LS Augmented Methods for Fluid and Porous Media Couplings

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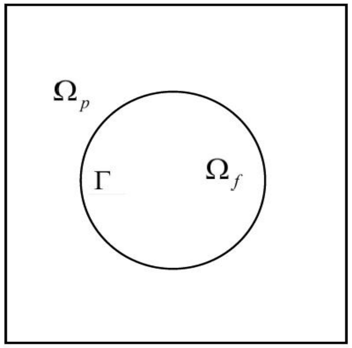
Outline

- ***Introduction/Problem:***
  - **Fluid and porous media (Stokes/Darcy or *Navier-Stokes/Darcy*)**
  - Different governing equations on different regions
  - Interface conditions (including BJ or BJS)



- ***Augmented IIM for Stokes-Darcy coupling***
  - Stokes/Darcy → *three Poisson Solvers*
- ***Least squares augmented IIM for NSE-D coupling***
  - Least squares: why? For the equivalence of two systems
- ***Numerical experiments & Conclusion***
  - *Analytic, flow problems, corners, multi-particles, …*

Some Comments: IB, IIM, IFEM

- **IB → IIM** (regularized to sharp interface)→ structured mesh, IIM is *second order point-wise* **both** in solution & ***Gradient*** !
- **IIM** (LeVeque/Li) → **IFEM** (Z. Li, T. Lin, X. Wu, FDM→FEM)

  T-F. Chan & B. Engquist: "*A new FDM should have a FEM version & vise versa*".
- **IFEM** (1D 98), **2D** (00-03, Numer. Math, Z. Li, T. Lin & X. Wu, R. Becker, high order, J. Guzman, Lin^2, .. Compared with X-FEM,...

  The structure and DoF remain unchanged! No bubble functions.
- ***Augmented IIM***, ***IFEM***, IB, and …
  - Can *decouple* problems
  - Can deal with more *general BC's*
  - Can utilize *fast solvers*
  - Can *preconditioning* the system …

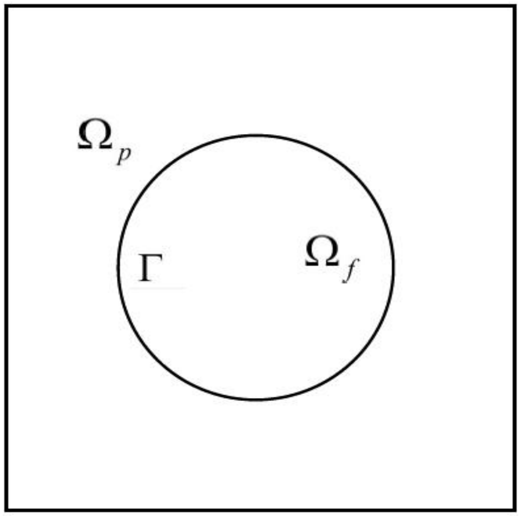
Fluid and Darcy Coupling

- Fluid flow: Stokes eqns or ***N-S eqns***

$$\nabla p = \nabla \bullet \mu(\nabla u + \nabla u^T) + g$$
$$\nabla \bullet u = 0$$

$$\rho\left(\frac{\partial u}{\partial t} + u \bullet \nabla u\right) + \nabla p = \mu \Delta u + g$$
$$\nabla \bullet u = 0$$

Navier-Stokes/Darcy



- Porous media flow: Darcy's law

$$u = -\frac{K}{\mu}\nabla p$$

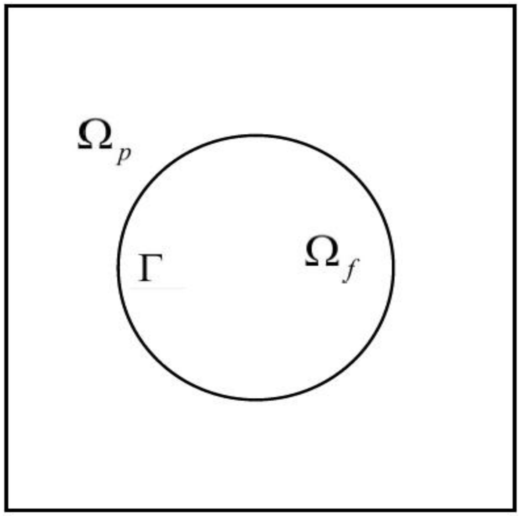
$$\nabla \bullet u = 0 \quad \text{or} \quad \nabla \bullet u = \phi$$

Interface Conditions

$$u_s \bullet n = u_D \bullet n \quad \text{(normal velocity is continuous)}$$

$$[p] = p_s - p_D = 2\mu(n \bullet T_s \bullet n), \quad T_s = (\nabla u_s + \nabla u_s^T)/2$$

$$n \bullet T_s \bullet \tau = -\frac{\sqrt{K}}{\mu\alpha}(u_s - u_D) \bullet \tau \ \text{(BJ) or } -\frac{\sqrt{K}}{\mu\alpha} u_s \bullet \tau \ \text{(BJS)}$$



Applications

- Flows across interfaces between soil and surface
- Oil reservoir
- Bio-medicine & *cell deformation*
- Blood motion in lungs, solid tumors and vessels
- Heat transfer in walls with fibrous insulation (firefighter)

Literature Review

- Regularity: W. Layton (SINUM, 40, 2003) *et al.*
- Numerical methods
  - **FEM** with domain decomposition, monolithic,…

$$\lambda_D K\nabla p_D \bullet n + g p_D = \eta_D$$

    Robin-Robin domain decomposition

  - **Fictitious domain approach**: the interface conditions are built into the weak form & *affects convergence order, larger system of saddle point system*

Literature Review -- FDM

- ***Cartesian mesh*** method (BEM & Stokelet by R. Cortetz for a circular interface)
- Cartesian method with local modified mesh (Z. Wang/Li, 2015), …
- Almost no ***non-trivial analytic solutions*** with curved interfaces
- Most simulations are for straight interfaces.

The weak form

Assume ϕ_p and $\mathbf{u}_f$ are 0 on the boundary $\partial\Omega$ and define the following functional spaces

$$
\begin{aligned}
H_f &= \{\mathbf{v}_f \in (H^1(\Omega_f))^d | \mathbf{v}_f = 0 \text{ on } \partial\Omega_f \backslash \Gamma\}, && (8)\\
Q &= L^2(\Omega_f), && (9)\\
H_p &= \{\psi_p \in H^1(\Omega_p) | \psi_p = 0 \text{ on } \partial\Omega_p \backslash \Gamma\}. && (10)
\end{aligned}
$$

The following bilinear forms are defined as

$$
\begin{aligned}
a_f(\mathbf{u}_f, \mathbf{v}_f) &= 2\nu(\frac{1}{2}(\nabla \mathbf{u}_f + \nabla^T \mathbf{u}_f), \frac{1}{2}(\nabla \mathbf{v}_f + \nabla^T \mathbf{v}_f)) \quad \text{on } \Omega_f, && (11)\\
a_p(\phi_p, \psi_p) &= (\mathbf{K}\nabla\phi_p, \nabla\psi_p) \quad \text{on } \Omega_p, && (12)\\
b_f(\mathbf{v}_f, p_f) &= -(\nabla \cdot \mathbf{v}_f, p_f) \quad \text{on } \Omega_f. && (13)
\end{aligned}
$$

Idea for Stokes-Darcy coupling

- **Step 1**: Get Poisson equation for the pressure.

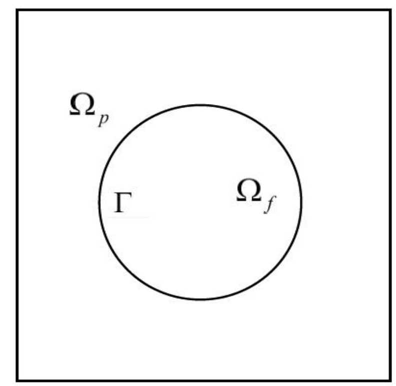
$$\nabla p = \nabla \bullet \mu(\nabla u + \nabla u^T) + g; \qquad u = -\frac{K}{\mu}\nabla p$$

$$\nabla \bullet u = 0 \qquad \nabla \bullet u = 0$$

$$\Delta p = \nabla \bullet g \quad \text{in Stokes}$$

$$\Delta p = 0 \quad \text{in Darcy}$$

$$[p] = q_1, \quad [p_n] = q_2, \quad \text{(new unknown)}$$



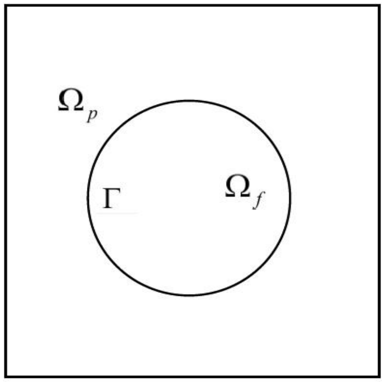
$$\Delta p = f(x) + \int_\Gamma q_2(s)\delta(x - X(s))\,ds + \int_\Gamma q_1(s)\delta'(x - X(s))\,ds$$

Idea for Stokes-Darcy coupling, II

- Step 2: Solve for the velocity

$$\nabla p = \nabla \bullet \mu(\nabla u + \nabla u^T) + g \qquad u = -\frac{K}{\mu}\nabla p$$

$$\nabla \bullet u = 0 \qquad \nabla \bullet u = 0$$



$$\Delta u = \begin{cases} (p_x - g_1)/\mu \\ -K\Delta p_x/\mu \end{cases}; \quad \Delta v = \begin{cases} (p_y - g_2)/\mu \\ -K\Delta p_y/\mu \end{cases},$$

$$[u_n] = q_3, \quad [v_n] = q_4, \quad \text{(new unknown)}$$

$$[u \bullet n] = 0, \quad [u \bullet \tau] = q_5 \Rightarrow \text{ get } [u] \ \& \ [v]$$

Elliptic Interface Problems

- IIM, or IFEM, or others

$$-\Delta u = f - \int_\Gamma C(s)\delta(x - X(s))ds, \quad x \subset \Omega$$

$$\text{or} \quad -\Delta u = f, \quad [u] = 0, \quad [u_n] = C(s), \quad x \subset \Omega^+ \cap \Omega^- \setminus \Gamma$$

$$\text{or} \quad -\Delta u = f, \quad [u] = w, \quad [u_n] = C(s), \quad x \subset \Omega^+ \cap \Omega^- \setminus \Gamma$$

- Numerical Methods: ***A U = F - BQ.***
  - One fast Poisson solver (FFT, Fishpack, multigrid,..)
  - Second order convergence (pointwise) in both of the ***solution*** AND ***gradient*** (T. Beale/A. Layton; Z. Li, …)

Original Augmented Idea

- Preconditioning PDE

$$-\nabla\cdot(\beta\nabla u) = f, \quad [u]=0, \ [\beta u_n] = C(s), \quad x \subset \Omega^+ \cap \Omega^- \setminus \Gamma$$

$$-\Delta u = \frac{f}{\beta}, \quad [u]=0, \ [u_n] = q(s), \quad x \subset \Omega^+ \cap \Omega^- \setminus \Gamma$$

$$AU + BQ = F \qquad \text{global}$$

Flux condition $[\beta u_n] = C(s)$,

$$CU + DQ = F \qquad \text{local}$$

Schur complement:

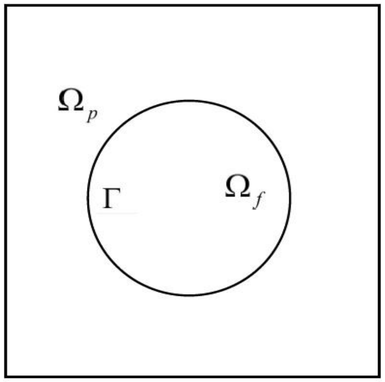
$$(D - CA^{-1}B)Q = F_2 \qquad \text{smaller Eqn.}$$

- Second order convergence for both in solution & ***gradient; convergence is*** independent of $\boldsymbol{\beta}$!
- Have been applied to IFEM (Ji/Li/Chen …)

Idea for Stokes-Darcy coupling III

- Introduce 5 (or 6) interface variables (from the primary variables), we get ***three Poisson equations*** for the pressure and velocity (***decoupled*** the PDEs)
- interface (augmented) variables

$$[p] = q_1, \quad [p_n] = q_2,$$
$$[u_n] = q_3, \quad [v_n] = q_4,$$
$$[u \bullet \tau] = q_5$$



Other Interface Conditions

- We set up **5** augmented variables, q_1-q_5, we need 5 augmented equations to close the system

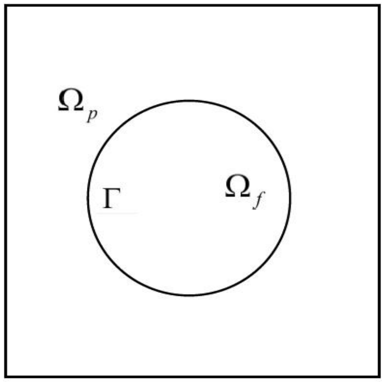
$$[p] = 2\mu(n \bullet T_s \bullet n)$$

$$u_s \bullet n = u_D \bullet n$$

$$n \bullet T_s \bullet \tau = -\frac{\sqrt{K}}{\mu\alpha}(u_s - u_D) \bullet \tau \quad \text{or} \quad -\frac{\sqrt{K}}{\mu\alpha} u_s \bullet \tau$$

$$\frac{\partial p_s}{\partial n} = \mu \Delta u_s \bullet n + g \bullet n$$

$$\frac{\partial p_D}{\partial n} = -\frac{\mu}{K} u_D \bullet n$$



Outer Boundary Conditions

- Darcy-Stokes (or Navier-Stokes)

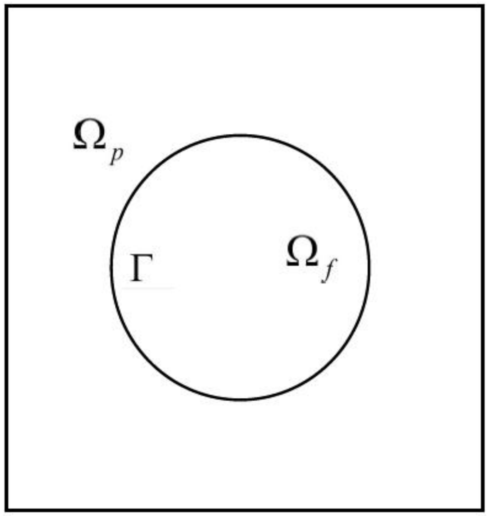
$$u_D \bullet n \ \text{ is given,} \quad \frac{\partial p_D}{\partial n} = -\frac{u_D \bullet n}{K} \quad \text{on } \partial\Omega$$

$$u_D = -K \nabla p_D, \quad \text{Dirichlet BC for } (u, v) \ \text{ on } \partial\Omega$$

- Stokes (or Navier-Stokes)-Darcy

$$u_S \ \text{ is given,} \quad \frac{\partial p_S}{\partial n} = \mu \Delta u_S \bullet n + F \bullet n \quad \text{on } \partial\Omega$$

$$\text{set} \quad \frac{\partial p_S}{\partial n} = q_6 \quad \text{another augmented variable}$$



Discretization & Schur Complement

- Discretization of ***three Poisson Eqns*** with jump conditions (linear problem)

$$AU + BQ = F$$

- Discretization of the *physical interface conditions*

$$CU + DQ = F_2$$

- ***Q*** is along the boundary ***O(5N)***
- Schur Complement (direct or *GMRES*)

$$(D - C A^{-1} B)Q = F_2 - C A^{-1} F, \quad SQ = F_3$$

Matrix-vector multiplication

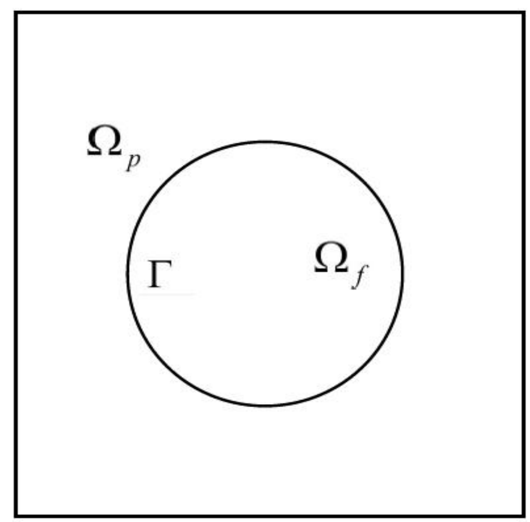
- It is easy to get the matrix multiplication for a given augmented vector ***Q*** to get ***SQ***
  - Solve three Poisson equations ***AU + BQ = F***
  - Find the residual of interface conditions

    R(Q)=CU + DQ -F₂ via some interpolation
- LU (SVD) or GMRES
  - Fixed interface, time independent, form the matrix S and use LU (SVD)
  - Moving interface *GMRES + precondioning*

Validation for Stokes/Darcy

- It is challenging to construct non-trivial analytic soln for curved interface (***r=1***). In Darcy region, ***u=0, v=0, p=1, with slip jump***. In fluid:

$$
\begin{aligned}
u(x,y) &= y(x^2+y^2-1)-2y, \\
v(x,y) &= -x(x^2+y^2-1)+2x, \\
p(x,y) &= x^2+y^2, \\
F_1(x,y) &= -8y+2x \\
F_2(x,y) &= 8x+2y,
\end{aligned}
$$



Validation for Stokes/Darcy II

- Average convergence rate: 1.8334 & 2.1388

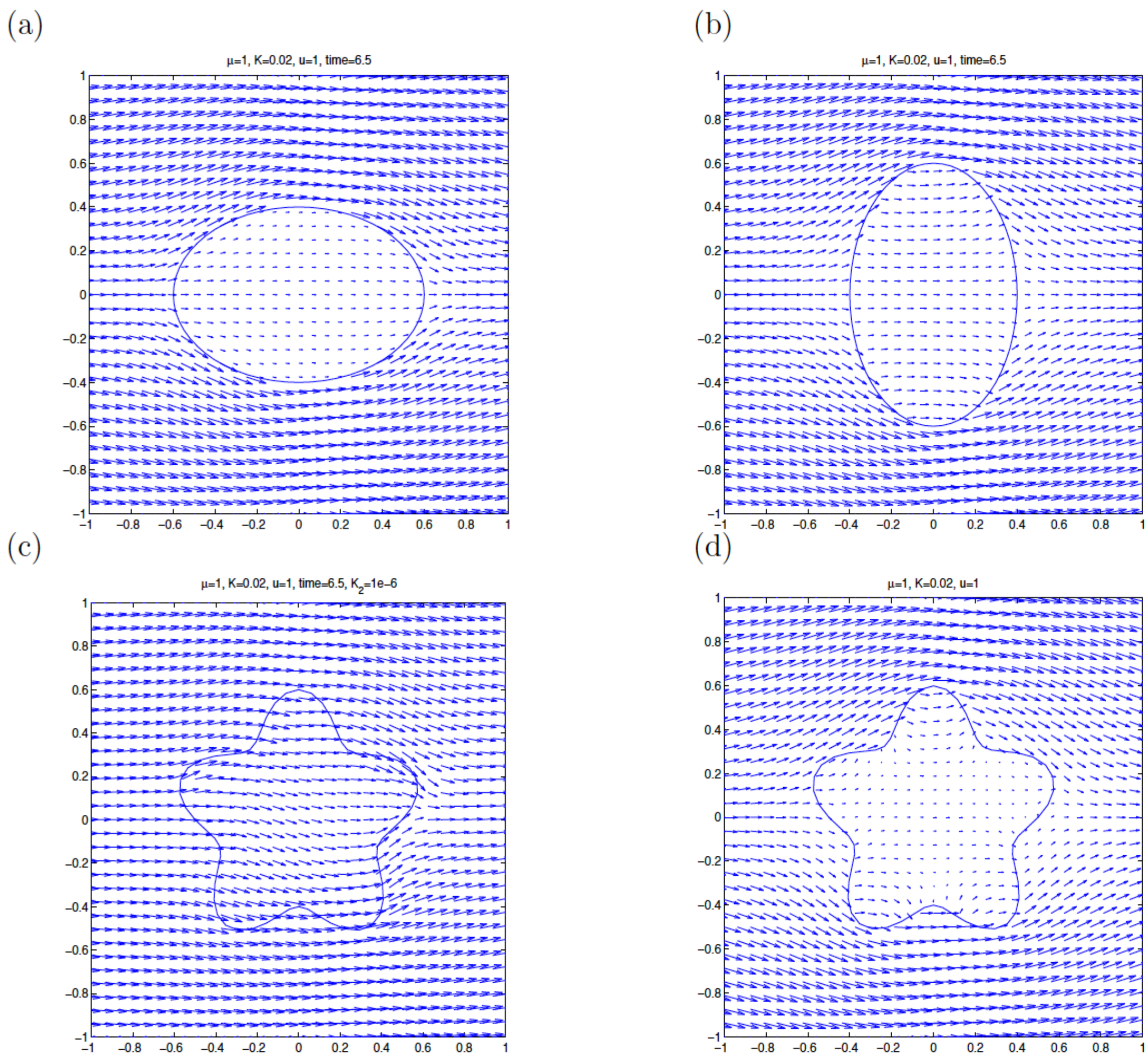
$$\mathbf{n} = [x,\ y]^T, \quad \boldsymbol{\tau} = [-y,\ x]^T, \quad p_s = 1, \quad p_D = 1, \quad u_s = -2y, \quad v_s = 2x,$$

$$u_D = 0, \quad v_D = 0, \quad \frac{\partial u_s}{\partial n} = 0, \quad \frac{\partial v_s}{\partial n} = 0, \quad \frac{\partial u_s}{\partial \boldsymbol{\tau}} = -2x, \quad \frac{\partial v_s}{\partial \boldsymbol{\tau}} = -2y,$$

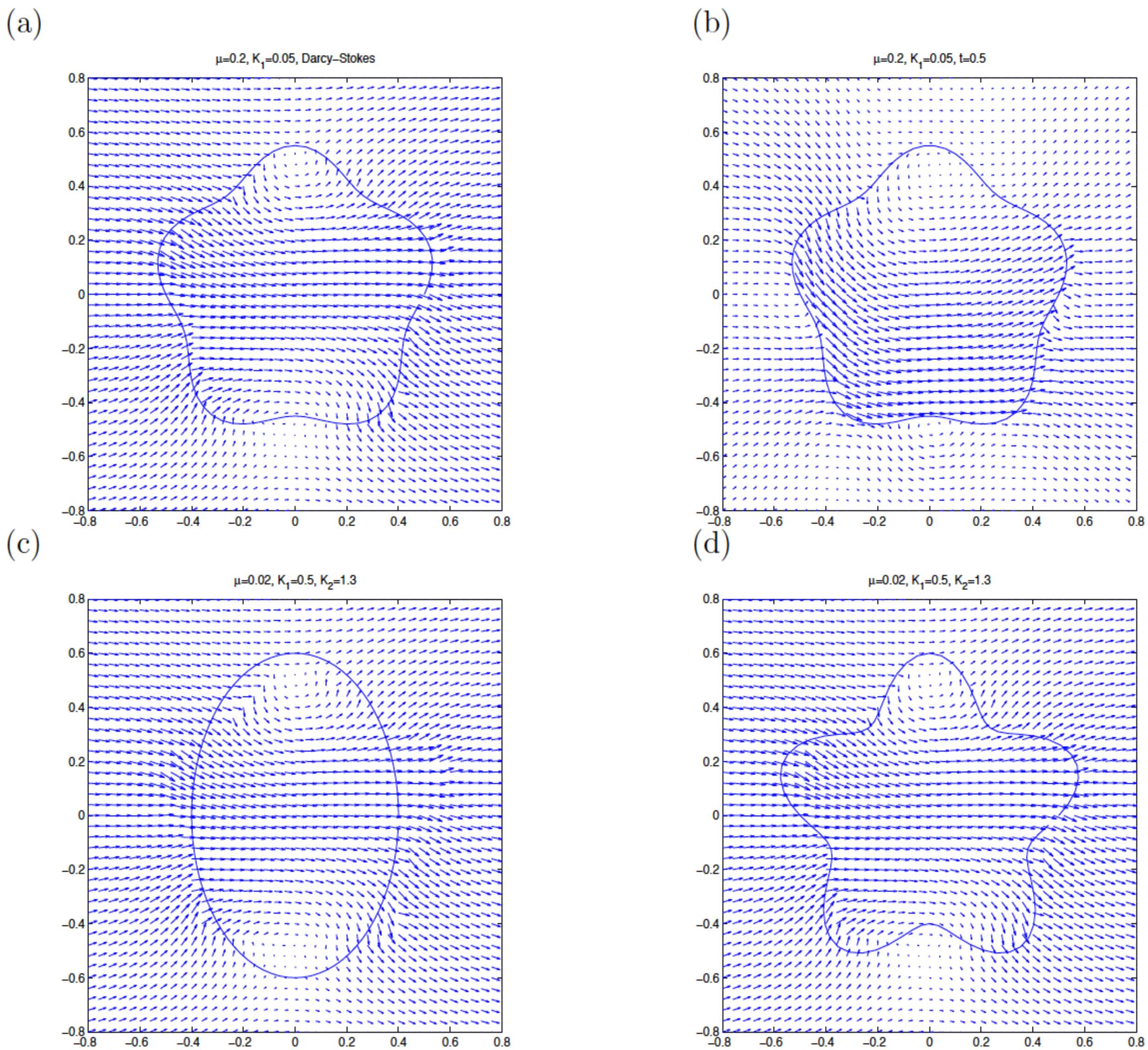
$$\mathbf{D}_s \mathbf{n} \cdot \mathbf{n} = 0 \qquad \mathbf{D}_s \boldsymbol{\tau} \cdot \mathbf{n} = -2.$$

N	$\|E_p\|_\infty$	$order$	$\|E_u\|_\infty$	order
16	$2.2299e-01$		$2.8516e-01$	
32	$2.7892e-02$	2.9991	$3.2203e-02$	3.1465
64	$7.8690e-03$	1.8256	$1.4188e-02$	1.1825
128	$6.0515e-03$	0.37889	$2.6563e-03$	2.4172
256	$3.2848e-03$	0.88149	$1.2963e-03$	1.0350
512	$9.6182e-04$	1.7720	$2.5613e-04$	2.3395

Flow Test I


(a)
μ=1, K=0.02, u=1, time=6.5
(b)
μ=1, K=0.02, u=1, time=6.5
(c)
μ=1, K=0.02, u=1, time=6.5, K2=1e−6
(d)
μ=1, K=0.02, u=1

Flow Test (fluid inside)



Flow tests: Stokes-Darcy

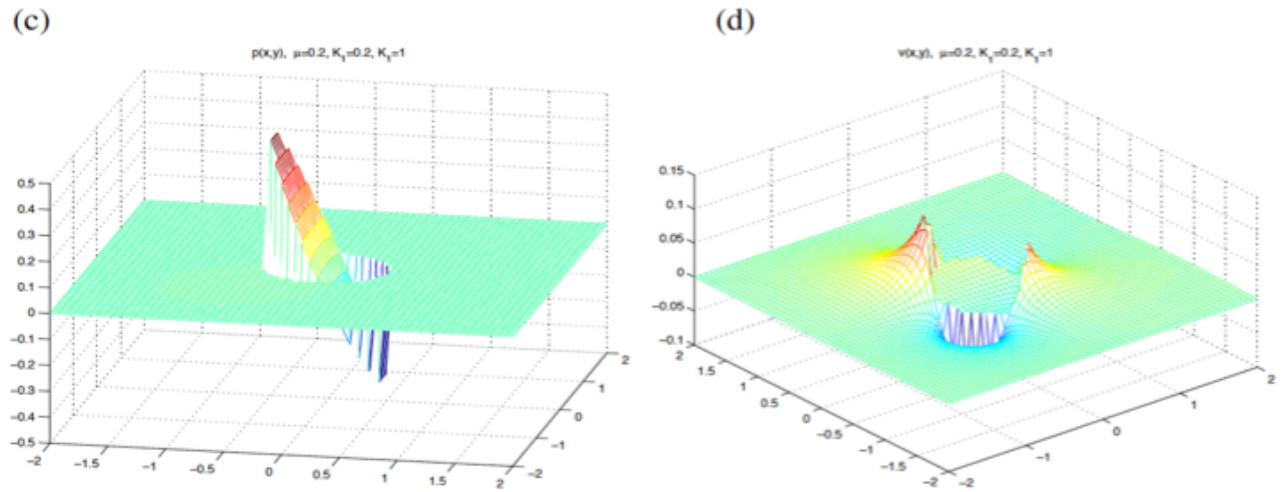
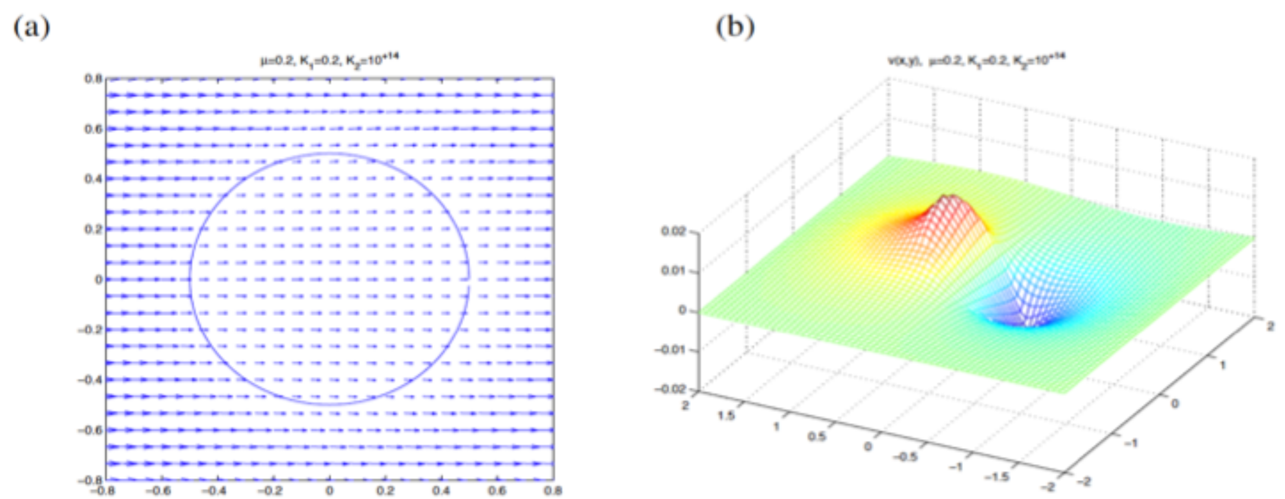


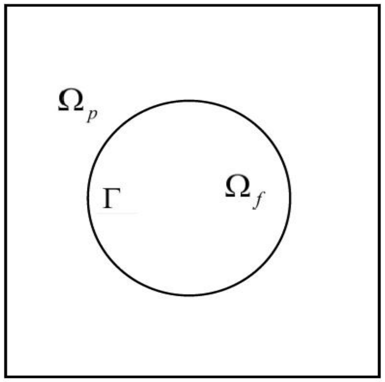
Figure 2. (a)-(b): Velocity plot of a porous media and a fluid (outside the interface $x^2 + y^2 = 0.5^2$) interaction with different parameters. (a): $K_1 = 1$, $\mu = 1$, $K_2 = 1$; (b): $K_1 = 0.2$, $\mu = 0.2$, $K_2 = 1$. (c): The mesh plot of the pressure corresponding to (b). (d): The mesh plot of the v component of the velocity corresponding to (b).



Equivalence of two systems

- **Original system** (Stokes/Darcy) —> via augmented (interface) variables —> **New system** (*three Poisson eqns*):

$$[p]=q_1,\quad [p_n]=q_2,$$
$$[u_n]=q_3,\quad [v_n]=q_4,$$
$$[u\bullet\tau]=q_5$$

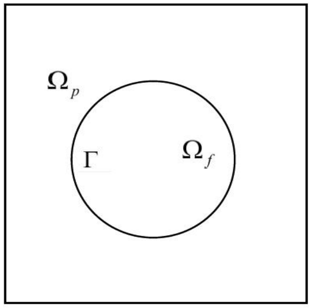


- The soln to the ***original*** is also a soln to the ***new.***

Equivalence of two systems

Is the solution to the new system also the soln to the original system?

- The momentum equations is satisfied
- The Darcy's law in porous media region is also enforced.
- For Stokes and Darcy's coupling, it works fine



Why not work well for NSE/Darcy?

(c) Comparison of the condition number.

$N = N_b$	$cond^{6eq}$ (present)	$cond^{5eq}$ ([27])
32	$5.1733e+02$	$1.6935e+06$
64	$3.1495e+03$	$1.8021e+06$
128	$3.3617e+04$	$1.2956e+08$
256	$2.5181e+05$	$2.2210e+09$
512	$2.3641e+06$	$1.5831e+11$

Why & where Least Squares

- The method (5 aug. variables) works fine for Stokes-Darcy's coupling, but barely works for Navier-Stokes & Darcy's coupling.
- Why? Is the velocity divergence free in flow region? We have ***Δ(div(u))=0 → div(u)=0? Yes, if it true along*** Γ & $\partial\Omega$.
- Our solution: enforce the divergence condition on Γ & $\partial\Omega$: $5\ aug\ viariables, six\ eqn.$ Least squares!
- Soln exists & it is unique if the original problem is well-posed!

Algorithm for NSE/Darcy

- Time marching + pressure incremental

$$\Delta p^{k+1} = \begin{cases} \nabla \cdot \mathbf{F}^{k+1/2} - \rho \nabla \cdot (\mathbf{u} \cdot \nabla \mathbf{u})^{k+1/2}, & \mathbf{x} \in \Omega_f, \\ 0 & \mathbf{x} \in \Omega_D, \end{cases}$$

$$[p^{k+1}] = q_1^{k+1}, \qquad \left[\frac{\partial p^{k+1}}{\partial n}\right] = q_2^{k+1}, \qquad \text{on } \Gamma.$$

Algorithm for NSE/Darcy II

$$\Delta u^* - \frac{2\rho}{\mu\Delta t}u^* = \begin{cases} \frac{2}{\mu}\left(p_x^{k+1} - \frac{\rho}{\Delta t}u^k\right) - \Delta u^k + \frac{2}{\mu}\left(\rho(\mathbf{u}\cdot\nabla u)^{k+1/2} - F_1^{k+1/2}\right), & \mathbf{x}\in\Omega_f, \\ -K_1\Delta p_x^{k+1} + \frac{2K_1\rho}{\mu\Delta t}p_x^{k+1}, & \mathbf{x}\in\Omega_D, \end{cases} \tag{26}$$

$$[u^*] = -q_5^{k+1}\sin\theta, \qquad \left[\frac{\partial u^*}{\partial n}\right] = q_3^{k+1}, \qquad \text{on } \Gamma;$$

$$\Delta v^* - \frac{2\rho}{\mu\Delta t}v^* = \begin{cases} \frac{2}{\mu}\left(p_y^{k+1} - \frac{\rho}{\Delta t}v^k\right) - \Delta v^k + \frac{2}{\mu}\left(\rho(\mathbf{u}\cdot\nabla v)^{k+1/2} - F_2^{k+1/2}\right), & \mathbf{x}\in\Omega_f, \\ -K_1\Delta p_y^{k+1} + \frac{2K_1\rho}{\mu\Delta t}p_y^{k+1}, & \mathbf{x}\in\Omega_D, \end{cases} \tag{27}$$

$$[v^*] = -q_5^{k+1}\cos\theta, \qquad \left[\frac{\partial v^*}{\partial n}\right] = q_4^{k+1}, \qquad \text{on } \Gamma;$$

Important Details

- How to make the Helmholtz *eqn* have the same magnitude in both fluid & porous media region? Add an artificial term in the Darcy's domain $\mathbf{u^*/\Delta t}$
- Reinforce the Darcy's law after solving the Helmholtz eqns.
- With and without projection step $(O(h^2))$.
- Given $\mathbf{Q}^{k+1}$, solve to get:

$$\mathbf{A U^{k+1} + B\, Q^{k+1} = F^k}$$

Algorithm Revisit II

- Discretization of ***three Poisson Eqns (ignore k)***

$$AU + BQ = F$$

- Discretization of the *physical interface conditions*

$$CU + DQ = F_2$$

- ***Q*** is along the boundary ***O(5N)***
- Schur Complement (direct or *GMRES*)

$$(D - C A^{-1} B)Q = F_2 - C A^{-1} F, \quad SQ = F_3$$

- For fixed Γ, h, ***S*** is a constant matrix.

Matrix-vector form at one step

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \tilde{u}^{\,k+1} \\ \mathbf{Q}^{k+1} \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{F}}_1^{k+1} \\ \tilde{\mathbf{F}}_2^{k+1} \end{bmatrix}.$$

Therefore, the Schur complement for $\mathbf{Q}^{k+1}$ is

$$(D - CA^{-1}B)\mathbf{Q}^{k+1} = \tilde{\mathbf{F}}_2^{k+1} - CA^{-1}\tilde{\mathbf{F}}_1^{k+1} = \bar{\mathbf{F}}^{k+1}, \qquad \text{or} \qquad S\mathbf{Q}^{k+1} = \bar{\mathbf{F}}^{k+1}.$$

Validation of NSE/Darcy

$$u_f = g(t)\left(y(x^2+y^2-1)+2y \right),$$
$$v_f = g(t)\left(-x(x^2+y^2-1)-2x \right),$$
$$p_f = g(t)\left(x^2+y^2 \right),$$

$$\mathbf{n}=[x,\ y]^T,\quad \boldsymbol{\tau}=[-y,\ x]^T,\quad p_f=g(t),\quad p_D=g(t),\quad u_f=2y\,g(t),\quad v_f=-2x\,g(t),$$

$$u_D=0,\quad v_D=0,\quad \frac{\partial u_f}{\partial n}=4y\,g(t),\quad \frac{\partial v_f}{\partial n}=-4x\,g(t),\quad \frac{\partial u_f}{\partial \tau}=2x\,g(t),\quad \frac{\partial v_f}{\partial \tau}=2y\,g(t),$$

$$\mathbf{n}\cdot\mathbf{D}_f\cdot\mathbf{n}=0 \qquad \boldsymbol{\tau}\cdot\mathbf{D}_f\cdot\mathbf{n}=0.$$

Grid refinement analysis

- Tangent slip, pressure is constant in Darcy. Average convergence: 2.0221 & 3.2110

(a) Comparison of the pressure error and accuracy order.

$N = N_b$	$\|E_p^{6eq}\|_\infty^N$ (present)	order	$\|E_p^{5eq}\|_\infty^N$ ([27])	order
32	$9.1289e-03$		$2.7892e-02$	
64	$1.9969e-03$	2.1927	$6.0515e-03$	0.37889
128	$9.9901e-04$	0.9918	$3.2848e-03$	0.88149
256	$3.6806e-04$	1.4406	$3.2848e-03$	0.88149
512	$6.4588e-05$	2.5106	$9.6182e-04$	1.7720

(b) Comparison of velocity error and accuracy order.

N	$\|E_{\mathbf{u}}^{6eq}\|_\infty^N$ (present)	order	$\|E_{\mathbf{u}}^{5eq}\|_\infty^N$ ([27])	order
32	$2.1160e-02$		$3.2203e-02$	
64	$5.8094e-03$	1.8649	$1.4188e-02$	1.1825
128	$1.4762e-03$	1.9765	$2.6563e-03$	2.4172
256	$3.3356e-04$	2.1459	$1.2963e-03$	1.0350
512	$9.0142e-05$	1.8877	$2.5613e-04$	2.3395

Another example

A *continuous tangential* velocity but *discontinuous pressure* along the interface. More importantly, the velocity and pressure are *non-trivial* in both regions and the normal derivatives of the velocity components are also discontinuous across the interface.

$$u_f = g(t)\left(y(x^2+y^2-1)+2x \right),$$

$$v_f = g(t)\left(-x(x^2+y^2-1)-2y \right),$$

$$p_f = 3g(t)\left(x^2-y^2 \right),$$

Grid Refinement Analysis

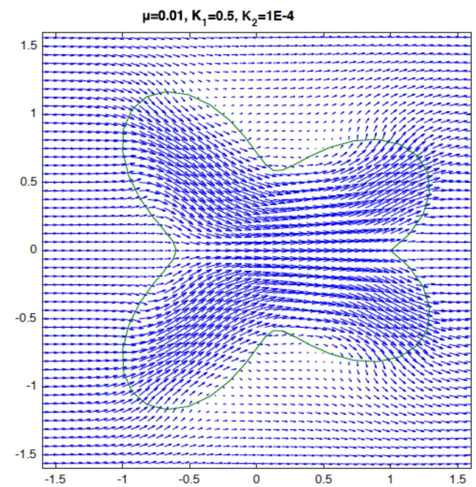
N	N_b	$\|E_p\|_\infty^N$	order	$\|E_{\mathbf{u}}\|_\infty^N$	order	cond-6eq	cond-5eq
16	16	$1.8890e-00$		$1.9839e-01$		$8.0806e+01$	$1.2104e+02$
32	22	$5.7256e-01$	1.7221	$2.2640e-02$	3.1314	$2.0234e+02$	$7.3228e+02$
64	30	$1.4790e-01$	1.9528	$1.7365e-03$	3.7046	$7.7313e+02$	$6.7254e+03$
128	42	$3.5359e-02$	2.0645	$1.5333e-04$	3.5015	$3.0480e+03$	$8.4892e+04$
256	68	$6.9394e-03$	2.3492	$2.6986e-05$	2.5064	$1.0844e+04$	$1.0888e+06$

N	$\|E_1\|_\infty^N$	$\|E_2\|_\infty^N$	$\|E_3\|_\infty^N$	$\|E_4\|_\infty^N$	$\|E_5\|_\infty^N$	$\|E_6\|_\infty^N$
32	$3.4972e-03$	$1.8503e-03$	$2.8119e-03$	$4.7202e-03$	$9.7248e-04$	$3.3883e-03$
64	$1.2280e-03$	$5.2899e-04$	$5.1238e-04$	$2.1042e-03$	$2.7033e-04$	$1.0480e-03$
128	$2.0427e-04$	$8.3986e-05$	$8.6997e-05$	$5.9738e-04$	$7.5299e-05$	$1.5937e-04$
256	$2.6319e-05$	$1.2165e-05$	$2.1739e-05$	$9.6036e-05$	$1.5473e-05$	$1.8553e-05$
order	2.5026	2.5091	2.2540	2.0640	2.0642	2.6538

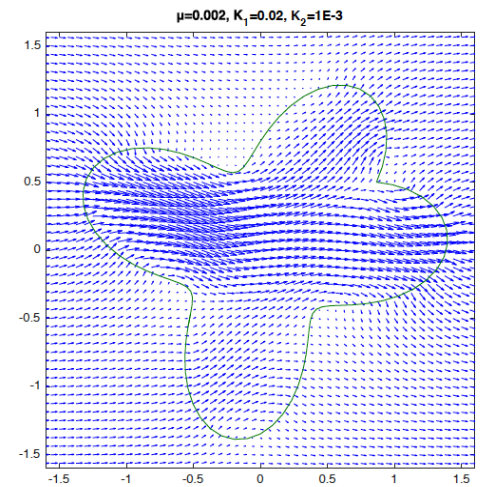
Table 2: The residual of the six interface conditions (15)-(20) of the computed solution. The last row is the average convergence order of the six interface equations.

Zhilin LI

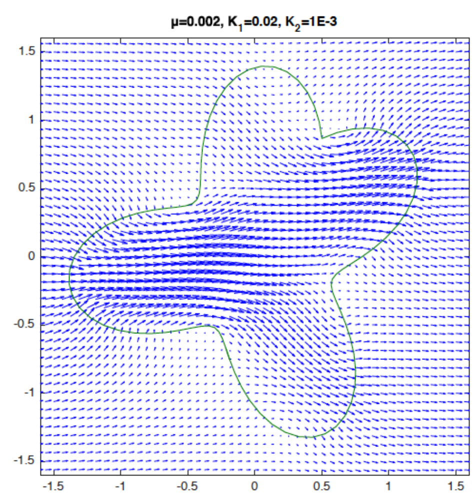
Orientation Effect (flow inside)

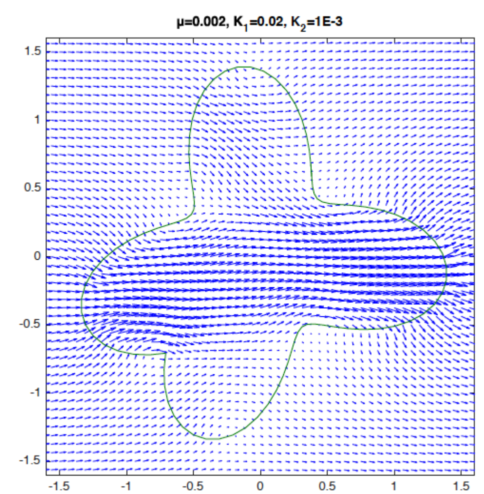


(c)



(d)





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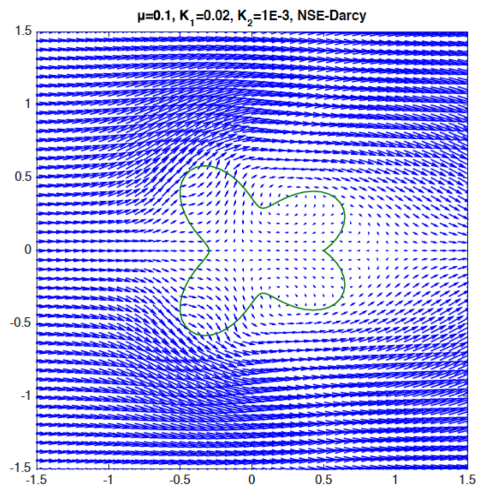
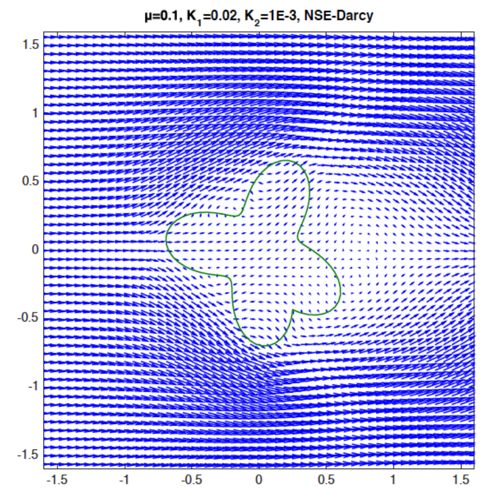
Orientation Effect (flow outside)

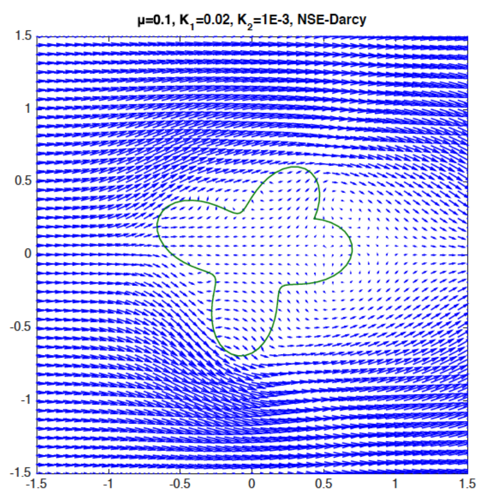
(a)



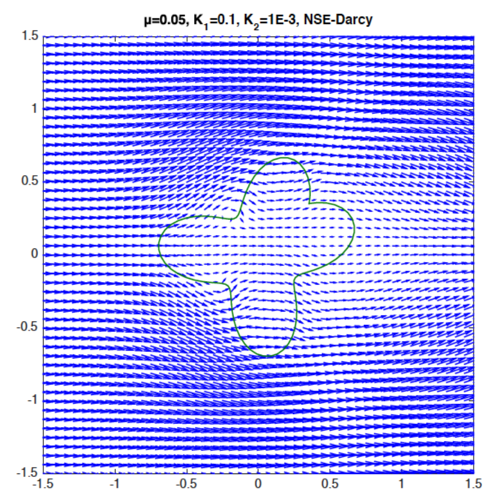
(b)



(c)



(d)



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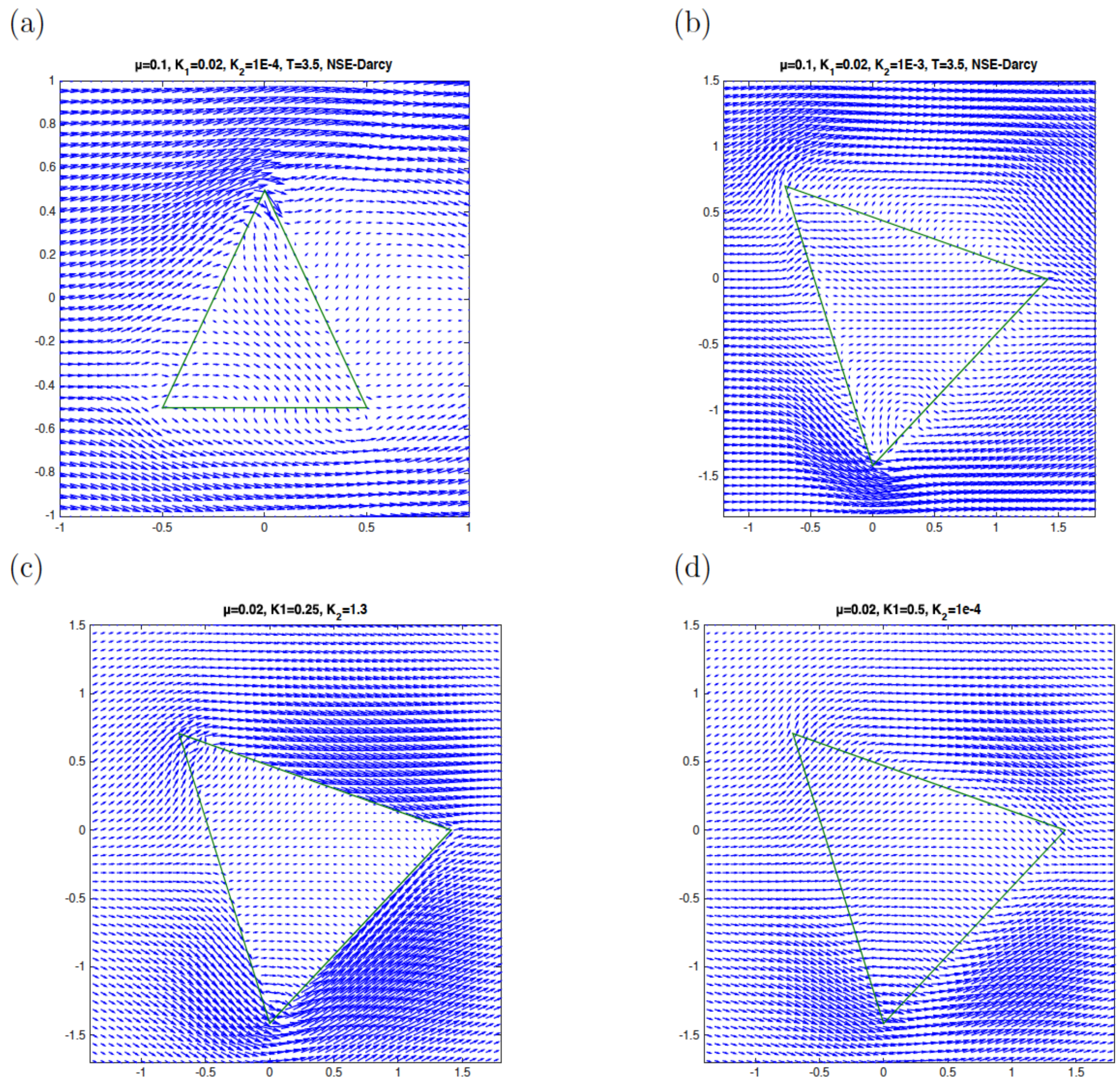
Corner Effect (flow outside)



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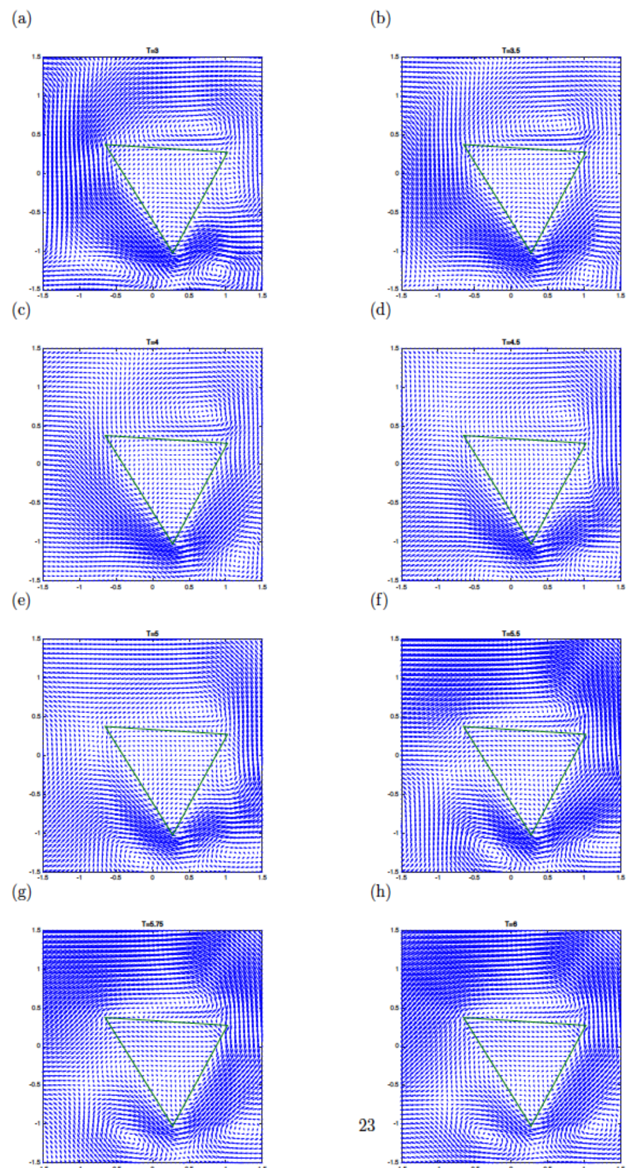
Transient Behaviors



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More on flow test with corners

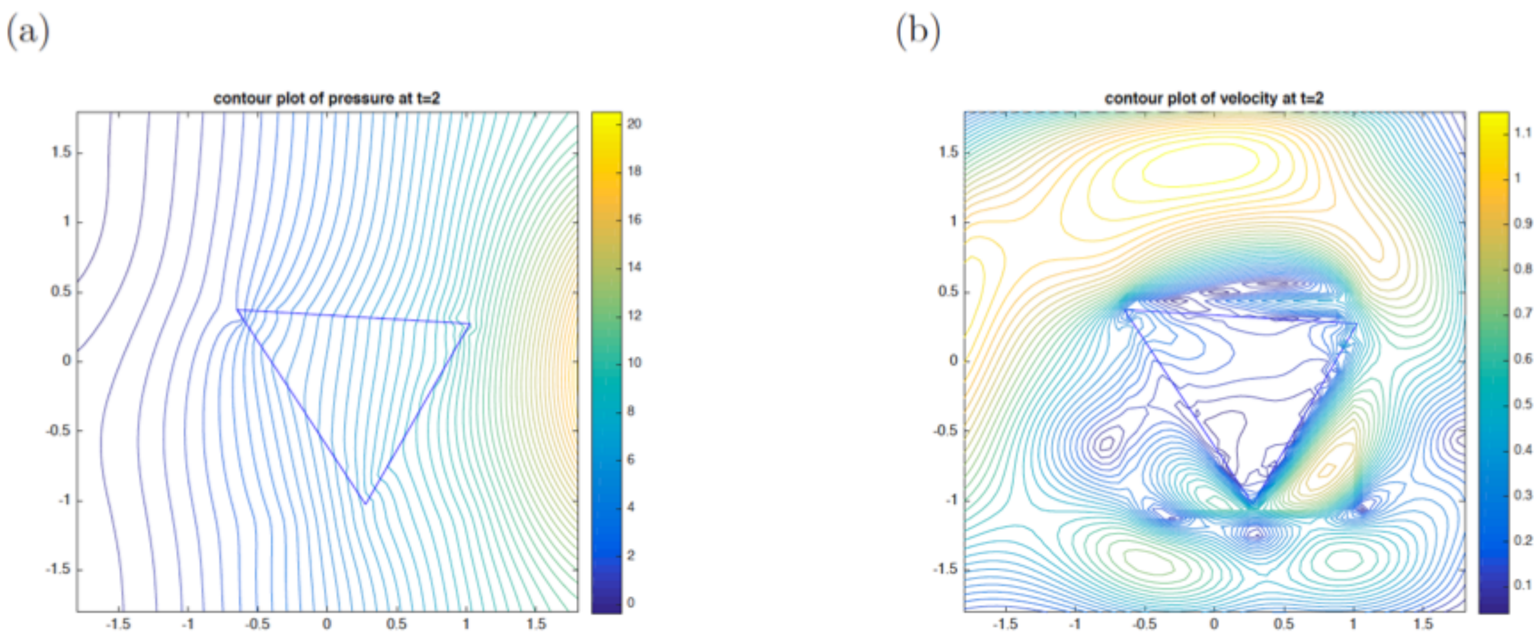
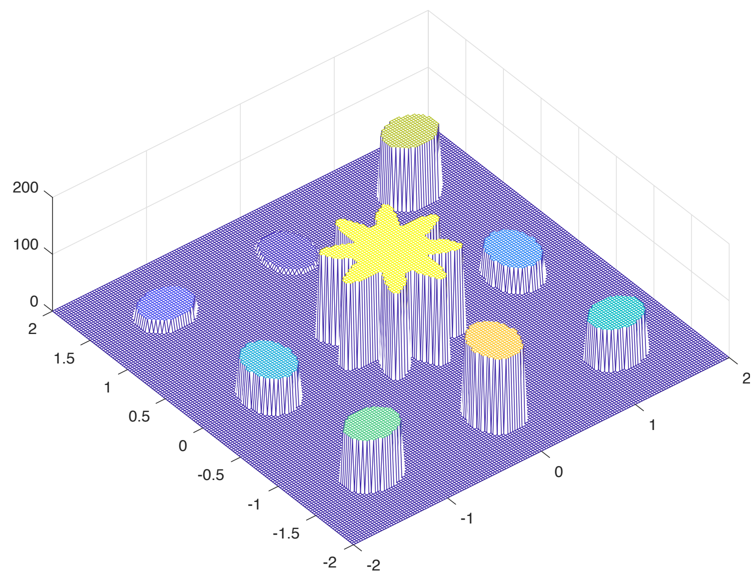
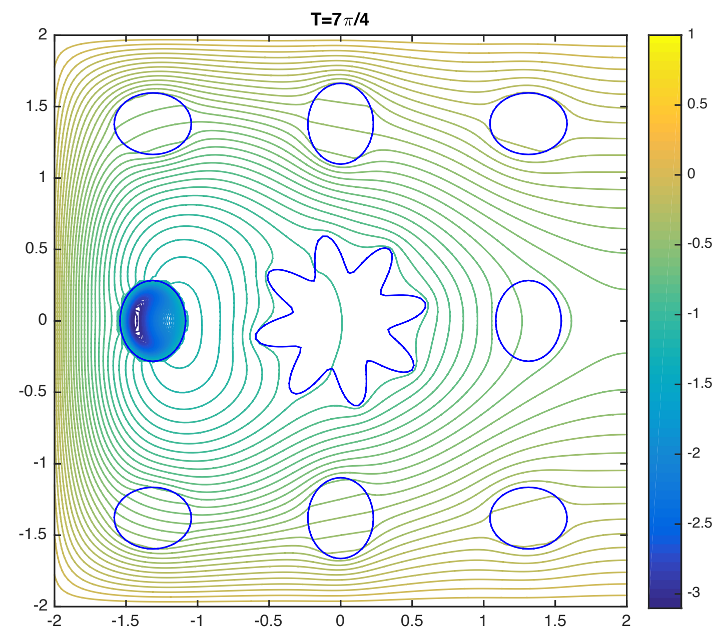


Figure 9: Contour plots of the pressure and the magnitude of the velocity at $t = 2$ with the set-up in Figure 8.

FSI (Fluid and Porous Media)

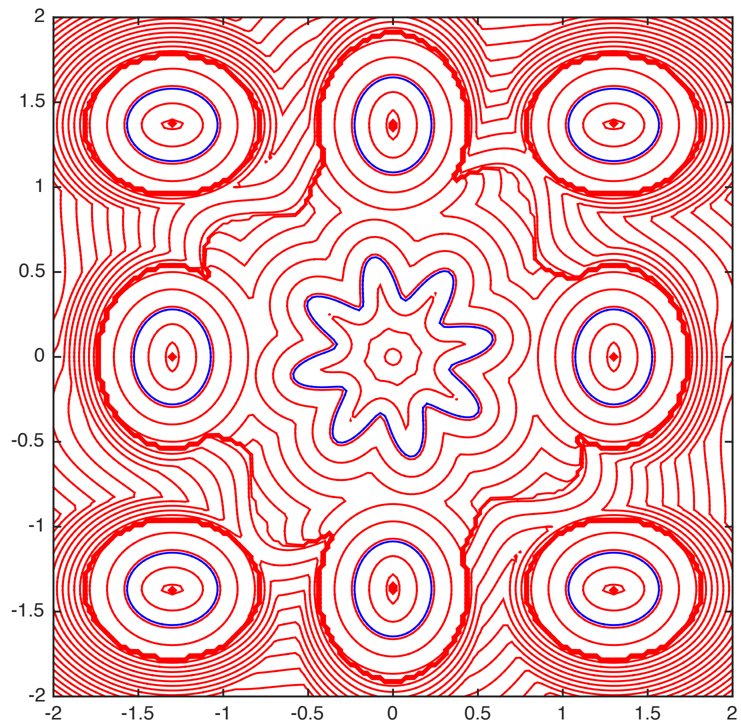
- Model: incompressible Navier-Stokes or Stokes equations coupled with Darcy's law. Multi-connected domain





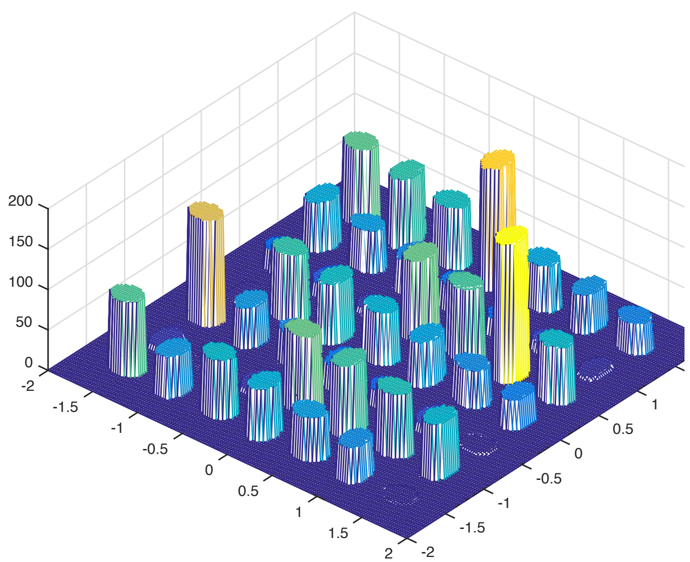
Signed distance function

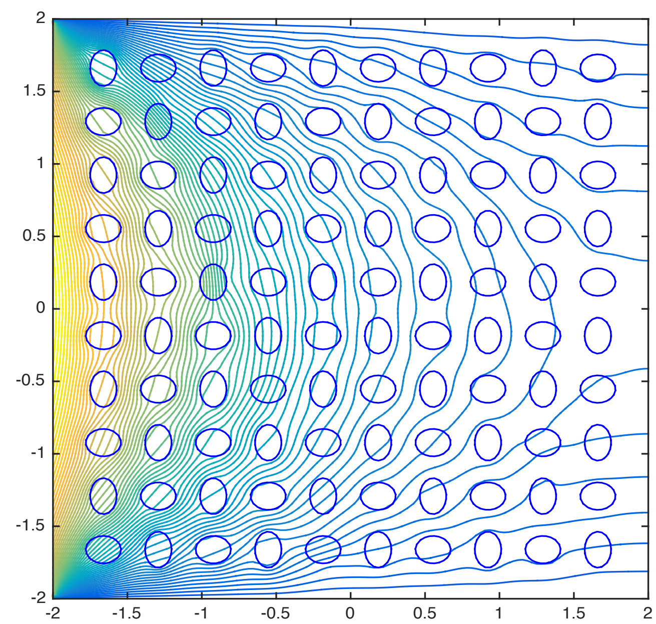
- Re-initialization of a 2D level set function in a tube: *blue line* is the interface. Treat boundary is challenging!



Multi-particles & multi-scales

- Many inclusions with different permeability. Flow from right





Conclusions

New augmented methods for fluid flow and porous media (***Stokes-Darcy or NES-Darcy***)

- Different governing equations are transformed to the same type equations ***via augmented interface variables***
- ***Three Poisson equations*** with jump in the soln and normal derivative
- Use ***least squares*** to get equivalent systems
- Can utilize the FFT based ***fast Poisson solver***

Conclusions

- Second order accurate in both ***pressure*** and ***velocity***
- Equivalence has been proved under stronger regularity assumptions
- What are the best augmented variables that have the same No. of unknowns and equations?

Why augmented approach?

- Can make the solver ***faster***, e.g, fast IIM for elliptic interface problems with piecewise constant coefficient, IIM for irregular domains.
- Can ***decouple problems***, e.g., the Stokes or NSE equations with discontinuous viscosity, the augmented approach enable us to decouple the jump conditions in the pressure and the velocity
- For some problems, it is the only way to get accurate discritization
- No need to have the Green functions, independent of BC, source terms, domains *etc*.
- Can ***couple problems***: deal with Stokes-Darcy coupling

Key Words & Related to others

- IB (Immersed Boundary) →IIM (Immersed Interface) 1st to 2nd velocity & pressure
- Cartesian meshes & Fast Helmoltz/Poisson solvers
- Efficiency and accuracy; augmented IIM
- Applications: Fluid & porous media; shape identification (inverse problem), moving contact lines; fast solver on irregular domains (elliptic, discontinuous viscosity, pressure BC, drop spreading, motion of voids of electric voids …)

Thank you!