p-adic L-functions for $\mathrm{GL}(n+1) \times \mathrm{GL}(n)$ IV

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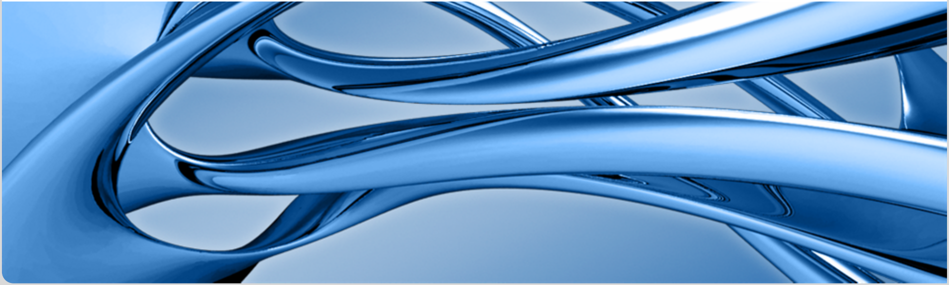

Overview

Previous lectures:

- $\mathrm{GL}(2)/F$ adelically (Kenichi)
- Rankin-Selberg *L*-functions following Jacquet, Piatetski-Shapiro and Shalika
- The relative modular symbol and algebraicity of special values
- Archmedean periods: Non-vanishing and period relations
- *p*-adic distributions attached to finite slope classes
  (Kazhdan-Mazur-Schmidt, Schmidt, J.)
- Boundedness in the nearly ordinary case
  (Schmidt, J.)
- Functional equation
  (J.)
- Manin congruences and independence of weight
  (J.)

This lecture:

- Interpolation formulae
  (Schmidt, J.)

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Hecke algebras

p fixed rational prime
$F_{\mathfrak{p}}/\mathbf{Q}_p$ finite extension, $\mathcal{O}_{\mathfrak{p}} \subseteq F_{\mathfrak{p}}$ valuation ring
$\mathfrak{p} \subseteq \mathcal{O}_{\mathfrak{p}}$ maximal ideal, $\varpi \in \mathfrak{p}$ uniformizer
$n \geq 1$, $e = (e_i)_i \in \mathbf{Z}^n$ *dominant* if

$$e_1 \geq e_2 \geq \cdots \geq e_n$$

$$\begin{aligned}
B_n &= T_n U_n \subset \mathrm{GL}_n \quad \text{upper triangular Borel} \\
I^n_{\alpha',\alpha} &= \{k \in \mathrm{GL}_n(F_{\mathfrak{p}}) \mid k \in B_n(\mathcal{O}_{\mathfrak{p}}/\mathfrak{p}^{\alpha}) \text{ and } k \in U_n(\mathcal{O}_{\mathfrak{p}}/\mathfrak{p}^{\alpha'})\} \\
\varpi^e &= \mathrm{diag}(\varpi^{e_1}, \varpi^{e_2}, \ldots, \varpi^{e_n}) \in \mathrm{GL}_n(F_{\mathfrak{p}}) \\
t_{\varpi} &= \varpi^{2\rho_n^{\vee}+(n)} = \mathrm{diag}(\varpi^n, \varpi^{n-1}, \ldots, \varpi) \in \mathrm{GL}_n(F_{\mathfrak{p}}) \\
U^e_{\varpi} &= I^n_{\alpha',\alpha} \varpi^e I^n_{\alpha',\alpha} = \bigsqcup_{u \in U_n(\mathcal{O}_{\mathfrak{p}})/\varpi^e U(\mathcal{O}_{\mathfrak{p}})\varpi^{-e}} u \varpi^e I^n_{\alpha',\alpha} \\
\mathcal{H}_{\alpha',\alpha} &= \mathbf{Z}[\{I^n_{\alpha',\alpha} \varpi^e \varepsilon I^n_{\alpha',\alpha} \mid e \text{ dominant and } \varepsilon \in T_n(\mathcal{O}_{\mathfrak{p}})\}] \\
&= \mathcal{H}_{0,\alpha}[T_n(\mathcal{O}_{\mathfrak{p}}/\mathfrak{p}^{\alpha'})] \quad \text{commutative } \mathbf{Z}\text{-algebra}
\end{aligned}$$

Hecke algebras

$$\omega_\nu = (\underbrace{1,\dots,1}_{\nu},\underbrace{0,\dots,0}_{n-\nu}) \quad \nu\text{-th fundamental weight}$$

$$U_{\varpi}^{\omega_\nu} \qquad \text{for } 1 \leq \nu \leq n \text{ generate } \mathcal{H}_{0,\alpha}$$

$$U_{\mathfrak{p}} = \prod_{\nu=1}^{n} U_{\varpi}^{\omega_\nu}$$

Parabolic Hecke algebra

$$I_{\alpha'}^{B_n} = I_{\alpha',\alpha} \cap B_n(F_{\mathfrak{p}})$$

$$\mathcal{H}_{\alpha'}^{B_n} = \mathbf{Z}[\{ I_{\alpha'}^{B_n} \varpi^e \varepsilon I_{\alpha'}^{B_n} \mid e \in \mathbf{Z} \text{ and } \varepsilon \in T_n(\mathcal{O}_{\mathfrak{p}})\}]$$

$$\widetilde{U}_i = I_{\alpha'}^{B_n} \begin{pmatrix} \mathbf{1}_{i-1} & 0 & 0 \\ 0 & \varpi & 0 \\ 0 & 0 & \mathbf{1}_{n-i} \end{pmatrix} I_{\alpha'}^{B_n} \in \mathcal{H}_{\alpha'}^{B_n}$$

$$\mathcal{H}_{\alpha',\alpha} \subseteq \mathcal{H}_{\alpha'}^{B_n} \quad \text{and}$$

$$q^{\frac{\nu(\nu-1)}{2}} \cdot U_{\varpi}^{\omega_\nu} = \widetilde{U}_1 \widetilde{U}_2 \cdots \widetilde{U}_\nu$$

p-stabilization: spherical case

Spherical Hecke algebra

$$T_\nu \;=\; I_{0,0}\varpi^{\omega_\nu} I_{0,0} \;\in\; \mathcal{H}_{0,0}$$

The reciprocal Hecke polynomial

$$H(X) \;=\; \sum_{\nu=0}^{n} (-1)^\nu q^{\frac{\nu(\nu-1)}{2}} T_\nu X^{n-\nu} \;\in\; \mathcal{H}_A^n(0,0)[X]$$

admits a factorization (Gritsenko)

$$H_F(X) \;=\; \prod_{i=1}^{n} (X - \widetilde{U}_i) \tag{1}$$

Using (1), we can p-stabilize in *spherical* representations $\Pi_\mathfrak{p}$:
Fix n Hecke roots $\alpha_1, \dots, \alpha_{n-1}, \alpha_n \in \mathbf{C}$, i.e.

$$H(\alpha_i) \cdot \Pi_\mathfrak{p}^{\mathrm{GL}_n(\mathcal{O}_\mathfrak{p})} \;=\; 0$$

p-stabilization: spherical case

Consider the operator

$$P_{\alpha_1,\dots,\alpha_{n-1}} = \prod_{i=1}^{n} \prod_{\substack{j=1 \\ j\neq i}}^{n-1} (\alpha_i q^{1-j} U_{\varpi}^{\omega_{j-1}} - U_{\varphi}^{\omega_j})$$

Proposition (Kazhdan-Mazur-Schmidt)

For any $W_{\mathfrak{p}} \in \mathscr{W}(\Pi_{\mathfrak{p}}, \psi_{\mathfrak{p}})^{\mathrm{GL}_n(\mathcal{O}_{\mathfrak{p}})}$*:*

$$U_{\varpi}^{\omega_\nu} \cdot P_{\alpha_1,\dots,\alpha_{n-1}} \cdot W_{\mathfrak{p}} = q^{-\frac{\nu(\nu-1)}{2}} \prod_{i=1}^{\nu} \alpha_i \cdot P_{\alpha_1,\dots,\alpha_{n-1}} \cdot W_{\mathfrak{p}}$$

$$P_{\alpha_1,\dots,\alpha_{n-1}} \cdot W_{\mathfrak{p}}(\mathbf{1}_n) \neq 0 \quad \textit{(explicit formula)}$$

Problem: If $\Pi_{\mathfrak{p}}$ is not spherical, we may still apply $P_{\alpha_1,\dots,\alpha_{n-1}}$ to essential vectors, but the result vanishes at $\mathbf{1}_n$ if ramification is too deep.
(use Miyauchi's, Kondo-Yasuda's and Matringe's computations)

p-stabilization: general case

Assume Π is a regular algebraic cuspidal representation of $\mathrm{GL}_n(\mathbf{A}_F)$

Fix an embedding $\mathbf{Q}(\Pi) \to \overline{\mathbf{Q}}_p$, assume $\left(\Pi_{\mathfrak{p}}^{I^n_{\alpha',\alpha}}\right)_{\mathrm{ord}} \neq 0$

Proposition (Hida)

- *There is a character* $\lambda : T_n(F_{\mathfrak{p}}) \to \mathbf{Q}(\Pi)^{\times}$ *with*

$$\Pi_{\mathfrak{p}} = {}^{\mathrm{a}}\operatorname{Ind}_{B_n(F_{\mathfrak{p}})}^{\mathrm{GL}_n(F_{\mathfrak{p}})}(\widetilde{\lambda})$$

  where $\widetilde{\lambda} = |\cdot|^{n-1}\lambda_1 \otimes |\cdot|^{n-2}\lambda_2 \otimes \cdots \otimes \lambda_n$

-
$$J_{B_n}(\Pi_{\mathfrak{p}}) = \bigoplus_{\omega \in W(\mathrm{GL}_n, T_n)} \widetilde{\lambda^{\omega}}$$

- *For* φ *in the* $\widetilde{\lambda^{\omega}}$*-isotypic component of* $J_{B_n}(\Pi_{\mathfrak{p}})$*:*

$$U_{\varpi}^{\omega_\nu} \cdot \varphi = q^{-\frac{\nu(\nu-1)}{2}} \lambda^{\omega}(\varpi^{\omega_\nu}) \cdot \varphi$$

p-stabilization: general case

Building on Hida's observation, it is not difficult to show

Proposition

- *The U_p-ordinary vectors in $\mathscr{W}(\Pi_{\mathfrak{p}}, \psi_{\mathfrak{p}})^{U_n(\mathcal{O}_{\mathfrak{p}})}$ lie in a unique line*
- $\exists! \omega \in W(\mathrm{GL}_n, T_n)$: *For every U_p-ordinary $W_{\mathfrak{p}} \in \mathscr{W}(\Pi_{\mathfrak{p}}, \psi_p)^{U_n(\mathcal{O}_{\mathfrak{p}})}$:*

$$U_{\varpi}^{\omega_\nu} \cdot W_{\mathfrak{p}} = q^{-\frac{\nu(\nu-1)}{2}} \lambda^{\omega}(\varpi^{\omega_\nu}) \cdot W_{\mathfrak{p}}$$

- *For every non-zero U_p-ordinary $W_{\mathfrak{p}} \in \mathscr{W}(\Pi_{\mathfrak{p}}, \psi_{\mathfrak{p}})^{U_n(\mathcal{O}_{\mathfrak{p}})}$:*

$$W_{\mathfrak{p}}(\mathbf{1}_n) \neq 0$$

Remark: In the Shalika case of $\mathrm{GL}(2n)$ we face several complications:

- $U_p = I_{\alpha',\alpha}^{2n} \varpi^{\omega_n} I_{\alpha',\alpha}^{2n}$, $\omega_n = (1, \ldots, 1, 0, \ldots, 0) \in X(T_{2n})$ is **not** regular
- $W_{\mathfrak{p}}(\mathbf{1}_{2n}) \neq 0$ requires careful study of intertwining operators
- Current methods need $W_{\mathfrak{p}}(\mathrm{diag}(\mathbf{1}_n, w_n)) \neq 0$

Local Birch Lemma

Back in the Rankin-Selberg context:

$$\begin{array}{rcl} W_{\mathfrak{p}} & \in & \mathscr{W}(\Pi_{\mathfrak{p}}, \psi_{\mathfrak{p}})^{I^{n+1}_{\alpha',\alpha}} \\ W'_{\mathfrak{p}} & \in & \mathscr{W}(\Sigma_{\mathfrak{p}}, \psi_{\mathfrak{p}}^{-1})^{I^{n}_{\alpha',\alpha}} \\ \theta & : & T_{n+1}(F_{\mathfrak{p}}) \to \mathbf{C}^\times \\ \theta' & : & T_{n}(F_{\mathfrak{p}}) \to \mathbf{C}^\times \\ \rightsquigarrow \theta \otimes \theta' & : & I_{\alpha',\alpha} = I^{n+1}_{\alpha',\alpha} \times I^{n}_{\alpha',\alpha} \to \mathbf{C}^\times \end{array}$$

Assume $W_{\mathfrak{p}}$ and $W'_{\mathfrak{p}}$ are of *Nebentypus* θ and θ', i.e.

$$\begin{array}{rrcl} \forall r \in I^{n+1}_{\alpha} : & W_{\mathfrak{p}}(-r) & = & \theta(r) \cdot W_{\mathfrak{p}}(-) \\ \forall r' \in I^{n}_{\alpha} : & W'_{\mathfrak{p}}(-r') & = & \theta'(r') \cdot W'_{\mathfrak{p}}(-) \end{array}$$

$\chi : F_{\mathfrak{p}}^\times \to \mathbf{C}^\times$ quasi-character. If for all $1 \leq \nu \leq \mu \leq n$:
The conductors $\mathfrak{f}_{\chi\theta^{w_\mu}\theta'} = (f_{\chi\theta^{\omega_\mu}\theta'})$ of $\chi\theta_\nu^{w_\mu}\theta'_\nu$ are non-trivial, and all agree
We say that $\chi\theta_\nu^{w_\mu}\theta'_\nu$ have *fully supported constant conductor*.
This is always the case whenever χ is sufficiently ramified.

Local Birch Lemma

Theorem (Local Birch Lemma, Schmidt 2001, J., 2009, 2017)

Let $W_{\mathfrak{p}} \otimes W'_{\mathfrak{p}} \in \mathscr{W}(\Pi_{\mathfrak{p}}, \psi_{\mathfrak{p}})^{I^{n+1}_{\alpha',\alpha}} \otimes W'_{\mathfrak{p}} \in \mathscr{W}(\Sigma_{\mathfrak{p}}, \psi_{\mathfrak{p}}^{-1})^{I^{n}_{\alpha',\alpha}}$ *of Nebentypus* $\theta \otimes \theta'$

Assume that for $1 \leq \nu \leq \mu \leq n$, $\chi\theta_{\nu}^{w_\mu}\theta'_{\nu}$ *have* fully supported constant conductor $\mathfrak{f}_{\chi\theta\theta'} \mid f = \varpi^{\alpha}$. *Then for every* $s \in \mathbf{C}$,

$$\int\limits_{U_n(F_{\mathfrak{p}})\backslash \mathrm{GL}_n(F_{\mathfrak{p}})} W_{\mathfrak{p}}\left(\mathrm{diag}(g,1)\cdot h_n \cdot \mathrm{diag}(t_{-f},1)\right) W'_{\mathfrak{p}}\left(g\cdot t_f\right)\chi(\det(g))\left|\det(g)\right|^{s-\frac{1}{2}} dg$$

$$= \prod_{\mu=1}^{n}\left(1-q^{-\mu}\right)^{-1} \cdot \mathfrak{N}(\mathfrak{f}_{\chi\theta\theta'})^{-\frac{(n+2)(n+1)n}{6}} \cdot \left|t_{\mathfrak{f}_{\chi\theta\theta'}}\right|^{\frac{1}{2}-s}$$

$$\times \prod_{\mu=1}^{n}\prod_{\nu=1}^{\mu}\left[\theta_{\nu}^{w_\mu}\theta'_{\nu}(f_{\chi\theta\theta'})\cdot G(\chi\theta_{\nu}^{w_\mu}\theta'_{\nu})\right]$$

$$\times W_{\mathfrak{p}}(\mathrm{diag}(t_{ff^{-1}_{\chi\theta\theta'}},1))\cdot W'_{\mathfrak{p}}(t_{ff^{-1}_{\chi\theta\theta'}})$$

Local Birch Lemma

Proof: (rough sketch)

1) Put $J_\ell^n = \ker\left[\mathrm{GL}_n(\mathcal{O}_\mathfrak{p}) \to \mathrm{GL}_n(\mathcal{O}_\mathfrak{p}/\mathfrak{f}^\ell)\right]$ and decompose:

$$\mathrm{GL}_n(F_\mathfrak{p}) = \bigsqcup_{\substack{e\in\mathbf{Z}^n \\ \omega\in W(\mathrm{GL}_n,T_n) \\ r\in\mathfrak{R}^\omega_{n,\ell(e)}}} U_n(F_\mathfrak{p})\varpi^e\omega r J^n_{\ell(e)}$$

with a system of representatives $\mathfrak{R}^\omega_{n,\ell(e)}$ for $I^n_{0,1}\cap\omega^{-1}B_n^-(\mathcal{O}_\mathfrak{p})\omega/J^n_\ell$

2) Fix a system of representatives $\mathcal{T}_{n,\ell}\subseteq T_n(\mathcal{O}_\mathfrak{p})$ for $T_n(\mathcal{O}_\mathfrak{p}/\mathfrak{f}^\ell)$.
Consider the corresponding 'partial integrals'

$$Z_n(s; W_\mathfrak{p}, W'_\mathfrak{p}, \delta, e, \omega, r) := \sum_{\gamma\in\mathcal{T}_{n,\ell}} \psi_\mathfrak{p}(\lambda_n^\delta(\varpi^e\omega^\gamma r))\cdot W_\mathfrak{p}\left(\varpi^e\omega^\gamma r\cdot D_n w_n\right)$$
$$\times W'_\mathfrak{p}(\varpi^e\omega^\gamma r)\cdot\chi(\varpi^e\omega^\gamma r)\cdot\left|\det(\varpi^e\omega^\gamma r)\right|^{s-\frac{1}{2}}$$

3) Prove the following technical Lemma

Local Birch Lemma

Lemma

Let $s \in \mathbf{C}$, $e \in \mathbf{Z}^n$, $\omega \in W(\mathrm{GL}_n, T_n)$, $\ell \in \mathbf{Z}$ *sufficiently large, and* $\delta \in \mathbf{Z}$*. Then*

(i) $Z_n(s; W_\mathfrak{p}, W'_\mathfrak{p}, \delta, e, \omega, r)$ *is independent of the choice of* $\mathcal{T}_{n,\ell}$*.*

(ii) *Assume that for all* $1 \leq \nu \leq n$ *and* $\mu = n + 1 - \nu$*,*

$$c(\chi\theta_\mu\theta'_\nu) \; > \; 0 \tag{2}$$

is satisfied. Then $Z_n(s; W_\mathfrak{p}, W'_\mathfrak{p}, \delta, e, \omega, r)$ *vanishes unless*

$$e_n \; = \; \delta + \alpha \cdot (n + 1 - \sigma(n)) - c(\chi\theta_{n+1-\sigma(n)}\theta'_{\sigma(n)}) \tag{3}$$

(iii) *If conditions* (2) *and* (3) *are satisfied, and if the exponent in* (2) *is independent of* ν*, then* $Z_n(s; W_\mathfrak{p}, W'_\mathfrak{p}, \delta, e, \omega, r)$ *vanishes unless* $\sigma(n) = n$ *and for* $1 \leq \nu \leq n$*,*

$$|r_{n\nu}| \; = \; \left|f^{n-\nu}\right| \tag{4}$$

(iv) *If the hypotheses of (iii) are satisfied, we may assume without loss of generality that*

$$r_{n1} = f^{n-1}, \quad \text{and} \quad r_{n\nu} \; = \; -f^{n-\nu} \text{ for } 2 \leq \nu \leq n. \tag{5}$$

If additionally, (2) *holds for all* $1 \leq \nu \leq \mu \leq n$*, then*

$$Z_n(s; W_\mathfrak{p}, W'_\mathfrak{p}, \delta, e, \omega, r) = \chi\theta^{w_n}\theta'(B_n) \cdot \prod_{\nu=1}^{n} \chi\theta_\nu^{w_n}\theta'_\nu \left(f_{\chi\theta_\nu^{w_n}\theta'_\nu}\right) \cdot \mathfrak{N}(\mathfrak{f}^\ell / \mathfrak{f}_{\chi\theta_\nu^{w_n}\theta'_\nu}) \cdot G(\chi\theta_\nu^{w_n}\theta'_\nu)$$

$$\times\, W_\mathfrak{p}\left(\varpi^e\omega r \cdot D_n w_n\right) \cdot W'_\mathfrak{p}(\varpi^e\omega r) \cdot \chi(\varpi^e\omega r) \cdot \left|\det(\varpi^e)\right|^{s-\frac{1}{2}}$$

Local Birch Lemma

4) The Lemma allows for an inductive argument:

$$Z_n(s; w, v, \delta, e, \omega, j_{n-1,0}(\tilde{r})C_n)$$

$$= |\varpi^{e_n}|^{s-\frac{1}{2}} \cdot \chi\theta^{w_n}\theta'(B_n) \cdot \prod_{\nu=1}^{n} \chi\theta_{n+1-\nu}\theta'_\nu \left(f_{\chi\theta_\nu^{w_n}\theta'_\nu}\right) \cdot \mathfrak{N}(\mathfrak{f}^\ell / \mathfrak{f}_{\chi\theta_\nu^{w_n}\theta'_\nu}) \cdot G(\chi\theta_\nu^{w_n}\theta'_\nu)$$

$$\times W_\mathfrak{p}\left(j_{n,e_n}(\varpi^{\tilde{e}}\tilde{\omega}\tilde{r})C_n \cdot D_n w_n\right) \cdot W'_\mathfrak{p}(j_{n,e_n}(\varpi^{\tilde{e}}\tilde{\omega}\tilde{r})C_n) \cdot \chi(j_{n,e_n}(\varpi^{\tilde{e}}\tilde{\omega}\tilde{r})C_n) \cdot \left|\varpi^{\tilde{e}}\right|^{s-\frac{1}{2}}$$

$$= \prod_{\nu=1}^{n} \chi\theta_{n+1-\nu}\theta'_\nu \left(f_{\chi\theta_\nu^{w_n}\theta'_\nu}\right) \cdot \mathfrak{N}(\mathfrak{f}^\ell / \mathfrak{f}_{\chi\theta_\nu^{w_n}\theta'_\nu}) \cdot G(\chi\theta_\nu^{w_n}\theta'_\nu)$$

$$\times |\varpi^{e_n}|^{s-\frac{1}{2}} \cdot \chi(\varpi^{e_n}) \cdot \chi\theta'(C_n) \cdot \chi\theta^{w_n}\theta'(B_n)$$

$$\times W_\mathfrak{p}\left(j_{n,e_n}(\varpi^{\tilde{e}}\tilde{\omega}\tilde{r})C_n \cdot D_n w_n\right) \cdot W'_\mathfrak{p}(j_{n,e_n}(\varpi^{\tilde{e}}\tilde{\omega}\tilde{r})) \cdot \chi(j_{n,e_n}(\varpi^{\tilde{e}}\tilde{\omega}\tilde{r})) \cdot \left|\varpi^{\tilde{e}}\right|^{s-\frac{1}{2}}$$

$$= \prod_{\nu=1}^{n} \chi\theta_{n+1-\nu}\theta'_\nu \left(f_{\chi\theta_\nu^{w_n}\theta'_\nu}\right) \cdot \mathfrak{N}(\mathfrak{f}^\ell / \mathfrak{f}_{\chi\theta_\nu^{w_n}\theta'_\nu}) \cdot G(\chi\theta_\nu^{w_n}\theta'_\nu)$$

$$\times |\varpi^{e_n}|^{s-\frac{1}{2}} \cdot \chi(\varpi^{e_n}) \cdot \chi\theta'\left(j_{n-1,0}(B_{n-1}^{-1})C_n\right) \cdot \chi\theta^{w_n}\theta'(B_n)$$

$$\times \psi(\lambda_{n-1}^{-e_n}(\varpi^{\tilde{e}}\tilde{\omega}\tilde{r}B_{n-1}) \times W_\mathfrak{p}\left(j_{n,e_n}(\varpi^{\tilde{e}}\tilde{\omega}\tilde{r}B_{n-1} \cdot D_{n-1}w_{n-1})\right) \cdot W'_\mathfrak{p}(j_{n,e_n}(\varpi^{\tilde{e}}\tilde{\omega}\tilde{r}B_{n-1}))$$

$$\times \chi(j_{n,e_n}(\varpi^{\tilde{e}}\tilde{\omega}\tilde{r}B_{n-1})) \cdot \left|\varpi^{\tilde{e}}\right|^{s-\frac{1}{2}}$$

Local Birch Lemma

$$= \prod_{\nu=1}^{n} \chi\theta_{n+1-\nu}\theta'_\nu \left(f_{\chi\theta_\nu^{w_n}\theta'_\nu}\right) \cdot \mathfrak{N}(\mathfrak{f}^\ell / \mathfrak{f}_{\chi\theta_\nu^{w_n}\theta'_\nu}) \cdot G(\chi\theta_\nu^{w_n}\theta'_\nu)$$
$$\times |\varpi^{e_n}|^{s-\frac{1}{2}} \cdot \chi(\varpi^{e_n}) \cdot \theta^{w_n}(B_n)$$
$$\times \psi(\lambda_{n-1}^{-e_n}(\varpi^{\tilde{e}}\tilde{\omega}\tilde{r}B_{n-1}) \cdot W_\mathfrak{p}\left(j_{n,e_n}(\varpi^{\tilde{e}}\tilde{\omega}\tilde{r}B_{n-1} \cdot D_{n-1}w_{n-1})\right) \cdot W'_\mathfrak{p}(j_{n,e_n}(\varpi^{\tilde{e}}\tilde{\omega}\tilde{r}B_{n-1}))$$
$$\times \chi(j_{n,e_n}(\varpi^{\tilde{e}}\tilde{\omega}\tilde{r}B_{n-1})) \cdot \left|\varpi^{\tilde{e}}\tilde{\omega}\tilde{r}B_{n-1}\right|^{s-\frac{1}{2}}$$

5) Using this, we can finally prove the following key lemma:

Lemma

Assume that $\chi\theta_\nu^{w_\mu}\theta'_\nu$ have fully supported constant conductor for $1 \leq \nu \leq \mu \leq n$. Define $\gamma := \alpha - c(\chi\theta_\nu^{w_n}\theta'_\nu)$. Then for all $e \in \mathbf{Z}^n$, $\omega \in W(\mathrm{GL}_n, T_n)$, $\ell \geq \max\{2n, n - e_1/\alpha, \dots, n - e_n/\alpha\}$ and $\delta \in \mathbf{Z}$, we have

$$\mathrm{vol}\left(U_n(\mathcal{O})\varpi^e J_\ell^n\right) \cdot \sum_{g \in \varpi^e\omega\mathfrak{R}_{n,\ell}^\omega} \psi(\lambda_n^\delta(g)) W_\mathfrak{p}(g \cdot D_n w_n) W'_\mathfrak{p}(g)\chi(\det(g)) \left|\det(g)\right|^{s-\frac{1}{2}}$$
$$= \mathfrak{N}(\mathfrak{f}_{\chi\theta\theta'})^{-\frac{(n+1)n(n-1)}{2}} \cdot \prod_{\mu=1}^{n} \theta^{w_\mu}(B_\mu) \cdot \prod_{\nu=1}^{\mu} \mathfrak{N}(\mathfrak{f}_{\chi\theta_\nu^{w_\mu}\theta'_\nu})^{-1} \chi\theta_\nu^{w_\mu}\theta'_\nu(f_{\chi\theta_\nu^{w_\mu}\theta'_\nu}) G(\chi\theta_\nu^{w_\mu}\theta'_\nu)$$
$$\times W_\mathfrak{p}(\varpi^e) W'_\mathfrak{p}(\varpi^e)\chi(\varpi^e) \left|\varpi^e\right|^{s-\frac{1}{2}} \quad \text{if } (e,\omega) = (d_{n,\gamma} + (\delta), \mathbf{1}_n) \text{ and } 0 \text{ otherwise.}$$

Local Birch Lemma

6) Relate the matrices B_n, C_n, $D_n w_n$ to $h_n \operatorname{diag}(t_{-f}, 1)$ and conclude the proof.

Global Birch Lemma

$F/\mathbf{Q}$ finite, p any rational prime
Π and Σ cuspidal automorphic representations of $\mathrm{GL}_{n+1}(\mathbf{A}_F)$ and $\mathrm{GL}_n(\mathbf{A}_F)$
Find a $\mathbf{Q}(\Pi,\Sigma)$-rational $W^{(p\infty)} \in \mathscr{W}(\Pi^{(p\infty)},\psi^{(p\infty)}) \otimes \mathscr{W}(\Sigma^{(p\infty)},\psi^{(p\infty),-1})$ satisfying

$$\Psi(s; W^{(p\infty)}) \;=\; L^{(p)}(s,\Pi\times\Sigma)$$

For $\chi : F^\times\backslash\mathbf{A}_F^\times \to \mathbf{C}^\times$, finite order, unramified outside $p\infty$,

$$\Psi(s; W^{(p\infty)}\cdot\chi^{(p\infty)}(\det{}_2 -)) \;=\; [\text{easy factor}]\cdot L^{(p)}(s,\Pi\times\Sigma\otimes\chi)$$

Recall: $G = \mathrm{res}_{F/\mathbf{Q}}\mathrm{GL}(n+1)\times\mathrm{GL}(n)$, $T = T_{n+1}\times T_n \subseteq G$ diagonal torus
$\vartheta : T(\mathbf{Q}_p)\to\mathbf{C}^\times$ quasi-character, Nebenypus of $\Pi_p\otimes\Sigma_p$ in the strict sense:
There is a non-zero $W_p \in \mathscr{W}(\Pi_p,\psi_p)\otimes W(\Sigma_p,\psi_p^{-1})^{I_{\alpha,\alpha}}$ satisfying

- $W_p(-r) = \vartheta(r)W_p(-)$
- $\forall\mathfrak{p}\mid p$: $\quad U_{\varpi}^{\omega_\nu}W_p = \vartheta(\varpi^{\omega_\nu})W_p$

Assume that $\chi\vartheta$ has *fully supported constant conductor at p* in the following sense:
For $1\leq\nu\leq\mu\leq n$ the conductors $\mathfrak{f}_{\chi\vartheta_{\mu,\nu}}$ are all equal and divisible by all $\mathfrak{p}\mid p$

Global Birch Lemma

For any $W_\infty \in \mathscr{W}(\Pi_\infty, \psi_\infty)\widehat{\otimes}\mathscr{W}(\Sigma_\infty, \psi_\infty^{-1})$ consider the inverse Fourier transform of $W = W_\infty \otimes W_p \otimes W^{(p\infty)}$

$$\varphi_W \in \Pi\widehat{\otimes}\Sigma \subseteq L^2(G(\mathbf{Q})\backslash G(\mathbf{A}))$$

Theorem (Global Birch Lemma, Schmidt, 2001, J., 2009, 2017)

For every $s \in \mathbf{C}$:

$$\int\limits_{H(\mathbf{Q})\backslash H(\mathbf{A})} \varphi_W(g \cdot ht_f)\,\chi(\det(g))\,|\det(g)|^{s-\frac{1}{2}}\,dg$$

$$= \Psi(s; W_\infty \cdot \chi_\infty(\det_2 -))\delta(W_p)\,\mathfrak{N}(\mathfrak{f})^{-\frac{(n+2)(n+1)n+(n+1)n(n-1)}{6}}\,\mathfrak{N}(\mathfrak{f}_{\chi\vartheta})^{-\frac{(n+1)n(n-1)}{6}}$$

$$\times \prod_{\mu=1}^{n}\prod_{\nu=1}^{\mu} G(\chi\vartheta_{\mu,\nu}) \cdot \vartheta(t_p^\alpha) \cdot \left|t_{f_{\chi\theta}}\right|^{\frac{1}{2}-s} \cdot L^{(p)}(s, \Pi \times \Sigma \otimes \chi)$$

where $\delta(W_p) = W_p(\mathbf{1}_n) \cdot \prod_{\mu=1}^{n}\prod_{\mathfrak{p}|p}\left(1 - q_{\mathfrak{p}}^{-\mu}\right)^{-1}.$

Abelian p-adic L-functions

Theorem (Schmidt, Kazhdan-Mazur-Schmidt, Kasten-Schmidt, Sun, J.)

Let $F/\mathbf{Q}$ be a number field, $\Pi\widehat{\otimes}\Sigma$ an irreducible regular algebraic cuspidal automorphic representation of $G(\mathbf{A})$ of cohomological weight λ. Assume:

(i) λ *is balanced.*

(ii) $\Pi\widehat{\otimes}\Sigma$ *is nearly ordinary at a prime p with p-Nebentypus $\vartheta : T(\mathbf{Q}_p) \to \mathbf{C}^\times$.*

Then there are complex periods $\Omega_j^\varepsilon \in \mathbf{C}^\times$ and a unique p-adic measure $\mu_{\Pi\widehat{\otimes}\Sigma} \in \mathcal{O}_{\mathbf{Q}(\Pi\widehat{\otimes}\Sigma)}[[C_F(p^\infty)]]$ with the following property.
For every $s_0 = \frac{1}{2} + j$ critical for $L(s, \Pi\widehat{\otimes}\Sigma)$, for all χ of finite order, unramified outside $p\infty$, such that $\chi_p\vartheta$ has fully supported constant conductor $\mathfrak{f}_{\chi\vartheta}$:

$$\int_{C_F(p^\infty)} \chi(x)\omega_F^j(x)\langle x\rangle_F^j d\mu_{\Pi\widehat{\otimes}\Sigma}(x) =$$

$$\mathfrak{N}(\mathfrak{f}_{\chi\vartheta})^{j\frac{(n+1)n}{2} - \frac{(n+1)n(n-1)}{6}} \cdot \prod_{\mu=1}^{n}\prod_{\nu=1}^{\mu} G(\chi\vartheta_{\mu,\nu}) \cdot \frac{L^{(p)}(s, \Pi\widehat{\otimes}\Sigma\otimes\chi)}{\Omega_j^{(-1)^j \operatorname{sgn}\chi}}$$

Non-abelian p-adic L-functions

$F/\mathbf{Q}$: CM or totally real
or assume existence of Galois representations for torsion classes for G
$\mathfrak{m}$: non-Eisenstein maximal ideal in Hida's universal $\mathbf{h}_{\mathrm{ord}}(K_{\infty,\infty};\mathcal{O})$
Assume K of full level outside p. Recall the canonical element

$$L_{p,\mathfrak{m}}^{\mathrm{univ}} = \int_{C(p^\infty)} d\mu^{\lambda,0} \in H_{\mathrm{ord}}^{q_0+l_0}(K_{\infty,\infty};\mathcal{O})_{\mathfrak{m}}$$

Theorem (J., 2017)

For every classical point $\xi \in \mathrm{Spec}\,\mathrm{h}_{\mathrm{ord}}^{q_0+l_0}(K_{\alpha',\alpha};\mathcal{O})_{\mathfrak{m}}(\overline{E})$ of regular balanced weight λ and p-Nebentyp ϑ, such that $s_0 = \frac{1}{2}$ is critical for $L(s,\Pi_\xi\widehat{\otimes}\Sigma_\xi)$:

$$\begin{aligned}\Omega_{\xi,p}^{-1}\cdot\xi(L_{p,\mathfrak{m}}^{\mathrm{univ}}) &= \int_{C_F(p^\infty)} d\mu_{\Pi_\xi\widehat{\otimes}\Sigma_\xi} \\ &= \mathfrak{N}(\mathfrak{f}_\vartheta)^{\frac{(n+1)n(n-1)}{6}}\cdot\prod_{\mu=1}^{n}\prod_{\nu=1}^{\mu} G(\vartheta_{\mu,\nu})\cdot\frac{L^{(p)}(\frac{1}{2},\Pi_\xi\widehat{\otimes}\Sigma_\xi)}{\Omega_\xi}\end{aligned}$$

The second identity is valid whenever ϑ has fully supported constant conductor.

Concluding remarks

- The same construction works in the **finite slope** case. Here we obtain a unique locally analytic distribution on $\mathbf{Z}_p^\times$, provided the slope is not larger than the number of critical values.
- Establishing the interpolation formula in the *fully supported* conductor case should be within reach.
- In the unramified case, the complete interpolation formula for $n = 2$ has been obtained by direct computation.
- By appropriate stabilization outside p it should be possible to allow for non-abelian interpolation with tame level.
- The recent progress on rational period relations is promising, but remains difficult.
- Integral period relations remain an open problem.
- Similar results and remarks apply in the Shalika case for $\mathrm{GL}(2n)$ (Ash-Ginzburg, Gehrmann, Dimitrov-J.-Raghuram).

The end.

Thank you for your attention.