Analysis of the finite element discretization of FSI with a fictitious domain approach

Lucia Gastaldi

Università di Brescia
http://lucia-gastaldi.unibs.it

Numerical Analysis of Coupled and Multi-Physics Problems with Dynamic Interfaces - Oaxaca, July 31, 2018

Outline

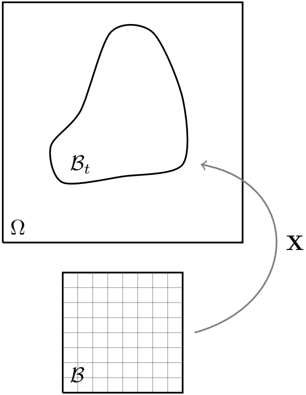
1. FSI
2. Fictitious domain
3. Saddle point problem analysis
4. Compressible solids

Main collaborators:
Daniele Boffi, Luca Heltai, Nicola Cavallini

Eulerian and Lagrangian frameworks

$\Omega \subset \mathbb{R}^d$, $d = 2, 3$
$\mathbf{x}$ Eulerian variable in Ω

$\mathcal{B}_t$ deformable structure domain
$\mathcal{B}_t \subset \mathbb{R}^m$, $m = d, d-1$
s Lagrangian variable in $\mathcal{B}$
$\mathbf{X}(\cdot, t) : \mathcal{B} \to \mathcal{B}_t$ position of the solid
$\mathbb{F} = \dfrac{\partial \mathbf{X}}{\partial s}$ deformation gradient



$\mathbf{u}(\mathbf{x}, t)$ material velocity

$$\mathbf{u}(\mathbf{x}, t) = \frac{\partial \mathbf{X}}{\partial t}(s, t) \quad \text{where} \quad \mathbf{x} = \mathbf{X}(s, t)$$

Model assumptions

Conservation of momenta, in absence of external forces,

$$\rho\dot{\mathbf{u}} = \rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}\,\mathbf{u}\right) = \operatorname{div}\boldsymbol{\sigma} \quad \text{in } \Omega$$

Mass conservation

$$\dot{\rho} + \rho\operatorname{div}\mathbf{u} = 0$$

Structural material with density ρ_s different from fluid density ρ_f

$$\rho = \begin{cases} \rho_f & \text{in } \Omega\setminus\mathcal{B}_t \\ \rho_s & \text{in } \mathcal{B}_t \end{cases}$$

In our case the Cauchy stress tensor has the following form

$$\boldsymbol{\sigma} = \begin{cases} \boldsymbol{\sigma}_f & \text{in } \Omega\setminus\mathcal{B}_t \\ \boldsymbol{\sigma}_s & \text{in } \mathcal{B}_t \end{cases}$$

Model assumptions

Thick incompressible solid

- **Incompressible fluid**: $\boldsymbol{\sigma}_f = -p_f \mathbb{I} + \nu_f \boldsymbol{\nabla}_{sym} \mathbf{u}_f$
  where $\boldsymbol{\nabla}_{sym} \mathbf{u} = \boldsymbol{\nabla} \mathbf{u} + (\boldsymbol{\nabla} \mathbf{u})^\top$ and p_f pressure
- **Visco-hyperelastic incompressible material**: $\boldsymbol{\sigma}_s = \boldsymbol{\sigma}_s^f + \boldsymbol{\sigma}_s^e$
  with $\boldsymbol{\sigma}_s^f = -p_s \mathbb{I} + \nu_s \boldsymbol{\nabla}_{sym} \mathbf{u}_s$ (p_s Lagrange multiplier associated with the incompressibility constraint) and $\boldsymbol{\sigma}_s^e$ elastic part of the stress

  The Piola–Kirchhoff stress tensor takes into account the change of variable

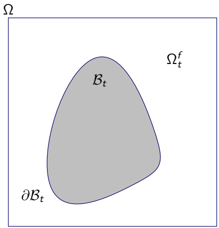
  $$\mathbb{P} = |\mathbb{F}| \boldsymbol{\sigma}_s^e \mathbb{F}^{-\top}$$

  and

  $$\mathbb{P}(\mathbb{F}) = \frac{\partial W}{\partial \mathbb{F}}$$

  where W is the potential energy density

FSI problem (thick incompressible solid)

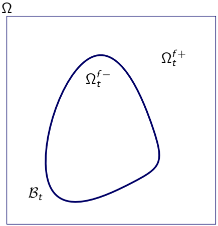


$$
\begin{aligned}
&\rho_f\left(\frac{\partial \mathbf{u}_f}{\partial t}+\mathbf{u}_f\cdot\nabla\,\mathbf{u}_f\right)=\operatorname{div}\boldsymbol{\sigma}_f && \text{in } \Omega\setminus\mathcal{B}_t\\
&\operatorname{div}\mathbf{u}_f=0 && \text{in } \Omega\setminus\mathcal{B}_t\\
&\rho_s\frac{\partial \mathbf{u}_s}{\partial t}=\operatorname{div}_s(|\mathbb{F}|\boldsymbol{\sigma}_s^f\mathbb{F}^{-\top}+\mathbb{P}(\mathbb{F})) && \text{in } \mathcal{B}\\
&\operatorname{div}_s\mathbf{u}_s=0 && \text{in } \mathcal{B}\\
&\mathbf{u}_f=\mathbf{u}_s && \text{on } \partial\mathcal{B}_t\\
&\boldsymbol{\sigma}_f\mathbf{n}_f=-(\boldsymbol{\sigma}_s^f+|\mathbb{F}|^{-1}\mathbb{P}\mathbb{F}^\top)\mathbf{n}_s && \text{on } \partial\mathcal{B}_t
\end{aligned}
$$

$$
\boldsymbol{\sigma}_f=-p_f\mathbb{I}+\nu_f\,\nabla_{sym}\,\mathbf{u}_f \qquad \boldsymbol{\sigma}_s^f=-p_s\mathbb{I}+\nu_s\,\nabla_{sym}\,\mathbf{u}_s \qquad \mathbf{u}_s=\frac{\partial\mathbf{X}}{\partial t}
$$

$+$ initial and boundary conditions

FSI problem (thin solid)



$$\rho_f\left(\frac{\partial \mathbf{u}_f}{\partial t}+\mathbf{u}_f\cdot\boldsymbol{\nabla}\,\mathbf{u}_f\right)=\operatorname{div}\boldsymbol{\sigma}_f \quad \text{in } \Omega\setminus\mathcal{B}_t$$

$$\operatorname{div}\mathbf{u}_f=0 \quad \text{in } \Omega\setminus\mathcal{B}_t$$

$$\rho_s\frac{\partial \mathbf{u}_s}{\partial t}=\operatorname{div}_s(\mathbb{P}(\mathbb{F}))+\mathbf{f} \quad \text{in } \mathcal{B}$$

$$\mathbf{u}_f=\mathbf{u}_s \quad \text{on } \mathcal{B}_t$$

$$\boldsymbol{\sigma}_f^+\mathbf{n}^+ + \boldsymbol{\sigma}_f^-\mathbf{n}^- = -\mathbf{f} \quad \text{on } \mathcal{B}_t$$

$$\boldsymbol{\sigma}_f=-p_f\mathbb{I}+\nu_f\,\boldsymbol{\nabla}_{sym}\,\mathbf{u}_f \qquad \mathbf{u}_s=\frac{\partial \mathbf{X}}{\partial t}$$

+ initial and boundary conditions

Variational formulation of IBM

- Navier–Stokes $\mathbf{u}(t) \in H_0^1(\Omega)^d \;\; p \in L_0^2(\Omega)$

$$\rho_f \frac{d}{dt}(\mathbf{u}(t), \mathbf{v}) + a(\mathbf{u}(t), \mathbf{v}) + b(\mathbf{u}(t), \mathbf{u}(t), \mathbf{v}) \; - (\operatorname{div} \mathbf{v}, p(t))$$
$$= \langle \mathbf{d}(t), v \rangle + \langle \mathbb{F}^{\mathrm{FSI}}(t), \mathbf{v} \rangle \qquad \forall \mathbf{v} \in H_0^1(\Omega)^d$$
$$(\operatorname{div} \mathbf{u}(t), q) = 0 \qquad \forall q \in L_0^2(\Omega)$$

- Excess Lagrangian mass density

$$\langle \mathbf{d}(t), v \rangle = -\delta_\rho \int_{\mathcal{B}} \frac{\partial^2 \mathbf{X}}{\partial t^2} \mathbf{v}(\mathbf{X}(s,t)) ds$$

- Load

$$\langle \mathbb{F}^{\mathrm{FSI}}(t), \mathbf{v} \rangle = -\int_{\mathcal{B}} \mathbb{P}(\mathbb{F}(s,t)) : \boldsymbol{\nabla}_s \mathbf{v}(\mathbf{X}(s,t))\, ds$$

- Body motion

$$\frac{\partial \mathbf{X}}{\partial t}(s,t) = \mathbf{u}(\mathbf{X}(s,t), t) \quad \forall s \in \mathcal{B}$$

- Initial conditions

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}) \; \forall \mathbf{x} \in \Omega, \quad \mathbf{X}(s, 0) = \mathbf{X}_0(s) \; \forall s \in \mathcal{B}.$$

Energy estimate

Stability estimate

<Boffi–Cavallini–G. '11>

$$\frac{\rho_f}{2}\frac{d}{dt}||\mathbf{u}(t)||_0^2 + \mu||\boldsymbol{\nabla}\,\mathbf{u}(t)||_0^2 + \frac{d}{dt}E(\mathbf{X}(t)) + \frac{1}{2}\delta_\rho\frac{d}{dt}\left\|\frac{\partial\mathbf{X}}{\partial t}\right\|_B^2 = 0$$

where $E\left(\mathbf{X}(t)\right) = \int_{\mathcal{B}} W(\mathbb{F}(s,t))\,ds$

<Boffi talk, this morning>

The energy estimate of the time-space discretization requires a CFL condition.

We extend the *fictitious approach* used successfully for interface problem.

Fictitious domain approach

Thick incompressible solid

<Boffi–Cavallini–G. '15>

- Fluid velocity and pressure are extended into the solid domain

$$\mathbf{u} = \begin{cases} \mathbf{u}_f & \text{in } \Omega \setminus \mathcal{B}_t \\ \mathbf{u}_s & \text{in } \mathcal{B}_t \end{cases} \qquad p = \begin{cases} p_f & \text{in } \Omega \setminus \mathcal{B}_t \\ p_s & \text{in } \mathcal{B}_t \end{cases}$$

- Body motion $\mathbf{u}(\mathbf{x}, t) = \dfrac{\partial \mathbf{X}}{\partial t}(s, t)$ for $\mathbf{x} = \mathbf{X}(s, t)$
- We introduce two functional spaces Λ and $\mathcal{Z}$ and a bilinear form $\mathbf{c} : \Lambda \times \mathcal{Z} \to \mathbb{R}$ such that

$$\mathbf{c}(\mu, \mathbf{z}) = 0 \quad \forall \mu \in \Lambda \qquad \Rightarrow \qquad \mathbf{z} = 0$$

Notation:

$$a(\mathbf{u}, \mathbf{v}) = (\nu\, \boldsymbol{\nabla}_{sym}\, \mathbf{u}, \boldsymbol{\nabla}_{sym}\, \mathbf{v}) \quad \text{with } \nu = \begin{cases} \nu_f & \text{in } \Omega \setminus \mathcal{B}_t \\ \nu_s & \text{in } \mathcal{B}_t \end{cases}$$

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \frac{\rho_f}{2} \left((\mathbf{u} \cdot \boldsymbol{\nabla}\, \mathbf{v}, \mathbf{w}) - (\mathbf{u} \cdot \boldsymbol{\nabla}\, \mathbf{w}, \mathbf{v}) \right)$$

$$\delta_\rho = \rho_s - \rho_f$$

Variational form with Lagrange multiplier

For $t \in [0, T]$, find $\mathbf{u}(t) \in H^1_0(\Omega)^d$, $p(t) \in L^2_0(\Omega)$, $\mathbf{X}(t) \in W^{1,\infty}(\mathcal{B})^d$, and $\boldsymbol{\lambda}(t) \in \mathbf{\Lambda}$ such that

$$\rho \frac{d}{dt}(\mathbf{u}(t), \mathbf{v}) + a(\mathbf{u}(t), \mathbf{v}) + b(\mathbf{u}(t), \mathbf{u}(t), \mathbf{v}) - (\operatorname{div} \mathbf{v}, p(t)) + \mathbf{c}(\boldsymbol{\lambda}(t), \mathbf{v}(\mathbf{X}(\cdot, t))) = 0 \qquad \forall \mathbf{v} \in H^1_0(\Omega)^d$$

$$(\operatorname{div} \mathbf{u}(t), q) = 0 \qquad \forall q \in L^2_0(\Omega)$$

$$\delta_\rho \int_{\mathcal{B}} \frac{\partial^2 \mathbf{X}}{\partial t^2}(t)\mathbf{z}\, ds + \int_{\mathcal{B}} \mathbb{P}(\mathbb{F}(t))\, \boldsymbol{\nabla}_s \mathbf{z}\, ds - \mathbf{c}(\boldsymbol{\lambda}(t), \mathbf{z}) = 0 \qquad \forall \mathbf{z} \in H^1(\mathcal{B})^d$$

$$\mathbf{c}\left(\boldsymbol{\mu}, \mathbf{u}(\mathbf{X}(\cdot, t), t) - \frac{\partial \mathbf{X}(t)}{\partial t}\right) = 0 \qquad \forall \boldsymbol{\mu} \in \mathbf{\Lambda}$$

Thin immersed solid

By integration by parts and the introduction of the Lagrange multiplier $\boldsymbol{\lambda} = \mathbf{f}$, we obtain the same variational form as before except for $\delta_\rho = \rho_s$ and the definition of Λ, $\mathcal{Z}$ and $\mathbf{c}$.

Definition of $\mathbf{c}$

Thick immersed solid

The fact that $\overline{\mathbf{X}} \in W^{1,\infty}(\mathcal{B})^d$ implies $\mathbf{v}(\overline{\mathbf{X}}(\cdot)) \in H^1(\mathcal{B})^d$

Case 1 $\mathcal{Z} = H^1(\mathcal{B})^d$, Λ dual space of $H^1(\mathcal{B})^d$, $\langle \cdot, \cdot \rangle$ duality pairing

$$\mathbf{c}(\boldsymbol{\lambda}, \mathbf{z}) = \langle \boldsymbol{\lambda}, \mathbf{z} \rangle \quad \boldsymbol{\lambda} \in \Lambda = (H^1(\mathcal{B})^d)', \ \mathbf{z} \in H^1(\mathcal{B})^d$$

Case 2 $\mathcal{Z} = H^1(\mathcal{B})^d$, $\Lambda = H^1(\mathcal{B})^d$

$$\mathbf{c}(\boldsymbol{\lambda}, \mathbf{z}) = \int_{\mathcal{B}} (\boldsymbol{\nabla}_s \boldsymbol{\lambda} \cdot \boldsymbol{\nabla}_s \mathbf{z} + \boldsymbol{\lambda} \cdot \mathbf{z}) \, ds \quad \boldsymbol{\lambda} \in \Lambda, \ \mathbf{z} \in H^1(\mathcal{B})^d$$

Thin immersed solid

In this case, $\mathbf{v}(\overline{\mathbf{X}}(\cdot))$ is the trace of $\mathbf{v}$ and it belongs to $H^{1/2}(\mathcal{B})^d$.

We set $\mathcal{Z} = H^{1/2}(\mathcal{B})^d$, Λ dual space of $H^{1/2}(\mathcal{B})^d$, $\langle \cdot, \cdot \rangle$ duality pairing

$$\mathbf{c}(\boldsymbol{\lambda}, \mathbf{z}) = \langle \boldsymbol{\lambda}, \mathbf{z} \rangle \quad \boldsymbol{\lambda} \in \Lambda = (H^{1/2}(\mathcal{B})^d)', \ \mathbf{z} \in H^{1/2}(\mathcal{B})^d$$

Time semi-discretization (Modified Backward Euler)

Given $\mathbf{u}_0 \in H^1_0(\Omega)^d$ and $\mathbf{X}_0 \in W^{1,\infty}(\mathcal{B})$, for $n = 1, \dots, N$ find $(\mathbf{u}^n, p^n) \in H^1_0(\Omega)^d \times L^2_0(\Omega)$, $\mathbf{X}^n \in H^1(\mathcal{B})^d$, and $\boldsymbol{\lambda}^n \in \mathbf{\Lambda}$, such that

$$\rho_f \left(\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t}, \mathbf{v} \right) + b(\mathbf{u}^n, \mathbf{u}^{n+1}, \mathbf{v}) + a(\mathbf{u}^{n+1}, \mathbf{v})$$

$$- (\operatorname{div} \mathbf{v}, p^{n+1}) + \mathbf{c}(\boldsymbol{\lambda}^{n+1}, \mathbf{v}(\mathbf{X}^n)) = 0 \qquad \forall \mathbf{v} \in H^1_0(\Omega)^d$$

$$(\operatorname{div} \mathbf{u}^{n+1}, q) = 0 \qquad \forall q \in L^2_0(\Omega)$$

$$\delta_\rho \left(\frac{\mathbf{X}^{n+1} - 2\mathbf{X}^n + \mathbf{X}^{n-1}}{\Delta t^2}, \mathbf{z} \right)_{\mathcal{B}} + (\mathbb{P}(\mathbb{F}^{n+1}), \boldsymbol{\nabla}_s \mathbf{z})_{\mathcal{B}}$$

$$- \mathbf{c}(\boldsymbol{\lambda}^{n+1}, \mathbf{z}) = 0 \qquad \forall \mathbf{z} \in H^1(\mathcal{B})^d$$

$$\mathbf{c}\left(\boldsymbol{\mu}, \mathbf{u}^{n+1}(\mathbf{X}^n) - \frac{\mathbf{X}^{n+1} - \mathbf{X}^n}{\Delta t} \right) = 0 \qquad \forall \boldsymbol{\mu} \in \mathbf{\Lambda}$$

$$\mathbf{u}^0 = \mathbf{u}_0, \qquad \mathbf{X}^0 = \mathbf{X}_0$$

Energy estimate

Proposition

We assume that W is a C^1 convex function over the set of second order tensors, then

$$\frac{\rho_f}{2\Delta t}\left(\|\mathbf{u}^{n+1}\|_0^2 - \|\mathbf{u}^n\|_0^2\right) + \nu\|\boldsymbol{\nabla}_{\mathrm{sym}}\,\mathbf{u}^{n+1}\|_0^2$$
$$+\frac{\delta_\rho}{2\Delta t}\left(\left\|\frac{\mathbf{X}^{n+1}-\mathbf{X}^n}{\Delta t}\right\|_{0,\mathcal{B}}^2 - \left\|\frac{\mathbf{X}^{n}-\mathbf{X}^{n-1}}{\Delta t}\right\|_{0,\mathcal{B}}^2\right)$$
$$+\frac{E(\mathbf{X}^{n+1})-E(\mathbf{X}^n)}{\Delta t} \le 0$$

Finite element discretization

We consider

- Background grid $\mathcal{T}_h$ for the domain Ω (meshsize h_x)
- $(V_h, Q_h) \subseteq H^1_0(\Omega)^d \times L^2_0(\Omega)$ stable pair for the Stokes equations
- Grid $\mathcal{S}_h$ for $\mathcal{B}$ (meshsize h_s)
- $S_h \subseteq H^1(\mathcal{B})^d$ continuous Lagrange elements

$$S_h = \{\mathbf{z} \in C^0(\mathcal{B};\Omega) : \mathbf{z} \in \mathbf{P}^1(T)\ \forall T \in \mathcal{S}_h\}$$

- $\Lambda_h \subseteq \Lambda$ continuous Lagrange elements. We consider $\Lambda_h = S_h$

Remark

- *If* $\mathbf{c}$ *is the duality pairing between* $H^1(\mathcal{B})^d$ *(or* $H^{1/2}(\mathcal{B})^d$*) and its dual space, we can represent it by the scalar product in* $L^2(\mathcal{B})$ *when* $\boldsymbol{\mu} \in L^2$
- *Stabilized* $P1 - P1$ *elements for Stokes could also be used*

<Annese, Phd Thesis '17>

Fully discrete problem

Given $\mathbf{u}_{0h} \in V_h$ and $\mathbf{X}_{0h} \in S_h$, for $n = 0, \dots, N-1$ find $(\mathbf{u}_h^n, p_h^n) \in V_h \times Q_h$, $\mathbf{X}_h^n \in S_h$, and $\boldsymbol{\lambda}_h^n \in \boldsymbol{\Lambda}_h$, such that

$$
\begin{aligned}
&\rho_f \left(\frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{\Delta t}, \mathbf{v} \right) + b(\mathbf{u}_h^n, \mathbf{u}_h^{n+1}, \mathbf{v}) + a(\mathbf{u}_h^{n+1}, \mathbf{v}) \\
&\qquad - (\operatorname{div} \mathbf{v}, p_h^{n+1}) + \mathbf{c}(\boldsymbol{\lambda}_h^{n+1}, \mathbf{v}(\mathbf{X}_h^n)) = 0 && \forall \mathbf{v} \in V_h \\
&(\operatorname{div} \mathbf{u}_h^{n+1}, q) = 0 && \forall q \in Q_h \\
&\delta_\rho \left(\frac{\mathbf{X}_h^{n+1} - 2\mathbf{X}_h^n + \mathbf{X}_h^{n-1}}{\Delta t^2}, \mathbf{z} \right)_{\mathcal{B}} + (\mathbb{P}(\mathbb{F}_h^{n+1}), \boldsymbol{\nabla}_s \mathbf{z})_{\mathcal{B}} \\
&\qquad - \mathbf{c}(\boldsymbol{\lambda}_h^{n+1}, \mathbf{z}) = 0 && \forall \mathbf{z} \in S_h \\
&\mathbf{c}\left(\boldsymbol{\mu}, \mathbf{u}_h^{n+1}(\mathbf{X}_h^n) - \frac{\mathbf{X}_h^{n+1} - \mathbf{X}_h^n}{\Delta t} \right) = 0 && \forall \boldsymbol{\mu} \in \boldsymbol{\Lambda}_h \\
&\mathbf{u}_h^0 = \mathbf{u}_{0h}, \qquad \mathbf{X}_h^0 = \mathbf{X}_{0h}
\end{aligned}
$$

where $\mathbb{F}_h^{n+1} = \boldsymbol{\nabla}_s \mathbf{X}_h^{n+1}$.

Energy estimate

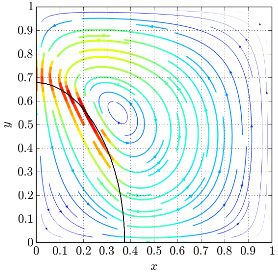
Proposition

We assume that W is a C^1 convex function over the set of second order tensors, then

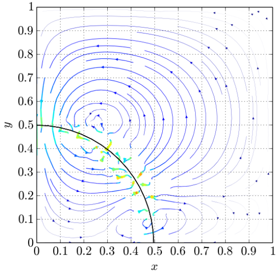
$$\frac{\rho_f}{2\Delta t}\left(\|\mathbf{u}_h^{n+1}\|_0^2 - \|\mathbf{u}_h^n\|_0^2\right) + \nu\|\boldsymbol{\nabla}_{\mathrm{sym}}\,\mathbf{u}_h^{n+1}\|_0^2 + \frac{\delta_\rho}{2\Delta t}\left(\left\|\frac{\mathbf{X}_h^{n+1}-\mathbf{X}_h^n}{\Delta t}\right\|_{0,\mathcal{B}}^2 - \left\|\frac{\mathbf{X}_h^{n}-\mathbf{X}_h^{n-1}}{\Delta t}\right\|_{0,\mathcal{B}}^2\right) + \frac{E(\mathbf{X}_h^{n+1})-E(\mathbf{X}_h^n)}{\Delta t} \leq 0$$

Numerical experiments

Codimension one structure position snapshots



(a) $t = 0.1$.

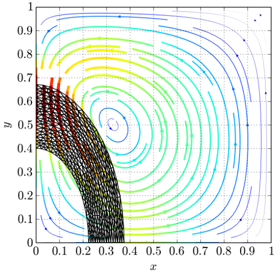


(b) $t = 2$.

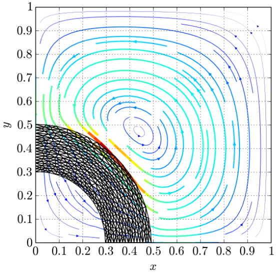
The pictures represent the velocity streamlines and the structure position for the first and final time steps. The streamline color pictures the velocity magnitude, red is the higher value.

Numerical experiments

Codimension zero structure position snapshots



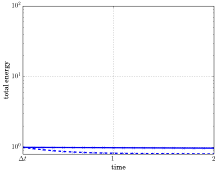
(a) $t = 0.1$.



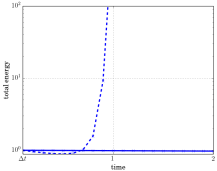
(b) $t = 2$.

The pictures represent the velocity streamlines and the structure position for the first and final time steps. The streamline color pictures the velocity magnitude, red is the higher value.

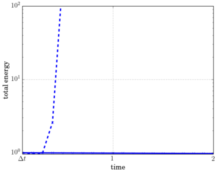
Energy ratio for codimension one structure



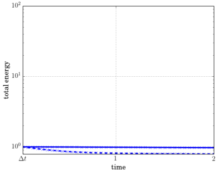
(a) $\Delta t = 10^{-1}$, $h_s = 1/8$.



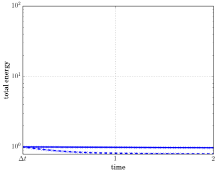
(b) $\Delta t = 10^{-1}$, $h_s = 1/16$.



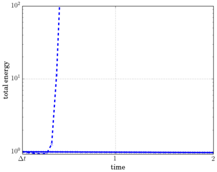
(c) $\Delta t = 10^{-1}$, $h_s = 1/32$.



(d) $\Delta t = 5 \cdot 10^{-2}$, $h_s = 1/8$.



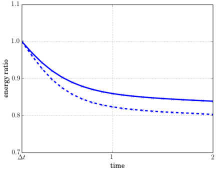
(e) $\Delta t = 5 \cdot 10^{-2}$, $h_s = 1/16$.



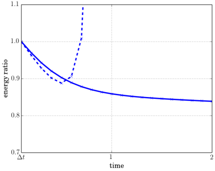
(f) $\Delta t = 5 \cdot 10^{-2}$, $h_s = 1/32$.

The structure elastic constant $\kappa = 5$, $h_x = 1/32$, the fluid viscosity $\nu = 1$, $\delta\rho = 0$. The solid line correspond to the DLM-IBM scheme, while the dashed line marks the energy for the FE-IBM scheme.

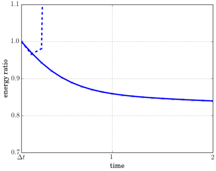
Energy ratio for codimension one structure



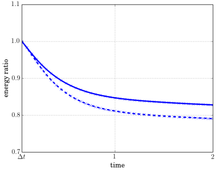
(a) $\Delta t = 10^{-1}$, $h_s = 1/8$.



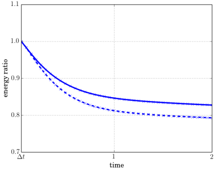
(b) $\Delta t = 10^{-1}$, $h_s = 1/16$.



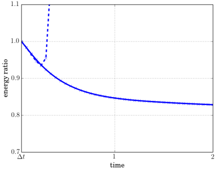
(c) $\Delta t = 10^{-1}$, $h_s = 1/32$.



(d) $\Delta t = 5 \cdot 10^{-2}$, $h_s = 1/8$.



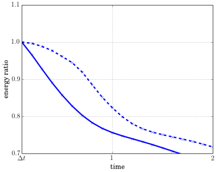
(e) $\Delta t = 5 \cdot 10^{-2}$, $h_s = 1/16$.



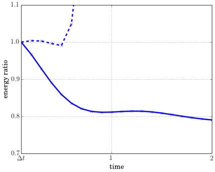
(f) $\Delta t = 5 \cdot 10^{-2}$, $h_s = 1/32$.

The structure elastic constant $\kappa = 5$, $h_x = 1/32$, the fluid viscosity $\nu = 1$, $\delta\rho = 0.3$. The solid line correspond to the DLM-IBM scheme, while the dashed line marks the energy for the FE-IBM scheme.

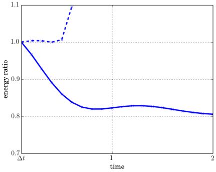
Energy ratio for codimension zero structure



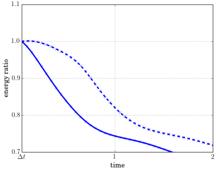
(a) $\Delta t = 10^{-1}$, $h_x = 1/4$.



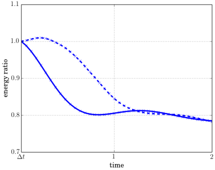
(b) $\Delta t = 10^{-1}$, $h_x = 1/8$.



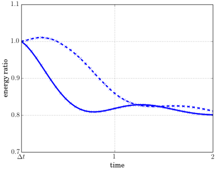
(c) $\Delta t = 10^{-1}$, $h_x = 1/16$.



(d) $\Delta t = 5 \cdot 10^{-2}$, $h_x = 1/4$.



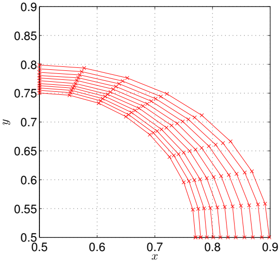
(e) $\Delta t = 5 \cdot 10^{-2}$, $h_x = 1/8$.



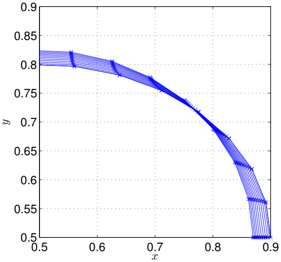
(f) $\Delta t = 5 \cdot 10^{-2}$, $h_x = 1/16$.

The structure elastic constant $\kappa = 1$, $h_s = 1/8$, the fluid viscosity $\nu = 0.05$, $\delta\rho = 0.3$. The solid line correspond to the DLM-IBM scheme, while the dashed line marks the energy for the FE-IBM scheme.

Mass conservation



(a) FE-IBM



(b) DLM-IBM

Mass conservation of the FE-IBM (left) and DLM-IBM (right) with higher order fluid element.

Stationary problem

At each time step we have to solve a stationary problem.

We consider a linear stress tensor

$$\mathbb{P}(\mathbb{F}) = \kappa\mathbb{F} = \kappa\,\boldsymbol{\nabla}_s\,\mathbf{X}$$

and set:

$$\begin{aligned}
&\mathbf{u} = \mathbf{u}^{n+1},\ p = p^{n+1},\ \mathbf{X} = \mathbf{X}^{n+1}/\Delta t,\ \boldsymbol{\lambda} = \boldsymbol{\lambda}^{n+1}\\
&\mathbf{f} = \frac{\rho_f}{\Delta t}\mathbf{u}^n\\
&\mathbf{g} = \frac{\delta_\rho}{\Delta t^2}\left(2\mathbf{X}^n - \mathbf{X}^{n-1}\right)\\
&\mathbf{d} = -\frac{1}{\Delta t}\mathbf{X}^n\\
&\alpha = \rho_f/\Delta t,\ \beta = \delta_\rho/\Delta t,\ \gamma = \kappa\Delta t
\end{aligned}$$

Saddle point problem

Problem

Let $\overline{\mathbf{X}} \in W^{1,\infty}(\mathcal{B})^d$ *be invertible with Lipschitz inverse and* $\overline{\mathbf{u}} \in L^\infty(\Omega)$*.*
Given $\mathbf{f} \in L^2(\Omega)^d$*,* $\mathbf{g} \in L^2(\mathcal{B})^d$*, and* $\mathbf{d} \in L^2(\mathcal{B})^d$*, find* $\mathbf{u} \in H^1_0(\Omega)^d$*,* $p \in L^2_0(\Omega)$*,* $\mathbf{X} \in H^1(\mathcal{B})^d$*, and* $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}$ *such that*

$$
\begin{aligned}
&\mathbf{a}_f(\mathbf{u},\mathbf{v}) - (\operatorname{div}\mathbf{v}, p) + \mathbf{c}(\boldsymbol{\lambda}, \mathbf{v}(\overline{\mathbf{X}})) = (\mathbf{f},\mathbf{v}) && \forall \mathbf{v} \in H^1_0(\Omega)^d \\
&(\operatorname{div}\mathbf{u}, q) = 0 && \forall q \in L^2_0(\Omega) \\
&\mathbf{a}_s(\mathbf{X},\mathbf{z}) - \mathbf{c}(\boldsymbol{\lambda},\mathbf{z}) = (\mathbf{g},\mathbf{z})_{\mathcal{B}} && \forall \mathbf{z} \in H^1(\mathcal{B})^d \\
&\mathbf{c}(\boldsymbol{\mu}, \mathbf{u}(\overline{\mathbf{X}}) - \mathbf{X}) = \mathbf{c}(\boldsymbol{\mu},\mathbf{d}) && \forall \boldsymbol{\mu} \in \boldsymbol{\Lambda}
\end{aligned}
$$

where

$$
\begin{aligned}
&\mathbf{a}_f(\mathbf{u},\mathbf{v}) = \alpha(\mathbf{u},\mathbf{v}) + a(\mathbf{u},\mathbf{v}) + b(\overline{\mathbf{u}},\mathbf{u},\mathbf{v}) && \forall \mathbf{u},\mathbf{v} \in H^1_0(\Omega)^d \\
&\mathbf{a}_s(\mathbf{X},\mathbf{z}) = \beta(\mathbf{X},\mathbf{z})_{\mathcal{B}} + \gamma(\boldsymbol{\nabla}_s\mathbf{X}, \boldsymbol{\nabla}_s\mathbf{z})_{\mathcal{B}} && \forall \mathbf{X},\mathbf{z} \in H^1(\mathcal{B})^d
\end{aligned}
$$

Discrete saddle point problem

Problem

Find $\mathbf{u}_h \in V_h$, $p_h \in Q_h$, $\mathbf{X}_h \in S_h$ *and* $\lambda_h \in \Lambda_h$ *such that*

$$
\begin{aligned}
& a_f(\mathbf{u}_h, \mathbf{v}) - (\operatorname{div} \mathbf{v}, p_h) + \mathbf{c}(\boldsymbol{\lambda}_h, \mathbf{v}(\overline{\mathbf{X}}(\cdot))) = (\mathbf{f}, \mathbf{v}) && \forall \mathbf{v} \in V_h \\
& (\operatorname{div} \mathbf{u}_h, q) = 0 && \forall q \in Q_h \\
& a_s(\mathbf{X}_h, \mathbf{z}) - \mathbf{c}(\boldsymbol{\lambda}_h, \mathbf{z}) = (\mathbf{g}, \mathbf{z})_{\mathcal{B}} && \forall \mathbf{z} \in S_h \\
& \mathbf{c}(\boldsymbol{\mu}, \mathbf{u}_h(\overline{\mathbf{X}}(\cdot)) - \mathbf{X}_h) = \mathbf{c}(\boldsymbol{\mu}, \mathbf{d}) && \forall \boldsymbol{\mu} \in \Lambda_h.
\end{aligned}
$$

Operator form of the stationary problem

The stationary problem has the following double saddle point structure

$$\left[\begin{array}{ccc|c} A_f & B_f^\top & 0 & C_f^\top \\ B_f & 0 & 0 & 0 \\ 0 & 0 & A_s & -C_s^\top \\ \hline C_f & 0 & -C_s & 0 \end{array}\right] \left[\begin{array}{c} \mathbf{u} \\ p \\ \mathbf{X} \\ \hline \boldsymbol{\lambda} \end{array}\right] \left[\begin{array}{c} \mathbf{f} \\ 0 \\ \mathbf{g} \\ \hline \mathbf{d} \end{array}\right]$$

In view of the analysis, it is useful to rearrange the variables as follows

$$\left[\begin{array}{ccc|c} A_f & 0 & C_f^\top & B_f^\top \\ 0 & A_s & -C_s^\top & 0 \\ C_f & -C_s & 0 & 0 \\ \hline B_f & 0 & 0 & 0 \end{array}\right] \left[\begin{array}{c} \mathbf{u} \\ \mathbf{X} \\ \boldsymbol{\lambda} \\ \hline p \end{array}\right] \left[\begin{array}{c} \mathbf{f} \\ \mathbf{g} \\ \mathbf{d} \\ \hline 0 \end{array}\right].$$

Theoretical results

<B.–Gastaldi '17>

This problem has been rigorously analyzed both at continuous and discrete level (existence, uniqueness, stability, and convergence)

Abstract saddle point formulation

Set: $\mathbb{V} = H^1_0(\Omega)^d \times H^1(\mathcal{B})^d \times \mathbf{\Lambda}$ and $\mathbf{V} = (\mathbf{v}, \mathbf{z}, \boldsymbol{\lambda}) \in \mathbb{V}$

$$\mathbb{A}(\mathbf{U}, \mathbf{V}) = \mathbf{a}_f(\mathbf{u}, \mathbf{v}) + \mathbf{a}_s(\mathbf{X}, \mathbf{z}) + \mathbf{c}(\boldsymbol{\lambda}, \mathbf{v}(\overline{\mathbf{X}}) - \mathbf{z}) - \mathbf{c}(\boldsymbol{\mu}, \mathbf{u}(\overline{\mathbf{X}}) - \mathbf{X})$$
$$\mathbb{B}(\mathbf{V}, q) = (\operatorname{div} \mathbf{v}, q)$$

Problem

Find $(\mathbf{U}, p) \in \mathbb{V} \times L^2_0(\Omega)$ *such that*

$$\begin{aligned} &\mathbb{A}(\mathbf{U}, \mathbf{V}) + \mathbb{B}(\mathbf{V}, p) = (\mathbf{f}, \mathbf{v}) + (\mathbf{g}, \mathbf{z})_{\mathcal{B}} + \mathbf{c}(\boldsymbol{\mu}, \mathbf{d}) && \forall \mathbf{V} \in \mathbb{V} \\ &\mathbb{B}(\mathbf{U}, q) = 0 && \forall q \in L^2_0(\Omega). \end{aligned}$$

and its discretization Set: $\mathbb{V}_h = V_h \times S_h \times \mathbf{\Lambda}_h$

Problem

Find $(\mathbf{U}_h, \boldsymbol{\lambda}_h) \in \mathbb{V}_h \times \mathbf{\Lambda}_h$ *such that*

$$\begin{aligned} &\mathbb{A}(\mathbf{U}_h, \mathbf{V}) + \mathbb{B}(\mathbf{V}, p_h) = (\mathbf{f}, \mathbf{v}) + (\mathbf{g}, \mathbf{z})_{\mathcal{B}} + \mathbf{c}(\boldsymbol{\mu}, \mathbf{d}) && \forall \mathbf{V} \in \mathbb{V}_h \\ &\mathbb{B}(\mathbf{U}_h, q) = 0 && \forall q \in Q_h. \end{aligned}$$

Main steps of the proof

Discrete case

Discrete inf-sup condition for $\mathbb{B}$

Since $V_h \times Q_h$ is stable for the Stokes equation, there exists a positive constant $\overline{\beta}_{\mathrm{div}}$ such that for all $q_h \in Q_h$

$$\sup_{\mathbf{V}_h \in \mathbb{V}_h} \frac{\mathbb{B}(\mathbf{V}_h, q_h)}{|||\mathbf{V}_h|||_{\mathbb{V}}} = \sup_{\mathbf{v}_h \in V_h} \frac{(\operatorname{div} \mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_1} \geq \overline{\beta}_{\mathrm{div}} \|q_h\|_0$$

The main issue is to show the invertibility of the operator matrix

$$\left[\begin{array}{cc:c} A_f & 0 & C_f^\top \\ 0 & A_s & -C_s^\top \\ \hdashline C_f & -C_s & 0 \end{array}\right]$$

on the discrete kernel of $\mathbb{B}$:

$$\mathbb{K}_{\mathbb{B},h} = \{\mathbf{V} \in \mathbb{V}_h : \mathbb{B}(\mathbf{V}, q) = 0 \ \forall q \in Q_h\}.$$

Main steps of the proof (cont'ed)

Discrete inf-sup for $\mathbb{A}$

There exists $\kappa_0 > 0$, independent of h_x and h_s, such that

$$\inf_{\mathbf{U}\in\mathbb{K}_{\mathbb{B},h}} \sup_{\mathbf{V}\in\mathbb{K}_{\mathbb{B},h}} \frac{\mathbb{A}(\mathbf{U},\mathbf{V})}{|||\mathbf{U}|||_{\mathbb{V}}|||\mathbf{V}|||_{\mathbb{V}}} \geq \kappa_0.$$

Proposition

There exists $\alpha_1 > 0$ independent of h_x and h_s such that

$$\mathbf{a}_f(\mathbf{u}_h,\mathbf{u}_h) + \mathbf{a}_s(\mathbf{X}_h,\mathbf{X}_h) \geq \alpha_1(\|\mathbf{u}_h\|_1^2 + \|\mathbf{X}_h\|_{1,\mathcal{B}}^2) \quad \forall(\mathbf{u}_h,\mathbf{X}_h)\in \mathbf{V}_{0,h}\times S_h$$

where

$$V_{0,h} = \{\mathbf{v}_h \in V_h : (\mathrm{div}\,\mathbf{v}_h, q_h) = 0\ \forall q_h \in Q_h\}$$

$$\mathbb{K}_h = \left\{(\mathbf{v}_h,\mathbf{z}_h)\in V_{0,h}\times S_h :\ \mathbf{c}(\boldsymbol{\mu}_h, \mathbf{v}_h(\overline{\mathbf{X}}) - \mathbf{z}_h) = 0\ \forall \boldsymbol{\mu}_h \in \boldsymbol{\Lambda}_h\right\}$$

Proposition (Thick immersed solid)

There exists a constant $\beta_1 > 0$ independent of h_x and h_s such that for all $\boldsymbol{\mu}_h \in \boldsymbol{\Lambda}_h$ it holds true

$$\sup_{(\mathbf{v}_h,\mathbf{z}_h)\in V_{0,h}\times S_h} \frac{\mathbf{c}(\boldsymbol{\mu}_h, \mathbf{v}_h(\overline{\mathbf{X}}) - \mathbf{z}_h)}{(\|\mathbf{v}_h\|_1^2 + \|\mathbf{z}_h\|_{1,\mathcal{B}}^2)^{1/2}} \geq \beta_1 \|\boldsymbol{\mu}_h\|_{\boldsymbol{\Lambda}}.$$

Proposition (Thin immersed solid)

We assume that the domain Ω is convex. If h_x/h_s is sufficiently small and the mesh $\mathcal{S}_h$ is quasi-uniform, then there exists a constant $\beta_1 > 0$ independent of h_x and h_s such that for all $\boldsymbol{\mu}_h \in \Lambda_h$ it holds true

$$\sup_{(\mathbf{v}_h,\mathbf{z}_h)\in V_{0,h}\times S_h} \frac{\mathbf{c}(\boldsymbol{\mu}_h, \mathbf{v}_h(\overline{\mathbf{X}}) - \mathbf{z}_h)}{(\|\mathbf{v}_h\|_1^2 + \|\mathbf{z}_h\|_{1,\mathcal{B}}^2)^{1/2}} \geq \beta_1 \|\boldsymbol{\mu}_h\|_{\boldsymbol{\Lambda}}.$$

The proof depends on the choice of $\mathbf{c}$.

Proof

Thick immersed solid

Case 1 $\mathbf{c}(\boldsymbol{\mu},\mathbf{z}) = \langle\boldsymbol{\mu},\mathbf{z}\rangle$ for $\boldsymbol{\mu}\in\boldsymbol{\Lambda}_h$ $\mathbf{z}\in S_h$

The above inf-sup condition holds true if the L^2-projection onto S_h is bounded in $H^1(\mathcal{B})^d$.

This can be proved by assuming that the mesh in $\mathcal{B}$ is quasi-uniform or satisfies weaker assumptions as in <Bramble–Pasciak–Steinbach '02> <Crouzeix–Thomée '87>

Case 2 $\mathbf{c}(\boldsymbol{\mu},\mathbf{z}) = \int_{\mathcal{B}}(\boldsymbol{\nabla}_s\,\boldsymbol{\mu}\,\boldsymbol{\nabla}_s\,\mathbf{z} + \boldsymbol{\mu}\mathbf{z})ds$ for $\boldsymbol{\mu}\in\boldsymbol{\Lambda}_h$ $\mathbf{z}\in S_h$

The result follows directly from the continuous inf-sup conditition.

Thin immersed solid

We use the continuous inf-sup condition, trace theorem and inverse inequality $\|\bar{\mathbf{u}}_h(\overline{\mathbf{X}}) - \bar{\mathbf{u}}(\overline{\mathbf{X}})\|_{0,\mathcal{B}} \le Ch_x^{1/2}\|\bar{\mathbf{u}}\|_1$ and $\|\boldsymbol{\mu}_h\|_{0,\mathcal{B}} \le Ch_s^{-1/2}\|\boldsymbol{\mu}_h\|_{\boldsymbol{\Lambda}}$.
Then

$$\begin{aligned}\mathbf{c}(\boldsymbol{\mu}_h,\bar{\mathbf{u}}_h(\overline{\mathbf{X}})) &\ge \frac{1}{2c}\|\boldsymbol{\mu}\|_{\boldsymbol{\Lambda}}\|\bar{\mathbf{u}}\|_1 - C\|\boldsymbol{\mu}_h\|_{0,\mathcal{B}}h_x^{1/2}\|\bar{\mathbf{u}}\|_1\\ &\ge \|\boldsymbol{\mu}\|_{\boldsymbol{\Lambda}}\|\bar{\mathbf{u}}\|_1\left(\frac{1}{2c} - C\left(\frac{h_x}{h_s}\right)^{1/2}\right)\end{aligned}$$

Error estimates

Theorem

The following error estimates hold true

$$\|\mathbf{u}-\mathbf{u}_h\|_{H^1_0(\Omega)^d}+\|p-p_h\|_{L^2(\Omega)}+\|\mathbf{X}-\mathbf{X}_h\|_{H^1(\mathcal{B})^d}+\|\lambda-\lambda_h\|_\Lambda$$
$$\leq C\inf_{(\mathbf{v},q,\mathbf{z},\mu)\in V_h\times Q_h\times S_h\times S_h}\Big(\|\mathbf{u}-\mathbf{v}\|_{H^1_0(\Omega)^d}+\|p-q\|_{L^2(\Omega)}$$
$$+\|\mathbf{X}-\mathbf{z}\|_{H^1(\mathcal{B})^d}+\|\lambda-\mu\|_\Lambda\Big)$$

The inflated balloon

Thin solid

$$\mathbf{u}(\mathbf{x},t) = \mathbf{0} \quad \forall \mathbf{x} \in \Omega, \quad \forall t \in]0,T[$$

$$p(\mathbf{x},t) = \begin{cases} \kappa(1/R - \pi R), & |\mathbf{x}| \le R \\ -\kappa\pi R, & |\mathbf{x}| > R \end{cases} \quad \forall t \in]0,T[$$

Time convergence

Δt	$\|\|\mathbf{X}_{ex} - \mathbf{X}_h\|\|_{L^2}$	L^2-rate	$\|\|\mathbf{u}_{ex} - \mathbf{u}_h\|\|_{L^2}$	L^2-rate
$1\cdot10^{-1}$	5.54945e-06	-	1.65152e-05	-
$5\cdot10^{-2}$	2.73334e-06	1.02	7.92803e-06	1.06
$2\cdot10^{-2}$	1.05724e-06	1.04	3.01373e-06	1.06
$1\cdot10^{-2}$	5.00445e-07	1.08	1.41808e-06	1.09

The inflated balloon

Spatial convergence

h_x	$\|\|p - p_h\|\|_{L^2}$	L^2-rate	$\|\|\mathbf{u} - \mathbf{u}_h\|\|_{L^2}$	L^2-rate
1/4	2.96063	-	0.02225	-
1/8	2.10271	0.49	0.01022	1.12
1/16	1.43488	0.55	0.00392	1.38
1/24	1.15722	0.53	0.00212	1.52
1/32	0.97502	0.60	0.00134	1.60
1/40	0.88740	0.42	0.00102	1.22
1/64	0.69442	0.52	0.00052	1.43

Compressible material

<Boffi-G.-Heltai '18>

- Mass conservation in Eulerian coordinates

$$\dot{\rho}_s + \rho_s \operatorname{div} \mathbf{u}_s = 0 \quad \text{in } \mathcal{B}_t$$

or, equivalently,

$$\operatorname{div} \mathbf{u}_s(\mathbf{x}, t) = \frac{\dot{J}(s,t)}{J(s,t)} \quad \text{for } \mathbf{x} = \mathbf{X}(s,t)$$

with $J = \det \mathbb{F}$ and thanks to $\rho_s(\mathbf{x}, t) = \rho_{s_0}(\mathbf{s})/J(\mathbf{s}, t)$

- Body motion $\mathbf{u}(\mathbf{x}, t) = \dfrac{\partial \mathbf{X}}{\partial t}(s, t)$ for $\mathbf{x} = \mathbf{X}(s, t)$
- Fluid velocity and pressure are extended into the solid domain

$$\mathbf{u} = \begin{cases} \mathbf{u}_f & \text{in } \Omega \setminus \mathcal{B}_t \\ \mathbf{u}_s & \text{in } \mathcal{B}_t \end{cases} \qquad p = \begin{cases} p_f & \text{in } \Omega \setminus \mathcal{B}_t \\ p_s = 0 & \text{in } \mathcal{B}_t \end{cases}$$

Pressure field has no physical meaning in the solid, it is imposed weakly to be zero.

Variational formulation

For almost every $t \in]0, T]$, find $(\mathbf{u}(t), p(t)) \in H_0^1(\Omega)^d \times L_0^2(\Omega)$, $\mathbf{w}(t) \in H^1(\mathcal{B})^d$, and $\boldsymbol{\lambda}(t) \in \boldsymbol{\Lambda}$ such that it holds

$$
\begin{aligned}
&\rho_f(\dot{\mathbf{u}}(t), \mathbf{v}) + a(\mathbf{u}(t), \mathbf{v}) - (\operatorname{div} \mathbf{v}, p(t)) \\
&\quad + \mathbf{c}(\boldsymbol{\lambda}(t), \mathbf{v}(\mathbf{X}(\cdot, t))) = (\mathbf{f}(t), \mathbf{v}) + (\boldsymbol{\tau}_g(t), \mathbf{v})_{\partial\Omega_N} && \forall \mathbf{v} \in H_0^1(\Omega)^d \\
&-(\operatorname{div} \mathbf{u}(t), q) + (J\, q(\mathbf{X}(\mathbf{s}, t))\mathbb{F}^{-\top}, \boldsymbol{\nabla}_s \dot{\mathbf{w}}(t))_{\mathcal{B}} \\
&\quad -\frac{1}{\kappa}(Jp(t), q)_{\mathcal{B}} = 0 && \forall q \in L_0^2(\Omega) \\
&(\delta_\rho \ddot{\mathbf{w}}(t), \mathbf{z})_{\mathcal{B}} + (\mathbb{P}(t), \boldsymbol{\nabla}_s \mathbf{z})_{\mathcal{B}} \\
&\quad + (Jp(\mathbf{X}(\mathbf{s}, t), t)\mathbb{F}^{-\top}, \boldsymbol{\nabla}_s \mathbf{z})_{\mathcal{B}} - \mathbf{c}(\boldsymbol{\lambda}(t), \mathbf{z}) = 0 && \forall \mathbf{z} \in H^1(\mathcal{B})^d \\
&\mathbf{c}\,(\boldsymbol{\mu}, \mathbf{u}(\mathbf{X}(\cdot, t), t) - \dot{\mathbf{w}}(t)) = 0 && \forall \boldsymbol{\mu} \in \boldsymbol{\Lambda} \\
&\mathbf{X}(\mathbf{s}, t) = \mathbf{s} + \mathbf{w}(\mathbf{s}, t) \quad \text{for } \mathbf{s} \in \mathcal{B} \\
&\mathbf{u}(0) = \mathbf{u}_0 \quad \text{in } \Omega, \qquad \mathbf{X}(0) = \mathbf{X}_0 \quad \text{in } \mathcal{B}.
\end{aligned}
$$

Properties of the solution

Proposition

Let $(\mathbf{u}, p, \mathbf{w}, \boldsymbol{\lambda})$ be a solution of the above problem. We have that

$$p(t) = 0 \quad \text{in } \mathcal{B}_t \quad \text{for } t \in]0, T]$$
$$(\operatorname{div} \mathbf{u}, q)_{\Omega \setminus \mathcal{B}_t} = 0 \quad \forall q \in L^2(\Omega \setminus \mathcal{B}_t).$$

Energy estimate

$$\frac{1}{2}\frac{d}{dt}\|\rho^{1/2}\mathbf{u}(t)\|^2_{0,\Omega} + \|\nu^{1/2}\,\boldsymbol{\nabla}_{\mathrm{sym}}\,\mathbf{u}(t)\|^2_{0,\Omega} + \frac{d}{dt}\int_B W(\mathbb{F}(t))\,d\mathbf{s}$$
$$\leq C\left(\|\mathbf{f}(t)\|^2_{0,\Omega} + \|\boldsymbol{\tau}_g(t)\|_{H^{-1/2}(\partial\Omega_N)}\right).$$

where

$$\rho = \begin{cases} \rho_f & \text{in } \Omega_t^f \\ \rho_s(\mathbf{x}, t) & \text{in } \Omega_t^s \end{cases}, \qquad \nu = \begin{cases} \nu_f & \text{in } \Omega_t^f \\ \nu_s & \text{in } \Omega_t^s \end{cases}.$$

Operator form of the stationary problem

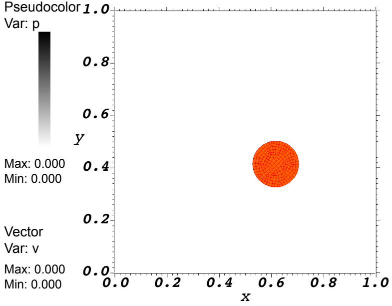
<B.–Gastaldi–Heltai '18>

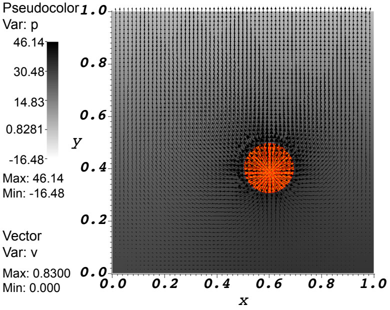
After some manipulations, the matrix form of the stationary problem is as follows

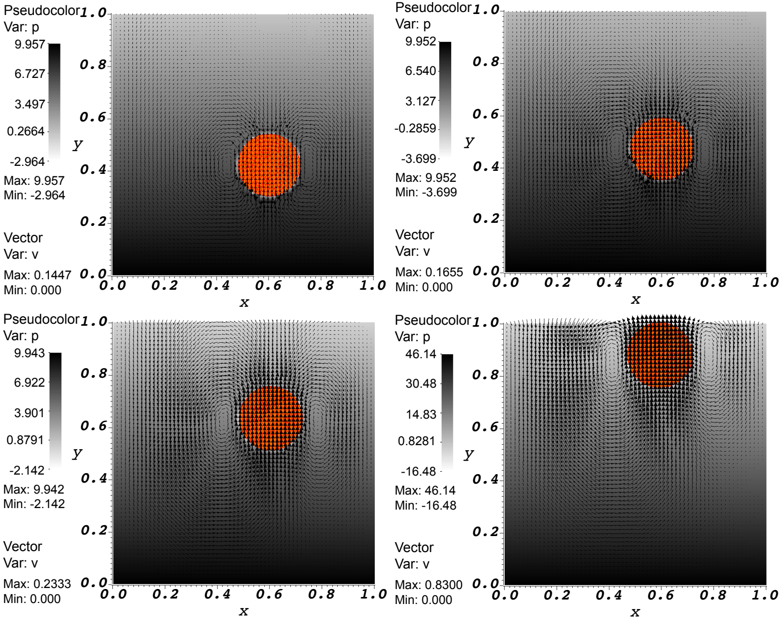
$$\begin{bmatrix} A & B^\top & 0 & C_1^\top \\ B & M_p & B_s^\top & 0 \\ 0 & B_s & A_s & C_2^\top \\ C_1 & 0 & C_2 & 0 \end{bmatrix} \begin{bmatrix} U \\ P \\ X \\ \Lambda \end{bmatrix} = \begin{bmatrix} F \\ 0 \\ G \\ D \end{bmatrix}$$

Preliminary numerical tests

$R = 0.125\,m$, $l = 1.0\,m$, $\rho_{s_0} = 0.8\,kg/m^3$, $\rho_f = 1.0\,kg/m^3$, $\mu^e = 20\,Pa$, $\mu_s = 2.0\,Pa{\cdot}s$, $\mu_f = 0.01\,Pa{\cdot}s$

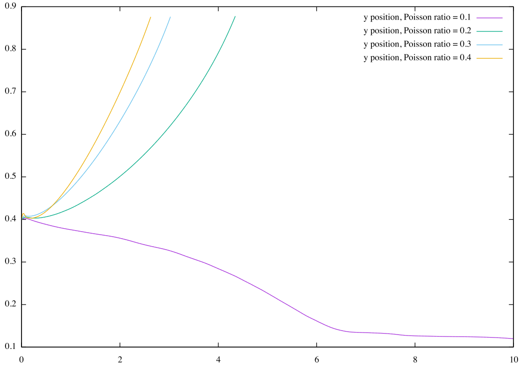





Pseudocolor
Var: p
9.957
6.727
3.497
0.2664
-2.964
Max: 9.957
Min: -2.964
Vector
Var: v
Max: 0.1447
Min: 0.000
Pseudocolor
Var: p
9.952
6.540
3.127
-0.2859
-3.699
Max: 9.952
Min: -3.699
Vector
Var: v
Max: 0.1655
Min: 0.000
Pseudocolor
Var: p
9.943
6.922
3.901
0.8791
-2.142
Max: 9.942
Min: -2.142
Vector
Var: v
Max: 0.2333
Min: 0.000
Pseudocolor
Var: p
46.14
30.48
14.83
0.8281
-16.48
Max: 46.14
Min: -16.48
Vector
Var: v
Max: 0.8300
Min: 0.000
x
y

Influhence of the Poisson ratio

Poisson ratio 0.1
Poisson ratio 0.2
Poisson ratio 0.3
Poisson ratio 0.4



Conclusions

- The use of the fictitious domain method with Lagrange multiplier can be successfully extended to FSI problems
- Resulting semi-implicit scheme is unconditionally stable in time
- Analysis of stationary problem provides optimal error estimates
- Compressible solids can be studied with analogous strategy

THANK YOU