Some Simple Preconditioners for Unfitted Nitsche methods of *high contrast interface* elliptic problems

Blanca Ayuso de Dios
Dipartimento di Matematica, Università di Bologna

Kyle Dunn & Marcus Sarkis (WPI, Worchester, US)
Simone Scacchi (UNIMI, Milano, Italy)

Numerical Analysis of Coupled and Multi-Physics Problems with Dynamic Interfaces
Oaxaca, July 30th-August 2nd, 2018

Why Solvers & Preconditioning?

PDE on Ω $\longrightarrow$ PDE Discretizations $\longrightarrow$ $Au = f$

- A is large, sparse, positive definite, ill-conditioned ($\kappa(A) = O(h^{-2})$)
- Solve Algebraic Linear Systems $Au = f$:
  - ▷ Direct Methods: CAUTION!! Cost=$O(N^3)$ $N \to \infty$
  - ▷ Iterative Methods ✓

Goal: Develop Uniformly Convergent Iterative methods for $Au = f$

- ▷ Find B such that $BAu = g$, $g = Bf$ easier (faster) than $Au = f$
- ▷ Good B: cheap, low storage, mesh/parameter independence..

(old) Domain Decomposition ideas [Bjorstad, Dryja, Widlund (86')]

- Idea : Divide and Conquer
- Possibility of dealing with bigger problems

Outline I

1. Model problem: an elliptic Inteface Problem
   - CutFEM Discretization for High-Contrast Problemt

2. CutFEM Solvers

3. Numerical Experiments

Model problem: an elliptic Inteface Problem

$\Omega = \Omega^- \cup \Omega^+ \subset \mathbb{R}^2$; $\Gamma := \partial\Omega^- \cap \partial\Omega^+ \in \mathcal{C}^2$

- Given $f \in L^2(\Omega)$ let $f^\pm = f_{|_{\Omega^\pm}}$ and Find u_* with $u_*^\pm := (u_*)_{|_{\Omega^\pm}}$:

$$\begin{cases} -\nabla \cdot \left(\rho^\pm \nabla u_*^\pm\right) = f^\pm & \text{in } \Omega^\pm \\ u_*^\pm = 0 & \text{on } \partial\Omega \\ [u_*] = 0 & \text{on } \Gamma \\ [\![\rho \nabla u_*]\!] = 0 & \text{on } \Gamma \end{cases}$$

- Notation:

$$[u] = u^+ - u^- \qquad [\![\rho \nabla u]\!] = \rho^+ \nabla u^+ \cdot \boldsymbol{n}^+ + \rho^- \nabla u^- \cdot \boldsymbol{n}^-$$

- Assumption: $0 < \rho^- \leq \rho^+$ both constants $\rho^\pm \in \mathbb{P}^0(\Omega^\pm)$

Model problem: an elliptic Inteface Problem

$\Omega = \Omega^- \cup \Omega^+ \subset \mathbb{R}^2$; $\Gamma := \partial\Omega^- \cap \partial\Omega^+ \in \mathcal{C}^2$

- Given $f \in L^2(\Omega)$ let $f^\pm = f_{|_{\Omega^\pm}}$ and Find u_* with $u_*^\pm := (u_*)_{|_{\Omega^\pm}}$:

$$\begin{cases} -\nabla \cdot \left(\rho^\pm \nabla u_*^\pm\right) = f^\pm & \text{in } \Omega^\pm \\ u_*^\pm = 0 & \text{on } \partial\Omega \\ [u_*] = 0 & \text{on } \Gamma \\ [\![\rho \nabla u_*]\!] = 0 & \text{on } \Gamma \end{cases}$$

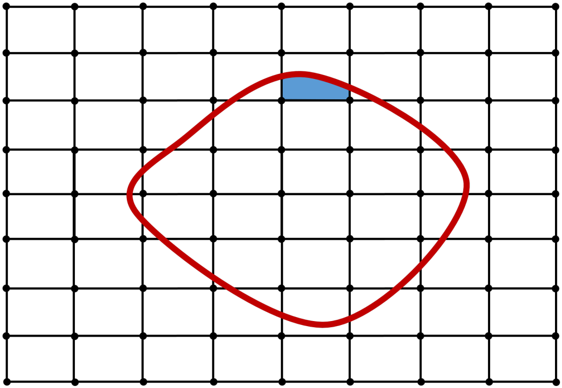
- Notation:

$$[u] = u^+ - u^- \qquad [\![\rho \nabla u]\!] = \rho^+ \nabla u^+ \cdot \boldsymbol{n}^+ + \rho^- \nabla u^- \cdot \boldsymbol{n}^-$$

- Assumption: $0 < \rho^- \leq \rho^+$ both constants $\rho^\pm \in \mathbb{P}^0(\Omega^\pm)$

- Notice: $u_*^\pm \in H^2(\Omega^\pm)$ but $u_* \in H^{3/2-\epsilon}(\Omega)$ for $\epsilon > 0$

Numerical Approximation to Interface Problem

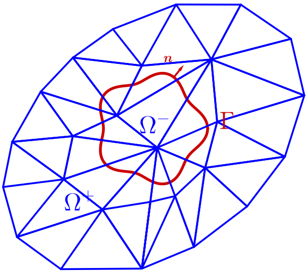


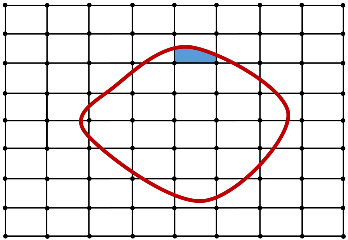
- No Mesh-Free approaches...
- Use *unfitted method*
  - ▷ *(eXtended) FEM*, Finite Cell Method (FCM), CutFEM, ...

Unfitted Methods (a brief (account of) history....)

- [Nitsche (1971)]
  - ▷ introduce penalties to weakly enforcing bc
- [Barrett & Elliot (1982—1987)] unfitted methods
  - ▷ Use of penalties for Curved boundaries & smooth interface
- [Belytschko (1999)..... Reusken & et al (2005...)] eXtended FEM
  - ▷ Generalized FEM, *enriched methods*, PUM
- [Hansbo & Hansbo (2002)] Nitsche method for interface problems
- [Parvizian & Düster & Rank, (2007)] Finite Cell Method (elasticity)
- [Burman& Hansbo (2012)] introduce CutFem
  - ▷ [Burman & Claus & Hansbo &Larsson& Massing (2014)]

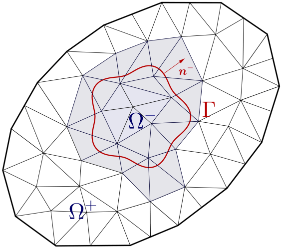
Unfitted Mesh along the Interface

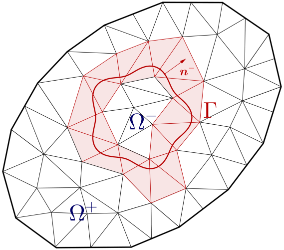




- $\mathscr{T}_h$ shape-regular & quasi-uniform
- cuts $\Gamma \cap \mathscr{T}_h$ *regular* cuts:
  - $\Gamma \cap K$ is either an edge or cuts exactly twice ∂K
  - 3D: [Guzman& Olshanskii (2018)] weaker assumptions

Unfitted Mesh along the Interface: Notation





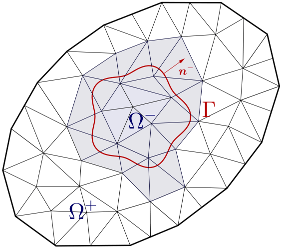
$$\mathscr{T}_h^{\pm} := \{T \in \mathcal{T}_h : T \cap \Omega^{\pm} \neq \emptyset\}, \qquad \mathscr{T}_h^{\Gamma} := \{T \in \mathcal{T}_h : T \cap \Gamma \neq \emptyset\}.$$

$$\Omega_h^{\pm} := \mathrm{Int}\Big(\bigcup_{T \in \mathcal{T}_h^{\pm}} \overline{T}\Big) \qquad \Omega_h^{\Gamma} := \mathrm{Int}\Big(\bigcup_{T \in \mathcal{T}_h^{\Gamma}} \overline{T}\Big).$$

$$\Omega_{h,0}^{\pm} = \Omega_h^{\pm} \backslash \overline{\Omega}_h^{\Gamma} \qquad \boxed{\Omega = \Omega_{h,0}^{+} \cup \overline{\Omega}_h^{\Gamma} \cup \Omega_{h,0}^{-}}$$

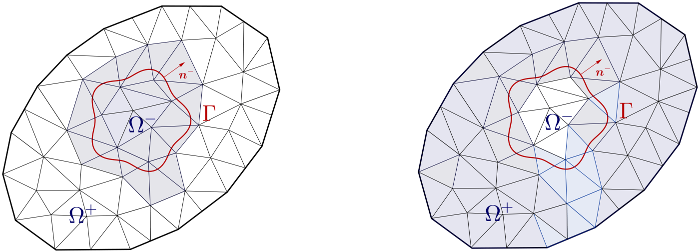
$$\mathcal{E}_h^{\Gamma,\pm} := \{e = \mathrm{Int}(\partial T_1 \cap \partial T_2) : T_1, T_2 \in \mathcal{T}_h^{\pm}, \text{ and } T_1 \in \mathcal{T}_h^{\Gamma} \text{ or/and } T_2 \in \mathcal{T}_h^{\Gamma}\}.$$

CutFEM Discretization for High-Contrast Problem



- Discrete Domain Ω_h^+ with ρ^--coefficient
  - ▷ $V^- := V_h(\Omega_h^-)$: conforming $\mathbb{P}^1(\mathscr{T}_h^{\,-}) \cap \mathcal{C}^0(\Omega_h^-)$ or $\mathbb{Q}^1(\mathscr{T}_h^{\,-}) \cap \mathcal{C}^0(\Omega_h^-)$

CutFEM Discretization for High-Contrast Problem

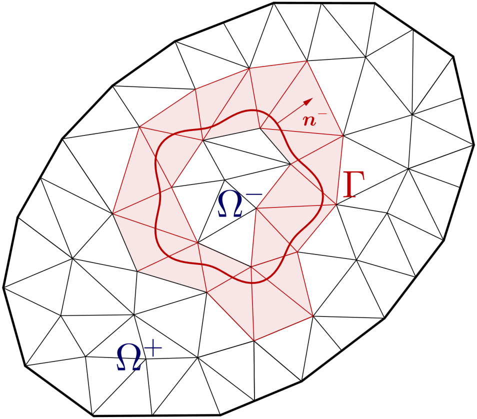


- Discrete Domain Ω_h^+ with ρ^--coefficient
  - ▷ $V^- := V_h(\Omega_h^-)$: conforming $\mathbb{P}^1(\mathscr{T}_h^-) \cap \mathcal{C}^0(\Omega_h^-)$ or $\mathbb{Q}^1(\mathscr{T}_h^-) \cap \mathcal{C}^0(\Omega_h^-)$
- Discrete Domain $\Omega_h^\pm$ with ρ^+-coefficient
  - ▷ $V^+ := V_h(\Omega_h^+)$: conforming $\mathbb{P}^1(\mathscr{T}_h^+) \cap \mathcal{C}^0(\Omega_h^+)$ or $\mathbb{Q}^1(\mathscr{T}_h^+) \cap \mathcal{C}^0(\Omega_h^+)$

▷ No-floating subdomain: functions are zero on $\partial\Omega \cap \partial\Omega^+$

CutFEM approximation

- Global space $V_h = V^- \times V^+$:
- *double-valued* on $\Omega_h^\Gamma := \{K \in \mathscr{T}_h \; : \; K \cap \Gamma \neq \emptyset\}$
- Nitsche-DG techniques to glue V_h^+ and V_h^- on Γ
- Flux edge stabilization on $\mathcal{E}_h^\Gamma = \{e \subset \partial K \; : \; K \in \Omega_h^\Gamma\}$ (difference with other techniques FCM...)



CutFEM approximation for High Contrast

[Burman, Guzmán, Sarkis (2018)]

Find $u_h = (u^+, u^-) \in V_h = V^+ \times V^-$, st

$$a_h(u_h, v_h) = (f^+, v^+)_{\Omega^+} + (f^-, v^-)_{\Omega^-} \quad \forall v_h = (v^+, v^-) \in V^+ \times V^-$$

$$a_h(u_h, v_h) = \int_{\Omega^-} \rho_- \nabla u^- \cdot \nabla v^- dx + \int_{\Omega^+} \rho_+ \nabla u^+ \cdot \nabla v^+ dx$$

$$+ \boxed{\int_\Gamma \left(\{\rho \nabla v_h\}_w \cdot \mathbf{n}^- [u_h] + \{\rho \nabla u_h\}_w \cdot \mathbf{n}^- [v_h] \right) ds + \frac{\gamma_\Gamma}{h} \{\rho\}_H \int_\Gamma [u_h][v_h]\, ds}$$

$$+ \gamma_2 \sum_{e \in \mathcal{E}_h^\Gamma} \left(|e| \int_e \rho_- [[\nabla u^-]]\, [[\nabla v^-]] + \rho_+ [[\nabla u^+]]\, [[\nabla v^+]] \right) ds,$$

$$\{\rho\}_H = \frac{2\rho^+ \rho^-}{\rho^+ + \rho^-}, \qquad \{\rho \nabla v_h\}_\omega := (\omega_- \rho^- \nabla v^- + \omega_+ \rho^+ \nabla v^+), \quad \omega_- + \omega_+ = 1$$

CutFEM approximation

$$a_h(u_h, v_h) = \int_{\Omega^-} \rho_- \nabla u^- \cdot \nabla v^- dx + \int_{\Omega^+} \rho_+ \nabla u^+ \cdot \nabla v^+ dx$$

$$+ \boxed{\int_\Gamma (\{\rho \nabla v_h\}_w \cdot \mathbf{n}^- [u_h] + \{\rho \nabla u_h\}_w \cdot \mathbf{n}^- [v_h]) \, ds + \frac{\gamma_\Gamma}{h} \{\rho\}_H \int_\Gamma [u_h] [v_h] \, ds}$$

$$+ \gamma_2 \sum_{e \in \mathcal{E}_h^\Gamma} \left(|e| \int_e \rho_- [\![\nabla u^-]\!] \, [\![\nabla v^-]\!] + \rho_+ [\![\nabla u^+]\!] \, [\![\nabla v^+]\!] \right) ds,$$

- Semi-Norms and Norms:

$$|v^\pm|^2_{V^\pm} := \rho_\pm \|\nabla v^\pm\|^2_{L^2(\Omega^\pm)} + \sum_{e \in \mathcal{E}_h^{\Gamma,\pm}} \gamma_\pm |e| \| \, [\![\nabla v^\pm]\!] \, \|^2_{L^2(e)} \qquad \forall v^\pm \in V^\pm .$$

$$\|v_h\|^2_{V_h} := |v^+|^2_{V^+} + |v^-|^2_{V^-} + \sum_{K \in \mathscr{T}_h^\Gamma} \frac{\gamma_\Gamma}{h_K} \{\rho\}_H \| \, [v_h] \, \|^2_{L^2(K \cap \Gamma)} \quad \forall v_h \in V_h = V^+ \times V^- .$$

- Stability $\quad a_h(v_h, v_h) \gtrsim \|v_h\|^2_{V_h}, \quad$ for all $\quad v_h \in V_h$
- Continuity $\quad |a_h(u_h, v_h)| \lesssim \|u_h\|_{V_h} \|v_h\|_{V_h}, \quad$ for all $\quad v_h, z_h \in V_h$.
- Constants independent of contrast & location of interface

CutFEM approximation

- Semi-Norms and Norms:

$$|v^{\pm}|^2_{V^{\pm}} := \rho_{\pm}\|\nabla v^{\pm}\|^2_{L^2(\Omega^{\pm})} + \sum_{e\in\mathcal{E}_h^{\Gamma,\pm}} \gamma_{\pm}|e|\| [[\nabla v^{\pm}]] \|^2_{L^2(e)} \qquad \forall v^{\pm} \in V^{\pm} .$$

$$\|v_h\|^2_{V_h} := |v^{+}|^2_{V^{+}} + |v^{-}|^2_{V^{-}} + \sum_{K\in\mathscr{T}_h^{\Gamma}} \frac{\gamma_\Gamma}{h_K}\{\rho\}_H\| [v_h] \|^2_{L^2(K\cap\Gamma)} \quad \forall v_h \in V_h = V^{+}\times V^{-} .$$

- Stability $\quad a_h(v_h, v_h) \gtrsim \|v_h\|^2_{V_h}, \quad$ for all $\quad v_h \in V_h$
- Continuity $\quad |a_h(u_h, v_h)| \lesssim \|u_h\|_{V_h}\|v_h\|_{V_h}, \quad$ for all $\quad v_h, z_h \in V_h.$
- Constants independent of contrast & location of interface

Ghost penalization provides:

$$\boxed{\|\nabla v^{\pm}\|^2_{L^2(\Omega_h^{\pm})} \lesssim \|\nabla v^{\pm}\|^2_{L^2(\Omega^{\pm})} + \sum_{e\in\mathcal{E}_h^{\Gamma,\pm}} \gamma_{\pm}|e|\| [[\nabla v^{\pm}]] \|^2_{L^2(e)} .}$$

$\Longrightarrow \qquad \kappa(A_h) = O\left(\frac{\rho_+}{\rho_-}h^{-2}\right)$ Cut cells do not degrade it!

Some Preconditioning Strategies for Unfitted Methods

▷ Old but Good idea: [Bank & Scott (1989)]
basis re-scaling (Diagonal smoother)

Linears 3D ✓
Linears in 2D: $\kappa(A_h) = O\left(N(1 + \log \left|\frac{h_{max}}{h_{min}}\right|)\right)$ ✓

- **XFem & Unfitted:** Diagonal scaling (Jacobi smoother)
  - ▷ [Lehrenfeld & Reusken (2017)] Schwarz method
- **FiniteCell Method:** Need of preconditioners for High order
  - ▷ [Prenter & Verhoosel & van Zwieten & E.H. van Brummelen (2017)]:
- **CutFem Method**
  - ▷ [Ludescher & Gross & Reusken (2018)] Multigrid (soft inclusion?)

  ....

Some Simple Preconditioners for CutFEM: outline

- Block-Jacobi: One Level method
  - ▷ Overlapping decomposition $\Omega_h^+ \cup \Omega_h^-$ (overlap in Ω_h^Γ)
- Dirichlet-Neuman:
  - ▷ Non- Overlapping decomposition $\Omega^+ \cup \Gamma \cup \Omega^- = \Omega_{h,0}^+ \cup \overline{\Omega_h^\Gamma} \cup \Omega_{h,0}^-$
  - One Level method & Two Level methods

Aim:

- Optimality wrt h
- Robustness w.r.t. ρ;
- robustness w.r.t $D^+ := \text{diam}(\Omega^+)$ for floating domain
- Scalable (result valid for many inclusions) ?

One-level Schwarz for CutFEM

- Restriction operators: $\mathcal{R}_{\pm} : V_h \longrightarrow \{V^{\pm}, 0\}$
- Local Solvers: $a^{\pm} : V^{\pm} \times V^{\pm} \longrightarrow \mathbb{R}$ are the restriction of $a_h(\cdot,\cdot)$ to the subspaces $\{V^{+} \times 0\}$ and $\{0 \times V^{-}\}$ respectively:

$$a^{\pm}(u^{\pm}, v^{\pm}) = a_h(R_{\pm}^{T} u^{\pm}, \mathcal{R}_{\pm}^{T} v^{\pm}) \qquad \forall\, u^{\pm},\, v^{\pm} \in V^{\pm} \ .$$

- Projection operators: $P^{\pm} = \mathcal{R}_{\pm}^{T} \widehat{P}_{\pm} : V_h \longrightarrow \mathcal{R}_{\pm}^{T} V^{\pm}$, with $\widehat{P}_{\pm} : V_h \longrightarrow V^{\pm}$:

$$a^{\pm}(\widehat{P}_{\pm} u_h, v^{\pm}) = a_h(u_h, \mathcal{R}_{\pm}^{T} v^{\pm}) \qquad \forall\, v^{\pm} \in V^{\pm} \ .$$

- one-level additive Schwarz operator: $\mathcal{B}_{jac}\mathcal{A} := P^{+} + P^{-}$

- Remark: $a^{+}(u^{+}, v^{+}) + a^{-}(u^{-}, v^{-}) \neq a_h(u, v)$

One-level Schwarz for CutFEM

One-level Schwarz for CutFEM

- Ω^+ floating and $\mathcal{B}_{jac}\mathcal{A} := P^+ + P^-$;

$$\kappa(\mathcal{B}_{jac}\mathcal{A}) \simeq \frac{\mathrm{diam}(\Omega^-)\gamma_\Gamma}{h}$$

- Robustness w.r.t. ρ;
- robustness w.r.t $D^+ := \mathrm{diam}(\Omega^+)$
- can be easily made Scalable (result valid for many inclusions);

One-level Schwarz for CutFEM

One-level Schwarz for CutFEM

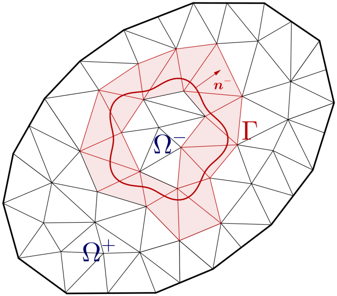
- Ω^+ floating and $\mathcal{B}_{jac}\mathcal{A} := P^+ + P^-$; $V^\pm = \mathbb{P}^p(\mathscr{T}_h) \cap \mathcal{C}^0(\Omega)$.

$$\kappa(\mathcal{B}_{jac}\mathcal{A}) \simeq \frac{\mathrm{diam}(\Omega^-)\gamma_\Gamma p^2}{h}$$

- Robustness w.r.t. ρ;
- robustness w.r.t $D^+ := \mathrm{diam}(\Omega^+)$
- can be easily made Scalable (result valid for many inclusions);

Neuman-Dirichlet preconditioner

Non- Overlapping decomp. $\Omega^+ \cup \Gamma \cup \Omega^- = \boxed{\Omega^+_{h,0} \cup \overline{\Omega^\Gamma_h} \cup \Omega^-_{h,0}}$



- local spaces on $\Omega^\pm_{h,0}$: $\quad V_0^\pm = \{v \in V^\pm \; : \; v|_K \equiv 0 \text{ on } \Omega^\Gamma_h\}.$
- Fat Trace spaces: $\quad W^\pm := \{v \in V^\pm \text{ restricted to } \Omega^\Gamma_h\}$

(towards..) Non-overlapping preconditioner

- Idea: orthogonal (w.r.t. a_h) splitting

$$\boxed{u_h = \mathcal{P}_h u + \mathcal{H}_h u \qquad \text{s.t.} \qquad a_h(\mathcal{H}_h u, \mathcal{P}_h u) = 0}$$

- $\mathcal{P}_h u = (\mathcal{P}^+ u^+, \mathcal{P}^- u^-)$ solution of local problems in $V_0^\pm$

  local solution operators $\mathcal{P}^\pm : V_h \longrightarrow V_0^\pm$ defined by

$$a^\pm(\mathcal{P}^\pm u^\pm, v^\pm) = (f^\pm, v^\pm)_{\Omega^\pm} \qquad \forall\, v^\pm \in V_0^\pm \,.$$

$$a^\pm(u^\pm, v^\pm) = \rho_\pm \left(\nabla u^\pm, \nabla v^\pm\right)_{\Omega^+} + \gamma_\pm \rho_\pm \langle |e| \left[\!\left[\nabla u^\pm\right]\!\right], \left[\!\left[\nabla v^\pm\right]\!\right] \rangle_{\mathcal{E}_h^{\Gamma,\pm}} \qquad u^\pm, v^\pm \in V_0^\pm$$

- $\mathcal{H}_h u = (\mathcal{H}^+ u^+, \mathcal{H}^- u^-)$ discrete *harmonic extension* (suitably defined...)

$$a_h(\mathcal{H}_h u_h, \mathcal{H}_h v_h) = (f, v_h)_\Omega - a_h(\mathcal{P}_h u_h, v_h) \qquad \forall\, v_h \in V_h$$

$\mathcal{H}_h u = u_h - \mathcal{P}_h u$ live on Fat Trace space $W^+ \times W^-$

(towards..) Non-overlapping preconditioner

- Idea: orthogonal (w.r.t. a_h) splitting

$$u_h = \mathcal{P}_h u + \mathcal{H}_h u \qquad \text{s.t.} \qquad a_h(\mathcal{H}_h u, \mathcal{P}_h u) = 0$$

- $\mathcal{P}_h u = (\mathcal{P}^+ u^+, \mathcal{P}^- u^-)$ solution of local problems in $V_0^\pm$

  local solution operators $\mathcal{P}^\pm : V_h \longrightarrow V_0^\pm$ defined by

$$a^\pm(\mathcal{P}^\pm u^\pm, v^\pm) = (f^\pm, v^\pm)_{\Omega^\pm} \qquad \forall\, v^\pm \in V_0^\pm \,.$$

$$a^\pm(u^\pm, v^\pm) = \rho_\pm \left(\nabla u^\pm, \nabla v^\pm\right)_{\Omega^+} + \gamma_\pm \rho_\pm \langle |e| \left[\left[\nabla u^\pm\right]\right], \left[\left[\nabla v^\pm\right]\right] \rangle_{\mathcal{E}_h^{\Gamma,\pm}} \qquad u^\pm, v^\pm \in V_0^\pm$$

- $\mathcal{H}_h u = (\mathcal{H}^+ u^+, \mathcal{H}^- u^-)$ discrete *harmonic extension* (suitably defined...)

$$a_h(\mathcal{H}_h u_h, \mathcal{H}_h v_h) = (f, v_h)_\Omega - a_h(\mathcal{P}_h u_h, v_h) \qquad \forall\, v_h \in V_h$$

  $\mathcal{H}_h u = u_h - \mathcal{P}_h u$ live on Fat Trace space $W^+ \times W^-$

Aim: build a preconditioner for the Schur complement: $\mathcal{S} : W_h \longrightarrow W_h$

$$< \mathcal{S}\eta, w >_{\ell^2(W^+)} := a_h(\mathcal{H}_h \eta, \mathcal{H}_h w) \qquad \forall\, \eta, w \in W_h$$

Algebraic formulation

- dofs for $V^{+} = \{V_0^{+}, W^{+}\}$
  - I^{+}: interior dofs V_0^{+}
  - W^{+} interface dofs for $V^{+} = \{V_0^{+}, W^{+}\}$
- all dofs for V^{-} (interior and on interface)

The linear system $\mathcal{A}\,\mathbf{U} = \mathbf{F}$ in block form:

$$\begin{bmatrix} \mathcal{A}_{I^+I^+} & \mathcal{A}_{I^+W^+} & 0 \\ \mathcal{A}_{W^+I^+} & \mathcal{A}_{W^+W^+} & \mathcal{A}_{W^+V^-} \\ 0 & \mathcal{A}_{V^-W^+} & \mathcal{A}_{V^-V^-} \end{bmatrix} \begin{bmatrix} \mathbf{U}_{I^+} \\ \mathbf{U}_{W^+} \\ \mathbf{U}_{V^-} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_{I^+} \\ \mathbf{F}_{W^+} \\ \mathbf{F}_{V^-} \end{bmatrix}$$

Algebraic formulation

- dofs for $V^+ = \{V_0^+, W^+\}$
  - I^+: interior dofs V_0^+
  - W^+ interface dofs for $V^+ = \{V_0^+, W^+\}$
- all dofs for V^- (interior and on interface)

The linear system $\mathcal{A}\,\mathbf{U} = \mathbf{F}$ in block form:

$$\begin{bmatrix} \mathcal{A}_{I^+I^+} & \mathcal{A}_{I^+W^+} & 0 \\ \mathcal{A}_{W^+I^+} & \boxed{\mathcal{A}^+_{W^+W^+} + \mathcal{A}^-_{W^+W^+}} & \mathcal{A}_{W^+V^-} \\ 0 & \mathcal{A}_{V^-W^+} & \mathcal{A}_{V^-V^-} \end{bmatrix} \begin{bmatrix} \mathbf{U}_{\mathbf{I}^+} \\ \mathbf{U}_{\mathbf{W}^+} \\ \mathbf{U}_{V^-} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_{\mathbf{I}^+} \\ \mathbf{F}_{\mathbf{W}^+} \\ \mathbf{F}_{V^-} \end{bmatrix}$$

- Elimination of the I^+ and V^- dofs $\Longrightarrow$ $\mathcal{S}\,\mathbf{U}_{W^+} = \mathbf{G}_{W^+}$

$$\boxed{\mathcal{S} = \mathcal{S}_+ + \mathcal{S}_-}$$

towards Neuman-Dirichlet preconditioner: local Schur Complement

$$\mathcal{A}_{W^+W^+} = \mathcal{A}^+_{W^+W^+} + \mathcal{A}^-_{W^+W^+}$$

Schur Complement: $\mathcal{S} = \mathcal{S}_+ + \mathcal{S}_-$

$$\mathcal{S}_+ = \mathcal{A}^+_{W^+W^+} - \mathcal{A}_{W^+I^+}\,\mathcal{A}^{-1}_{I^+I^+}\,\mathcal{A}_{I^+W^+}$$

$$\mathcal{S}_- = \mathcal{A}^-_{W^+W^+} - \mathcal{A}_{W^+V^-}\,\mathcal{A}^{-1}_{V^-V^-}\,\mathcal{A}_{V^-W^+}$$

$$\mathcal{S}\,\mathbf{U}_{W^+} = \mathbf{G}_{W^+}$$

$$\mathbf{G}_{W^+} = \mathbf{F}_{W^+} - \mathcal{A}_{W^+I^+}\,\mathcal{A}^{-1}_{I^+I^+}\,\mathbf{F}_{I^+} - \mathcal{A}_{W^+V^-}\,\mathcal{A}^{-1}_{V^-V^-}\,F_{V^-}$$

We recover $\mathbf{U}_{\mathbf{I}^+}$ and $\mathbf{U}_{I^-}$ via

$$\mathbf{U}_{\mathbf{I}^+} = \mathcal{A}^{-1}_{I^+I^+}(\mathbf{F}_{I^+} - \mathcal{A}_{I^+W^+}\,\mathbf{U}_{\mathbf{W}^+})$$

$$\mathbf{U}_{V^-} = \mathcal{A}^{-1}_{V^-V^-}(\mathbf{F}_{V^-} - \mathcal{A}_{V^-W^+}\,\mathbf{U}_{\mathbf{W}^+})$$

Neuman-Dirichlet preconditioner: Harmonic extension

Auxiliary forms : • $b^+(u^+, v^+) = (\rho^+ \nabla u^+, \nabla v^+)_{\Omega^+} + \gamma_+ \langle |e| \rho_+ [[\nabla u^+]], [[\nabla v^+]] \rangle_{\mathcal{E}_h^{\Gamma,+}}$

- $\mathcal{H}_h : W^+ \longrightarrow V_h$ discrete *harmonic extension* $\mathcal{H}_h \eta^+ := (\mathcal{H}^+ \eta^+, \mathcal{H}^- \eta^+)$
  - ▷ $\mathcal{H}_+ : W^+ \subset W_h \longrightarrow V^+$ discrete *harmonic* w.r.t. $b^+(\cdot, \cdot)$

$$b^+(\mathcal{H}_+ \eta^+, v^+) = 0 \qquad \forall\, v^+ \in V_0^+ , \qquad \mathcal{H}_+ \eta^+ = (\eta^+, 0) \quad \text{on } \Omega_h^\Gamma$$

Neuman-Dirichlet preconditioner: Harmonic extension

Auxiliary forms :
- $b^+(u^+, v^+) = (\rho^+\nabla u^+, \nabla v^+)_{\Omega^+} + \gamma_+\langle |e|\rho_+ [[\nabla u^+]], [[\nabla v^+]]\rangle_{\mathcal{E}_h^{\Gamma,+}}$
- $b^-(u^-, v^-) = (\rho^-\nabla u^-, \nabla v^-)_{\Omega^-} + \gamma_-\langle |e|\rho_- [[\nabla u^-]], [[\nabla v^-]]\rangle_{\mathcal{E}_h^{\Gamma,-}}$

$$+ \sum_{K\in\mathscr{T}_h{}^\Gamma} \frac{\gamma_\Gamma}{h_K}\{\rho\}_H \int_{K\cap\Gamma} [u^+ - u^-][0 - v^-]ds$$

- $\mathcal{H}_h : W^+ \longrightarrow V_h$ discrete *harmonic extension* $\mathcal{H}_h\eta^+ := (\mathcal{H}^+\eta^+, \mathcal{H}^-\eta^+)$
  - ▷ $\mathcal{H}_+ : W^+ \subset W_h \longrightarrow V^+$ discrete *harmonic* w.r.t. $b^+(\cdot,\cdot)$

$$b^+(\mathcal{H}_+\eta^+, v^+) = 0 \quad \forall\, v^+ \in V_0^+ , \qquad \mathcal{H}_+\eta^+ = (\eta^+, 0) \quad \text{on } \Omega_h^\Gamma$$

  - ▷ $\mathcal{H}_- : W^+ \subset W_h \longrightarrow V^-$ discrete *harmonic* w.r.t. $b^-(\cdot,\cdot)$

$$b^-(\mathcal{H}_-\eta^+, v^-) = 0 \quad \forall\, v^- \in V^- , \quad \mathcal{H}_-\eta^+ = (\eta^+, (\mathcal{H}_-\eta^+)^-) \in W^+ \times W^- \quad \text{on } \Omega_h^\Gamma .$$

Neuman-Dirichlet preconditioner: Harmonic extension

- $\mathcal{H}_h : W^+ \longrightarrow V_h$ discrete *harmonic extension* $\mathcal{H}_h\eta^+ := (\mathcal{H}^+\eta^+, \mathcal{H}^-\eta^+)$

Auxiliary forms :
- $b^+(u^+, v^+) = (\rho^+\nabla u^+, \nabla v^+)_{\Omega^+} + \gamma_+\langle |e|\rho_+ [\![\nabla u^+]\!], [\![\nabla v^+]\!]\rangle_{\mathcal{E}_h^{\Gamma,+}}$
- $b^-(u^-, v^-) = (\rho^-\nabla u^-, \nabla v^-)_{\Omega^-} + \gamma_-\langle |e|\rho_- [\![\nabla u^-]\!], [\![\nabla v^-]\!]\rangle_{\mathcal{E}_h^{\Gamma,-}}$

$$+ \sum_{K\in\mathcal{T}_h^\Gamma} \frac{\gamma_\Gamma}{h_K}\{\rho\}_H \int_{K\cap\Gamma} [u^+ - u^-][0 - v^-]ds$$

▷ $\mathcal{H}_+ : W^+ \subset W_h \longrightarrow V^+$ discrete *harmonic* w.r.t. $b^+(\cdot,\cdot)$

$$b^+(\mathcal{H}_+\eta^+, \mathcal{H}_+\eta^+) = \min_{v^+\in V_0^+} |v|^2_{V^+} \qquad \text{if } |\cdot|_{V^+} \text{ is a norm .}$$

▷ $\mathcal{H}_- : W^+ \subset W_h \longrightarrow V^-$ discrete *harmonic* w.r.t. $b^-(\cdot,\cdot)$

$$b^-(\mathcal{H}_-\eta^+, \mathcal{H}_-\eta^+) \asymp \min_{\substack{v^-\in V^- \\ (v^- - \mathcal{H}_-\eta^+)\in V_0^+}} \left(|v^-|^2_{V^-} + \sum_{K\in\mathcal{T}_h^\Gamma} \frac{\gamma_\Gamma}{h_K}\{\rho\}_H \|[\eta^+ - \mathcal{H}_-\eta^+]\|^2_{L^2(K\cap\Gamma)} \right) .$$

Neuman-Dirichlet preconditioner: local Schur complements

- $\mathcal{H}_h u = (\mathcal{H}^+ u^+, \mathcal{H}^- u^-)$ discrete *harmonic extension*

$$< \mathcal{S}\eta, w >_{\ell^2(W^+)} = a_h(\mathcal{H}_h\eta^+, \mathcal{H}_h w^+) \qquad \forall\, \eta^+, w^+ \in W^+ \;,$$

$$\begin{cases} < \mathcal{S}_+\eta, w >_{\ell^2(W^+)} := b^+(\mathcal{H}_+\eta^+, \mathcal{H}_+ w^+) & \forall\, \eta^+, w^+ \in W^+ \;, \\ < \mathcal{S}_-\eta, w >_{\ell^2(W^+)} := b^-(\mathcal{H}_-\eta^+, \mathcal{H}_- w^+) & \forall\, \eta^+, w^+ \in W^+ \;. \end{cases}$$

Obvious Lemma: $\mathcal{S} \simeq \mathcal{S}_+ + \mathcal{S}_-$.

- ND: Preconditioner for $\mathcal{S}$ based on $\mathcal{S}_+$ (largest coefficient)
  - Case 1: Ω^+ is“not” a floating subdomain
  - Case 2: Ω^+ is floating

Case 1: Ω^+ is"not" a floating subdomain

- Idea: Choose $\mathcal{S}_+^{-1}$ as preconditioner (recall $\rho^+ \geq \rho^-$)

- $\partial\Omega^+ \cap \partial\Omega \neq \emptyset \quad \Longrightarrow \quad |\cdot|_{V^+}$ is a norm (and $|\cdot|_{V^+} \asymp \sqrt{\rho_+}\|\cdot\|_{H^1(\Omega_h^+)}$)

$$< \eta^+, \mathcal{S}_+\eta^+ >_{\ell^2(W^+)} = b^+(\mathcal{H}_+\eta^+, \mathcal{H}_+\eta^+) = \min_{v^+\in V_0^+} |v|^2_{V^+}$$

$\Longrightarrow \mathcal{S}_+$ is invertible ✓

Theorem: Ω^+ is"not" a floating subdomain:

$$a_h(\mathcal{H}_h w^+, \mathcal{H}_h w^+) \lesssim b^+(\mathcal{H}_+ w, \mathcal{H}_+ w) \lesssim a_h(\mathcal{H}_h w^+, \mathcal{H}_h w^+)$$

$$\Longrightarrow \qquad \mathcal{S}_+ \simeq \mathcal{S} = \mathcal{S}_+ + \mathcal{S}_-$$

- Ingredient: Extension operator from [Burman, Guzman, Sarkis (2017)]

$\Longrightarrow \mathcal{S}_+^{-1}$ is Optimal and Robust preconditioner

Case 2: Ω^+ is a floating subdomain. **One Level**

- $\partial\Omega^+ \cap \partial\Omega \neq \emptyset \quad \Longrightarrow \quad |\cdot|_{V^+}$ is NOT a norm $\quad \Longrightarrow \quad \nexists\, \mathcal{S}_+^{-1}$ **XX**
- **One-Level method:** regularize $\widehat{\mathcal{S}}_{+,reg}$

$$< \widehat{\mathcal{S}}_{+,reg}\eta^+, w^+ >_{\ell^2(W^+)} = < \mathcal{S}_+\eta^+, w^+ >_{\ell^2(W^+)} + \epsilon < \eta^+, w^+ >_{\ell^2(W^+)} \quad \forall\, \eta^+, w^+ \in W^+ \,.$$

$$b_\Gamma^+(\mathcal{H}_+ w, \mathcal{H}_+ w) = \min_{\substack{v^+ \in V^+ \\ (v^+ - \mathcal{H}_+ w) \in V_0^+}} \left(|v^+|_{V^+} + \frac{\{\rho\}_H}{D_+} \|v^+\|^2_{L^2(\Gamma)} \right)$$

$$b_M^+(\mathcal{H}_+^M w, \mathcal{H}_+^M w) \quad \min_{\substack{v^+ \in V^+ \\ (v^+ - \mathcal{H}_+ w) \in V_0^+}} \left(|v^+|_{V^+} + \frac{\{\rho\}_H}{D_+^2} \|v^+\|^2_{L^2(\Omega_h^+)} \right)$$

Optimal & Robust preconditioner

$$\mathcal{S} \lesssim \mathcal{S}_+^\Gamma \lesssim C_0 \mathcal{S} \qquad \mathcal{S} \lesssim \mathcal{S}_+^M \lesssim \theta C_0 \mathcal{S} \qquad C_0 \simeq \frac{\mathrm{diam}(\Omega^-)}{\mathrm{diam}(\Omega^+)} \qquad \theta \leq 1$$

Case 2: Ω^+ is a floating subdomain. **Two-Level**

- $\partial\Omega^+ \cap \partial\Omega \neq \emptyset \quad \Longrightarrow \quad |\cdot|_{V^+}$ is NOT a norm $\quad \Longrightarrow \quad \nexists\, \mathcal{S}_+^{-1}$ **XX**
- **Two -Level method:** consider splitting $W^+ = \widetilde{W} \oplus W^0$
  - $W^0 = \ker(\mathcal{S}_+)$ (one dimensional coarse space)
  - $\widetilde{W} \simeq W^+ \setminus \mathbb{R}$
  - define $\widehat{\mathcal{S}}_+ = \mathcal{S}_+|_{\widetilde{W}} : \widetilde{W} \longrightarrow \widetilde{W}$

$$\mathcal{B}_{two} = \widehat{\mathcal{S}}_+^{-1} + \mathcal{S}_0^{-1}$$

with $\qquad (\mathcal{S}_0\eta_0, w_0)_{\ell^2(W^+)} = a_h(\mathcal{H}_h\eta_0, \mathcal{H}_h w_0) \qquad \forall\, \eta_0, w_0 \in W_0\ .$

Optimal & Robust preconditioner

$$\mathcal{S} \lesssim \widehat{\mathcal{S}}_+ + \mathcal{S}_0 \lesssim \mathcal{S}$$

$\longrightarrow$ classical Schwarz theory...

Ω^+ non-floating: Optimality wrt h

- $\Omega^+ = (0, 0.45) \times (0, 1)$ and $\Omega^- = (0.45, 1) \times (0, 1)$
- Q^1-elements. $\rho^+ = \rho^- = 1$
- PCG: 10^{-6} residual reduction stopping criteria

$1/h$	**full cg**		**schur noprec**		**schur ND prec**	
	κ_2	it	κ_2	it	κ_2	it
8	4.16e+2	48	62.20	16	2.05	6
16	1.63e+3	94	1.44e+2	25	2.04	6
32	6.49e+3	183	3.18e+2	40	2.03	6
64	2.59e+4	370	6.75e+2	62	2.01	5
128	1.03e+5	732	1.39e+3	91	2.01	5
256	4.14e+5	1422	2.84e+3	137	2.01	5

Ω^+ non-floating:: Robustness wrt ρ (*soft inclusion*)

- $\Omega^+ = (0, 0.45) \times (0, 1)$ and $\Omega^- = (0.45, 1) \times (0, 1)$
- Q^1-elements. $h = 1/64, \rho^- = 1$
- PCG: 10^{-6} residual reduction stopping criteria

ρ_+	**full cg**		**schur noprec**		**schur ND prec**	
	κ_2	it	κ_2	it	κ_2	it
1	2.59e+4	370	6.75e+2	62	2.01	5
10^2	4.41e+5	2247	1.30e+3	82	1.06	4
10^4	4.27e+7	11567	1.34e+3	83	1.01	3
10^6	4.27e+9	25685	1.35e+3	83	1.01	3

Ω^+ floating: Optimality w.r.t h

- Ω^+ a disk of radius 0.15 and $\Omega^- = (0,1)^2 \setminus \bar{\Omega}^+$
- Q^1-elements $\rho^+ = \rho^- = 1$
- PCG: 10^{-6} residual reduction stopping criteria

$1/h$	full cg		schur b_Γ		schur b_M	
	κ_2	it	κ_2	it	κ_2	it
8	6.38e+3	252	9.92	12	3.51	14
16	1.77e+4	520	10.54	14	2.11	14
32	5.83e+4	863	11.92	18	2.09	14
64	2.14e+4	1625	13.54	22	2.08	14
128	8.19e+5	3163	14.65	24	2.13	14
256	3.20e+6	6140	15.97	24	2.19	14

Ω^+ floating: Optimality w.r.t h

- Ω^+ a disk of radius 0.15 and $\Omega^- = (0,1)^2 \setminus \bar{\Omega}^+$
- Q^1-elements $\rho^+ = \rho^- = 1$
- PCG: 10^{-6} residual reduction stopping criteria

$1/h$	full cg		Two-Level		schur b_M	
	κ_2	it	κ_2	it	κ_2	it
8	6.38e+3	252	6.76	11	3.51	14
16	1.77e+4	520	6.39	15	2.11	14
32	5.83e+4	863	6.29	16	2.09	14
64	2.14e+4	1625	6.34	16	2.08	14
128	8.19e+5	3163	6.37	16	2.13	14
256	3.20e+6	6140	6.39	16	2.19	14

Ω^+ floating: Robustness wrt ρ (hard inclusion)

- Ω^+ a disk of radius 0.15 and $\Omega^- = (0,1)^2 \setminus \bar{\Omega}^+$
- Q^1-elements. $h = 1/64, \rho^- = 1$
- PCG: 10^{-6} residual reduction stopping criteria

ρ_+	full CG		schur b_Γ		schur b_M	
	κ_2	it	κ_2	it	κ_2	it
1	2.14e+5	1625	14.65	24	2.13	14
10^2	2.00e+7	12906	9.95	8	1.83	5
10^4	2.00e+9	$>$100000	9.93	5	1.83	4
10^6	5.70e+10	$>$100000	9.93	4	1.83	3
10^8	4.20e+12	$>$100000	9.93	3	1.83	3

Ω^+ floating: Robustness wrt ρ (hard inclusion)

- Ω^+ a disk of radius 0.15 and $\Omega^- = (0,1)^2 \setminus \bar{\Omega}^+$
- Q^1-elements. $h = 1/64, \rho^- = 1$
- PCG: 10^{-6} residual reduction stopping criteria

ρ_+	full CG		Two-Level		schur b_M	
	κ_2	it	κ_2	it	κ_2	it
1	2.14e+5	1625	6.37	16	2.13	14
10^2	2.00e+7	12906	6.33	6	1.83	5
10^4	2.00e+9	$>$100000	6.33	4	1.83	4
10^6	5.70e+10	$>$100000	6.33	3	1.83	3
10^8	4.20e+12	$>$100000	6.33	3	1.83	3

Ω^+ floating: Optimality while decreasing diam(Ω^+)

Ω^+ a disk of radius D^+ and $\Omega^- = (0,1)^2 \setminus \bar{\Omega}^+$

- $\mathbb{Q}^1$-elements $\rho^+ = \rho^- = 1$; h=1/64
- PCG: 10^{-6} residual reduction stopping criteria

diam(Ω^+)	full cg		schur b_Γ		schur b_M	
	κ_2	it	κ_2	it	κ_2	it
0.4	6.38e+3	252	4.91	21	7.31	18
0.2	1.77e+4	520	14.65	24	2.13	14
0.1	5.83e+4	863	22.88	29	2.21	14
0.05	2.14e+4	1625	28.34	41	3.20	15
0.02	8.19e+5	3163	33.65	54	5.67	15
0.01	3.20e+6	6140	38.89	59	10.46	17

Ω^+ floating: Optimality while decreasing diam(Ω^+)

Ω^+ a disk of radius D^+ and $\Omega^- = (0,1)^2 \setminus \bar{\Omega}^+$

- $\mathbb{Q}^1$-elements $\rho^+ = \rho^- = 1$; h=1/64
- PCG: 10^{-6} residual reduction stopping criteria

diam(Ω^+)	full cg		Two-Level		schur ND prec	
	κ_2	it	κ_2	it	κ_2	it
0.4	6.38e+3	252	21.46	19	7.31	18
0.2	1.77e+4	520	6.27	16	2.13	14
0.1	5.83e+4	863	3.75	14	2.21	14
0.05	2.14e+4	1625	2.65	14	3.20	15
0.02	8.19e+5	3163	2.49	14	5.67	15
0.01	3.20e+6	6140	3.25	11	10.46	17

Ω^+ floating: Optimality while decreasing diam(Ω^+)

- Ω^+ a disk of radius D^+ and $\Omega^- = (0,1)^2 \setminus \bar{\Omega}^+$
- $\mathbb{Q}^2$-elements $\rho^+ = \rho^- = 1$; h=1/64
- PCG: 10^{-6} residual reduction stopping criteria

diam(Ω^+)	Two level		schur ND prec	
	κ_2	it	κ_2	it
0.4	21.39	10	2.61	14
0.2	6.56	9	3.25	14
0.1	3.86	8	6.04	16
0.05	2.79	9	9.19	17
0.02	2.81	9	16.31	21
0.01	3.78	9	45.07	23

Concluding remarks & Outlook

- Balancing NN (using the whole fat trace space)
- extension to Stokes
- Space decomposition approach ?
- AMG ....?
- Still quite a few things to understand ?