A Simple Solver for Simulating Fluid-Structure Interactions in 2D

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(Joint work with Arin Gregorian)

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Numerical Analysis of Coupled and Multi–Physics Problems with Dynamic Interfaces

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Immersed Boundary Problems

- Fluid-structure interaction belong to a more general class of fluid problems with internal boundaries
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    - elasticity, *e.g.*, deformation of red blood cells
- Typically presented in Eulerian-Lagrangian formulation – numerically challenging

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  - still challenging...

Elasticity – Lagrangian formulation of $\mathbf{F}$

$$\rho \left(\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\, \mathbf{u} \right) - \nabla \cdot (\mu\, D(\mathbf{u})) + \nabla p = \mathbf{F}$$

$$\mathbf{F}(\mathbf{x}, t) = \int_{s_1}^{s_2} \mathbf{f}(s,t)\delta(\mathbf{x} - \mathbf{X}(s,t))\, ds$$

$\mathbf{f}(s,t)$– body force density with respect of measure ds

s – parametrization of interface satisfying $\dfrac{\partial \mathbf{X}(s,t)}{\partial t} = \mathbf{u}(\mathbf{X}(s,t), t)$

$\mathbf{f}(s,t) = \dfrac{\partial}{\partial s} \left(T(s,t)\, \tau(s,t) \right)$, T – tension, τ - unit tangent (Peskin, 1981)

Elasticity – Eulerian Formulation

Force derived from Energy (Cottet et. al., 2005 - 08):

$$\mathcal{E}_a(\phi) = \int_\Omega E\left(|\nabla\phi|\right) \frac{1}{\epsilon}\zeta\left(\frac{\phi}{\epsilon}\right) dx$$

where:

- E - stress-strain relationship: $E(r) = \lambda(r-1)$, it accounts for the response of the membrane to a change in area

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- $\phi(x,t)$ - level set function satisfying:
    - $\phi(x,0) = \Gamma_0$ – initial interface position ($\phi < 0$ inside, $\phi > 0$ outside)
    - $\phi_t + \mathbf{u} \cdot \nabla\phi = 0$
    - interface location $\Gamma_t = \{x \in \Omega : \phi(x) = 0\}$
    - ζ cut-off function: $|\nabla\phi|\frac{1}{\epsilon}\zeta\left(\frac{\phi}{\epsilon}\right) \to \delta_{\phi=0}$ as $\epsilon \to 0$

Elasticity – Eulerian Formulation (cont'd)

This leads to:

$$\frac{d}{dt}\mathcal{E}_a(\phi) = -\int_\Omega \mathbf{F}_a(\mathbf{x},t)\cdot \mathbf{u}\, ds$$

and using the divergence theorem, we arrive at:

$$\mathbf{F}_a(\mathbf{x},t) = \left\{ \nabla\left[E(|\nabla\phi|)\right] - \nabla\cdot\left[E(|\nabla\phi|)\frac{\nabla\phi}{|\nabla\phi|}\right]\frac{\nabla\phi}{|\nabla\phi|}\right\}|\nabla\phi|\frac{1}{\epsilon}\zeta\left(\frac{\phi}{\epsilon}\right)$$

Remark: Not the only expression for $\mathbf{F}$, it can be written/calculated in tangential plus normal components showing how curvature acts on normal direction

Summary of Eulerian Formulation

$$\begin{cases} \rho_\epsilon(\phi)\left(\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\,\mathbf{u}\right) - \nabla\cdot\left(\mu(\phi)\,D(\mathbf{u})\right) + \nabla p = \mathbf{F}_a(\phi) + \mathbf{F}_c(\phi) \\ \nabla\cdot\mathbf{u} = 0 \\ \phi_t + \mathbf{u}\cdot\nabla\phi = 0 \end{cases}$$

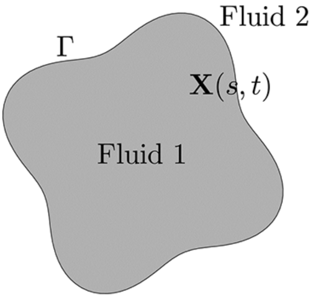
with the elastic and curvature forces

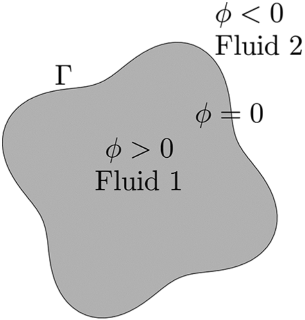
$$\mathbf{F}_a(\phi) = \left\{\nabla\left[E(|\nabla\phi|)\right] - \nabla\cdot\left[E(|\nabla\phi|)\frac{\nabla\phi}{|\nabla\phi|}\right]\frac{\nabla\phi}{|\nabla\phi|}\right\}|\nabla\phi|\frac{1}{\epsilon}\zeta\left(\frac{\phi}{\epsilon}\right),$$

coupling the two equations, and the density and viscosity convected by fluid velocity:

$$\rho_t + \mathbf{u}\cdot\nabla\rho = 0 \qquad \mu_t + \mathbf{u}\cdot\nabla\mu = 0$$

Lagrangian vs. Eulerian





$$\frac{\partial \mathbf{X}(s,t)}{\partial t} = \mathbf{u}(\mathbf{X}(s,t),t)$$

$$\phi_t + \mathbf{u} \cdot \nabla \phi = 0$$

Numerical Solvers – Previous Work

- Lagrangian formulation: Lee and LeVeque (2008):
  - Navier-Stokes: fractional step method with reprojection to enforce $\nabla \cdot \mathbf{u} = 0$
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  - UCLA group (Fedkiw, Merriman, Osher, mid 90s): high resolution ENO for level set, and different approaches for NS (*e.g.*, reprojection, vorticity)... but different physics (*e.g.*, multi fluid, gas bubbles)

Our Approach

- Navier-Stokes: use vorticity formulation...

$$\omega_t + \mathbf{u} \cdot \nabla\omega = \frac{\mu}{\rho}\Delta\omega + \nabla \times \left(\frac{\mathbf{F}}{\rho}\right)$$

$$\nabla \cdot \mathbf{u} = 0$$

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- Why vorticity formulation?
  - note that the vorticity formulation also has the form $\omega_t + H(\nabla\omega) = \ldots$
  - we can use the same scheme for both equations!!!

Our Approach – Other Observations

What about:

(1) $\nabla \cdot \mathbf{u} = 0$? We use stream function ψ:

$$\Delta\psi = -\omega$$

Then, recover $\mathbf{u}$ as:

$$u_{j,k} = \frac{\psi_{j,k+1} - \psi_{j,k-1}}{2\,\Delta y} \quad \text{and} \quad v_{j,k} = -\frac{\psi_{j+1,k} - \psi_{j-1,k}}{2\,\Delta x}$$

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(2) $\rho(\phi)$ and $\mu(\phi)$?... they are not constant!!!
but they remain constant inside and outside the interface
plus we regularize them with the cut-off function:

$$\rho_\epsilon(\phi) = \rho_1 + H\left(\frac{\phi}{\epsilon}\right)(\rho_2 - \rho_1) + \lambda_\theta \frac{1}{\epsilon}\zeta\left(\frac{\phi}{\epsilon}\right), \quad \mu_\epsilon(\phi) = \mu_1 + H\left(\frac{\phi}{\epsilon}\right)(\mu_2 - \mu_1)$$

where $H(r) = \int_{-\infty}^{r} \zeta(s)\,ds$, and λ_θ is the surface density in a reference configuration

Central Scheme for HJ (Kurganov–Tadmor, 2000)

2nd order semi-discrete Scheme for $\phi_t + H(\nabla\phi) = 0$:

$$\frac{d\phi_{j,k}}{dt} = -\frac{1}{4}\left[H(\phi_x^+,\phi_y^+) + H(\phi_x^+,\phi_y^-) + H(\phi_x^-,\phi_y^+) + H(\phi_x^-,\phi_y^-)\right]_{j,k} + \frac{a_{j,k}}{2}\left[(\phi_x^+ - \phi_x^-) + (\phi_y^+ - \phi_y^-)\right]_{j,k}$$

where $(\phi_x^\pm)_{j,k}$ and $(\phi_y^\pm)_{j,k}$ are non-oscillatory (minmod limiter) reconstruction of the first derivatives of ϕ, and

$$a_{j,k} = \max_{\pm} \sqrt{H^2_{\phi_x}(\phi_x^\pm,\phi_y^\pm)_{j,k} + H^2_{\phi_y}(\phi_x^\pm,\phi_y^\pm)_{j,k}}$$

evolved with 2nd order SSP RK scheme, under the CFL condition

$$\Delta t < c\,\frac{\min(\Delta x, \Delta y)}{\max_{j,k}\{a_{j,k}\}} \qquad c < \frac{1}{2} \qquad \text{(provided RHS of HJ is 0!)}$$

Scheme – Additional Details

- cut-off functions:

$$\zeta\left(\frac{r}{\epsilon}\right) = \begin{cases} \frac{1}{2}\left(1+\cos\frac{\pi r}{\epsilon}\right) & \text{if } |r| < \epsilon \\ 0 & \text{otherwise} \end{cases}, \quad H\left(\frac{r}{\epsilon}\right) = \begin{cases} 0 & \text{if } r < -\epsilon \\ \frac{r+\epsilon}{2\,\epsilon} + \frac{\sin\frac{\pi r}{\epsilon}}{2\pi} & \text{if } |r| < \epsilon \\ 1 & \text{if } r > \epsilon \end{cases}$$

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- Poisson equation solved with five-point formula using SOR
- No re-initialization of ϕ, better to regularize (Cottet et. al.), replace

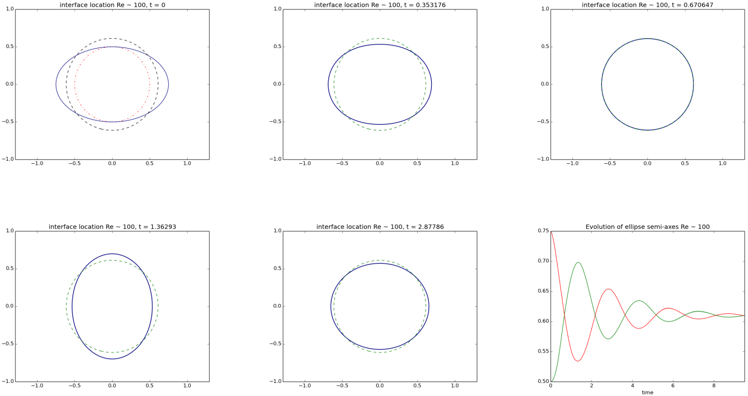
$$\frac{1}{\epsilon}|\nabla\phi|\zeta\left(\frac{\phi}{\epsilon}\right) \quad \text{by} \quad \frac{1}{\epsilon}\zeta\left(\frac{\phi}{\epsilon|\nabla\phi|}\right)$$

  $\phi/|\nabla\phi|$ behaves as distance and carries elasticity information (stretching) that would be lost with reinitialization

Elastic Membrane – $Re \sim 100$

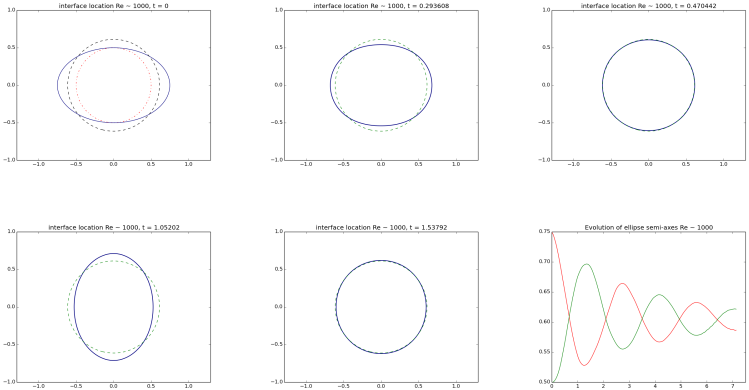
- massless elastic membrane $r = 0.5$ immersed in fluid at rest with $\rho = 1.0$, $\mu = 0.01$
- stretched into elliptical shape with semi-axes $a = 0.75$, $b = 0.5$
- membrane should go back to equilibrium: circle stretched by a factor of 1.262, $r = 0.6124$
- $\phi(\mathbf{x}, 0)$ – signed distance function to elipse multiplied by stretched factor
- Grid size 64×64

Elastic Membrane – $Re \sim 100$



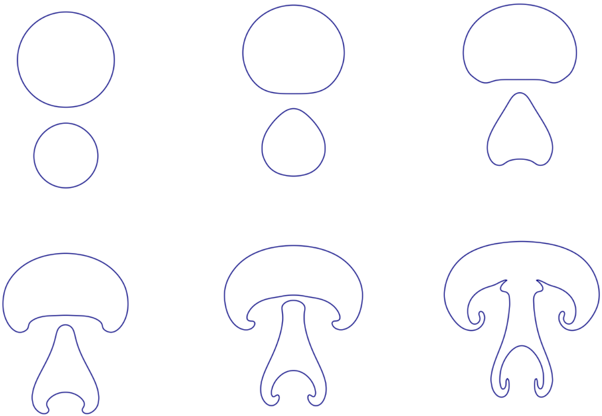
Elastic Membrane – $Re \sim 1000$

- Same as before with $\mu_2 = 0.001$ on a 32×32 grid

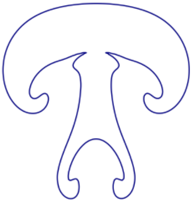


Coalescence of Two Gas Bubbles

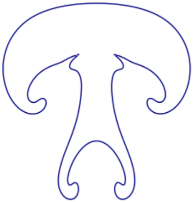
Two merging gas bubbles with $\rho_1 = 1$, $\rho_2 = 10$, $\eta_1 = 2.5 \times 10^{-4}$, $\eta_2 = 5 \times 10^{-4}$



Coalescence of Two Gas Bubbles – Re-Initialization of Level Set



with re-initialization



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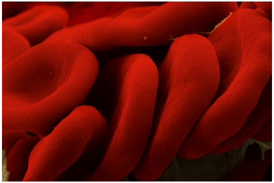
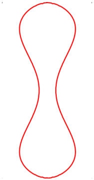
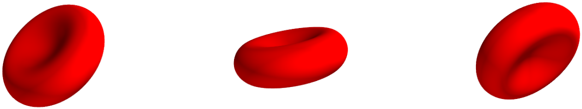
  - incompressible MHD, liquid metals

Work in Progress

We would like to simulate the deformation of (red blood) cells...







- these are the most abundant cells in the human body
- they have no nucleus $\Rightarrow$ challenging deformation mechanics

Thank you very much

Muchas Gracias