde{\mathrm{Sp}}_n(\mathbb{A})$ *is defined by*

$$\psi_\lambda\left[\begin{pmatrix} u & \\ & {}^t u^{-1}\end{pmatrix}\begin{pmatrix} 1_n & S \\ & 1_n\end{pmatrix}\right] = \psi\left(u_{1,2} + \cdots + u_{n-1,n} + \frac{\lambda}{2} s_{n,n}\right).$$

Then:

1. *Generic representation* $\Pi \otimes \chi_E$ *of* $\mathbb{G}(\mathbb{A})$ *is nearly equivalent to* π.
2. *We have* $\sigma = \Theta_n(\Pi, \psi^\lambda)$ *by Jiang & Soudry.*
3. *Since* $\Theta_n(\pi, \psi)$ *and* $\Theta_n(\Pi, \psi^\lambda)$ *are both non-zero and cuspidal, we have* $L(1/2, \pi) \neq 0$ *and* $L(1/2, \Pi) \neq 0$ *by Yamana.*
4. *Finally we may show that* $L(s, \Pi_v) = L(s, \pi_v \times \chi_v)$ *for any place* v *using Adams & Barbasch for archimedean and Gan & Savin for non-archimedean.*

Q.E.D. of Theorem 2

Refined Gross-Prasad conjecture

- Ichino & Ikeda: formulated a conjectural *explicit central L-value formula* by refining the Gross-Prasad conjecture in the co-dimension one case.
- Liu: succeeded in formulating the conjectural explicit central L-value formula in the arbitrary co-dimension case.

Set Up

- *π: an irreducible tempered cuspidal automorphic representation of $G(\mathbb{A})$ with $G \in \mathcal{G}_n$.*
- *All global measures are Tamagawa measures.*
- $\langle \phi_1, \phi_2 \rangle := \int_{G(F)\backslash G(\mathbb{A})} \phi_1(g)\, \overline{\phi_2(g)}\, dg$, *Petersson product on V_π.*
- $\langle\ ,\ \rangle_v$*: G_v-invariant Hermitian inner product on V_{π_v} such that*

$$\langle \phi_1, \phi_2 \rangle = \prod_v \langle \phi_{1,v}, \phi_{2,v} \rangle_v \quad \textit{for } \phi_i = \otimes_v \phi_{i,v} \in V_\pi .$$

Set Up (continued)

- dg_v: *measure on* G_v *such that* $\mathrm{Vol}\,(K_v, dg_v) = 1$ *for almost all* v.
- dt_v: *similarly taken measure on* $D_{E,v} := \mathrm{SO}\,(E)_v$.
- *Haar measure constants:* $dg = C_G \cdot \prod_v dg_v$, $dt = C_E \cdot \prod_v dt_v$.
- *Local integral* $\alpha_v\,(\phi_v, \phi'_v)$:

$$\alpha_v\,(\phi_v, \phi'_v) := \int_{D_{E,v}} \int_{S_v}^{\mathrm{st}} \left\langle \pi_v\,(s_v t_v)\,\phi_v, \phi'_v \right\rangle_v \psi_v^{-1}\,(s)\; dt_v\, ds_v.$$

  Here $\int_{S_v}^{\mathrm{st}}$ *denotes the stable integration on* S_v *defined by Liu.*

- *Liu showed that when* v *is "good," we have*

$$\alpha_v\,(\phi_v, \phi'_v) = \frac{L\left(\frac{1}{2}, \pi_v\right) L\left(\frac{1}{2}, \pi_v \times \chi_{E,v}\right) \prod_{j=1}^{n} \zeta_{F_v}\,(2j)}{L\,(1, \pi_v, \mathrm{Ad})\, L\,(1, \chi_{E,v})}.$$

Theorem 3 (F & Morimoto, arXiv)

- *F: totally real number field.*
- *$\pi = \otimes_v \pi_v$: irreducible cuspidal tempered automorphic representation of $G(\mathbb{A})$ for $G \in \mathcal{G}_n$.*
- *At any archimedean place v, π_v is a discrete series representation.*

Suppose that $B_E \not\equiv 0$ on V_π.
Then:

- *For any v, $\exists\, \phi'_v \in V_{\pi_v}$: $K_{G,v}$-finite vector such that $\alpha_v(\phi'_v, \phi'_v) \neq 0$.*
- *For any non-zero $\phi \in V_\pi$ of the form $\phi = \otimes_v \phi_v$, we have*

$$\frac{\left|B_E(\phi)\right|^2}{\langle \phi, \phi \rangle} = 2^{-\ell} C_E \times \frac{L\left(\frac{1}{2}, \pi\right) L\left(\frac{1}{2}, \pi \times \chi_E\right) \prod_{j=1}^{n} \xi_F(2j)}{L(1, \pi, \mathrm{Ad})\, L(1, \chi_E)} \cdot \prod_v \frac{\alpha_v^\natural(\phi_v, \phi_v)}{\langle \phi_v, \phi_v \rangle}.$$

(Recall that all L-functions are complete L-functions.)

(Theorem 3 continued) Here

$$\alpha_v^\natural(\phi_v, \phi_v) := \frac{L(1, \pi_v, \mathrm{Ad})\, L(1, \chi_{E,v})}{L(1/2, \pi_v)\, L(1/2, \pi_v \times \chi_{E,v}) \prod_{j=1}^n \zeta_{F_v}(2j)} \cdot \alpha_v(\phi_v, \phi_v)$$

and hence $\frac{\alpha_v^\natural(\phi_v,\phi_v)}{\langle \phi_v,\phi_v\rangle_v} = 1$ *for almost all* v.

- π *has a weak lift* Π *to* $\mathrm{GL}_{2n}(\mathbb{A})$, *i.e.* $\Pi = \otimes_v \Pi_v$ *is an irreducible automorphic representation of* $\mathrm{GL}_{2n}(\mathbb{A})$ *such that* Π_v *is a local Langlands lift of* π_v *at all archimedean and almost all non-archimedean* v. *Then* Π *is of the form* $\Pi = \boxplus_{i=1}^{\ell} \pi_i$ *(isobaric sum) such that*
  - π_i: *irreducible cuspidal automorphic representation of* $\mathrm{GL}_{2n_i}(\mathbb{A})$ *such that* $L(s, \pi_i, \wedge^2)$ *has a pole at* $s = 1$, $\sum_{i=1}^k n_i = n$, $\pi_i \not\simeq \pi_j$ $(i \neq j)$.

  (Indeed the existence of such Π *readily follows from Theorem 1.)*

When $n = 2$, Theorem 3 has been proved by Liu for endoscopic Yoshida lifts and by Corbett for non-endoscopic Yoshida lifts.

Skeleton of the proof of Theorem 3: ($A \underset{\text{a.a.}}{=} B$ implies that $A = B$ up to multiplication by a product of finitely many local factors.)

1. Global pull-back formula of Bessel periods by F.:

$$W(\tilde{\phi}; \psi_\lambda) \underset{\text{a.a.}}{=} C_G C_E^{-1} \cdot B_E(\phi) \quad \text{where } \tilde{\phi} := \theta_\psi^\varphi(\phi).$$

2. Explicit formula for metaplectic Whittaker periods by Lapid-Mao:

$$\frac{|W(\tilde{\phi}; \psi_\lambda)|^2}{< \tilde{\phi}, \tilde{\phi} >} \underset{\text{a.a.}}{=} 2^{-\ell} \cdot \frac{L(1/2, \pi \times \chi_E) \prod_{j=1}^n \xi_F(2j)}{L(1, \pi, \mathrm{Ad})}.$$

3. Precise Rallis inner product formula by Gan-Takeda:

$$\frac{< \tilde{\phi}, \tilde{\phi} >}{< \phi, \phi >} \underset{\text{a.a.}}{=} C_G \cdot \frac{L(1/2, \pi)}{\prod_{j=1}^n \xi_F(2j)}.$$

$\Longrightarrow$ We are reduced to proving a pull-back formula for the local metaplectic Whittaker pairing.

Böcherer's conjecture

Recall: $\mathbb{G}_2 = \mathrm{SO}(3,2) \simeq \mathrm{PGSp}(2)$.

- $k_1 \geq k_2 \geq 3$, $k_1 \equiv k_2 \pmod 2$.
- $\varrho := \mathrm{Sym}^{k_1-k_2} \otimes \det^{k_2}$ *and* V_ϱ *its space.*
- *A holomorphic function* $f : \mathfrak{H}_2 \to V_\varrho$ *is a Siegel cusp form of degree 2 of weight* ρ *with respect to* $\mathrm{Sp}_2(\mathbb{Z})$ *when*

$$f(\gamma\langle Z\rangle) = \varrho(CZ+D)\, f(Z) \ \text{ for } Z \in \mathfrak{H}_2,\ \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_2(\mathbb{Z})$$

and it has the Fourier expansion:

$$f(Z) = \sum_{T>0} a(T,f) \exp\left[2\pi\sqrt{-1}\,\mathrm{Tr}(TZ)\right], \ \ a(T,f) \in V_\varrho$$

where $T = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$, $a, b, c \in \mathbb{Z}$ *and* T *is positive definite.*

Böcherer's conjecture

- E: *imaginary* quadratic field, D_E: discriminant of E, h_E: class number of E.
- $-d_E$: square free integer such that $E = \mathbb{Q}\left(\sqrt{-d_E}\right)$.
- $S_E := \begin{pmatrix} 1 & \mathrm{Re}(\delta) \\ \mathrm{Re}(\delta) & \delta\bar{\delta} \end{pmatrix}$ where $\delta = \begin{cases} \sqrt{-d_E} & \text{if } -d_E \not\equiv 1 \pmod 4; \\ \frac{1+\sqrt{-d_E}}{2} & \text{if } -d_E \equiv 1 \pmod 4. \end{cases}$
- $T_E := \left\{ g \in \mathrm{GL}_2 \mid \det(g)^{-1} \cdot {}^t g\, S_E\, g = S_E \right\}$. Note: $T_E \simeq E^\times$.
- $\{t_i\}_{1 \le i \le h_E}$: representatives of

$$T_E(\mathbb{Q}) \backslash T_E(\mathbb{A}) / T_E(\mathbb{R}) \prod_{p<\infty} \left(T_E(\mathbb{Q}_p) \cap \mathrm{GL}_2(\mathbb{Z}_p) \right)$$

  such that $t_i \in \prod_{p<\infty} T_E(\mathbb{Q}_p)$ and let us write $t_i = \gamma_i m_i k_i$ where $\gamma_i \in \mathrm{GL}_2(\mathbb{Q})$, $m_i \in \mathrm{GL}_2^+(\mathbb{R})$, $k_i \in \mathrm{GL}_2(\mathbb{Z}_p)$.
- $S_i := \det(\gamma_i)^{-1} \cdot {}^t\gamma_i S_E \gamma_i$ for $1 \le i \le h_E$.

Definition (Special Bessel model in the Siegel modular setting)

For a Siegel cusp form $\varPhi$ of degree 2 of weight ϱ with respect to $\mathrm{Sp}_2(\mathbb{Z})$ which is a Hecke eigenform, let w_E be the number of roots of 1 in E and

$$B(\varPhi; E) := \frac{1}{w_E} \sum_{i=1}^{h_E} \varrho(\gamma_i)\left[a(S_i, \varPhi)\right].$$

Theorem 4 (F & Morimoto, Math. Ann.)

Let $\varPhi$ be a Siegel cusp form of degree 2 with respect to $\mathrm{Sp}_2(\mathbb{Z})$ of weight ρ, which is a Hecke eigenform.
Then

$$B(\varPhi; E) \neq 0 \iff L\left(\frac{1}{2}, \pi(\varPhi)\right) \cdot L\left(\frac{1}{2}, \pi(\varPhi) \times \chi_E\right) \neq 0$$

where $\pi(\varPhi)$ is the cuspidal representation of $\mathrm{PGSp}_2(\mathbb{A})$ attached to $\varPhi$.

Böcherer's conjecture

Conjecture (Böcherer (circa 1986, before Gross-Prasad))

Suppose that $k_1 = k_2 = k$, i.e. Φ is scalar valued.
Then there exists a constant C_Φ which depends only on Φ such that, for any imaginary quadratic field E,
we have

$$L\left(1/2, \pi\left(\Phi\right) \times \chi_E\right) = C_\Phi \cdot \left|D_E\right|^{-k+1} \cdot \left|B\left(\Phi; E\right)\right|^2.$$

Remark

- *Böcherer did not speculate the nature of the constant C_Φ.*
- *Böcherer verified the conjecture for Saito-Kurokawa lifts.*
- *Explicit formulas of $B\left(\Phi; E\right)$ for Yoshida lifts have been obtained by Böcherer & Schulze-Pillot, Böcherer, Dummigan & Schulze-Pillot and Hsieh & Namikawa.*

Explicit refinement of Böcherer's conjecture

Dickson, Pitale, Saha & Schmidt (arXiv:1512.07204) showed that Refined Gross-Prasad conjecture for Bessel periods on $\mathrm{SO}(5)$ implies *Böcherer's conjecture with the constant C_Φ explicitly determined.*

Thus:

Our Theorem 3, together with Dickson et al., yields the explicit refinement of Böcherer's conjecture.

Theorem 5 (F & Morimoto, arXiv)

Suppose that $k_1 = k_2 = k$, i.e. Φ is scalar valued. Let Φ be a Siegel cusp form of degree 2 of weight k with respect to $\mathrm{Sp}_2(\mathbb{Z})$, which is a Hecke eigenform. Suppose that Φ is not a Saito-Kurokawa lift.

Then we have

$$\frac{|B(\Phi; E)|^2}{\langle \Phi, \Phi \rangle} = |D_E|^{k-1} \cdot 2^{2k-5} \cdot \frac{L\left(\frac{1}{2}, \pi(\Phi)\right) L\left(\frac{1}{2}, \pi(\Phi) \times \chi_E\right)}{L(1, \pi(\Phi), \mathrm{Ad})}.$$

More generally:

- *N: odd square free integer such that $\left(\frac{D_E}{p}\right) = -1$ for $\forall p | N$.*
- *Φ: Siegel cusp form of degree 2 of weight k with respect to $\Gamma_0^{(2)}(N)$ which is a Hecke eigenform but not a Saito-Kurokawa lift.*

Then:

$$\frac{|B(\Phi; E)|^2}{\langle \Phi, \Phi \rangle} = |D_E|^{k-1} \cdot 2^{2k-5-c} \cdot \prod_{p|N} J_p \cdot \frac{L(1/2, \pi(\Phi)) L(1/2, \pi(\Phi) \times \chi_E)}{L(1, \pi(\Phi), \mathrm{Ad})}.$$

Here $c = 1$ or 0 depending on whether Φ is a Yoshida lift or not, and

$$J_p = \begin{cases} \left(1 + p^{-2}\right)\left(1 + p^{-1}\right) & \text{if } \pi(\Phi)_p \text{ is of type IIIa;} \\ 2\left(1 + p^{-2}\right)\left(1 + p^{-1}\right) & \text{if } \pi(\Phi)_p \text{ is of type VIb;} \\ 0 & \text{otherwise.} \end{cases}$$

Remark

- *"type" refers to the representation types in Roberts & Schimidt.*
- *Work in progress: extensions to* $\begin{cases} \textit{non-special Bessel model case}; \\ \textit{when } \Phi \textit{ is vector valued}; \\ \textit{when } k = 2. \end{cases}$

We mention one of the immediate consequences of Theorem 5.

Theorem 6 (Algebraicity of central values of spinor L-functions)

- *Φ: Siegel cusp form of degree 2 of weight k with respect to $\mathrm{Sp}_2(\mathbb{Z})$, which is a Hecke eigenform but not a Saito-Kurokawa lift.*
- *We may normalize Φ so that all Fourier coefficients $a(T, \Phi)$ of Φ are in $\bar{\mathbb{Z}}$, the set of algebraic integers.*

Then for any imginary quadratic field E,

$$w(E)^2 \cdot D_E^{k-1} \cdot 2^{2k-5} \cdot \frac{L\left(\frac{1}{2}, \pi(\Phi)\right) L\left(\frac{1}{2}, \pi(\Phi) \times \chi_E\right)}{L(1, \pi(\Phi), \mathrm{Ad})} \cdot \langle \Phi, \Phi \rangle \in \bar{\mathbb{Z}}.$$

Remark

According to a conjecture concerning Whittaker periods, by Ichino in the GSp_2 case and by Lapid & Mao in more general cases,

$$L\left(1, \pi\left(\Phi\right), \mathrm{Ad}\right)$$

above may be essentially replaced by

$$\langle \Phi_{\mathrm{gen}}, \Phi_{\mathrm{gen}} \rangle$$

where Φ_{gen} is an automorphic form in the space of the generic representation in the same L-packet as $\pi\left(\Phi\right)$.