Analysis of Immersed Elastic Filaments in Stokes Flow

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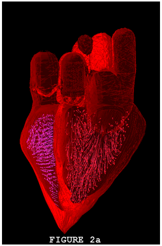
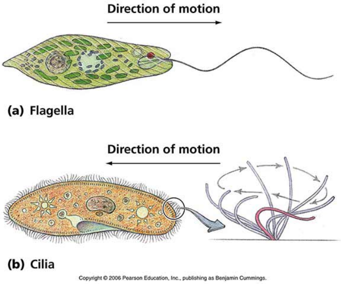
Filaments in Stokes Fluid



McQueen and Peskin



Pearson Education

Motion:

Motivation:

- Fluid structure interaction problems abound.
- Many computational studies. Fewer studies on analysis/numerical analysis.

We study the following simple settings:

- *Peskin Problem* (with Analise Rodenberg and Dan Spirn): Elastic string in a 2D Stokes fluid. The full dynamic problem is studied.
- *Slender Body Problem* (with Laurel Ohm and Dan Spirn): A thin filament in a 3D Stokes fluid. The stationary problem is studied.

Outline

Outline

1. Peskin Problem
   - Setup
   - Local Existence/Regularity
2. Slender Body Theory
   - Introduction
   - Setup
   - Well-posedness and Error Estimates

Peskin Problem (Jump Formulation)

We consider the *Peskin problem*.

$$\mu\Delta \boldsymbol{u} - \nabla p = 0, \quad \nabla\cdot\boldsymbol{u} = 0 \text{ for } \mathbb{R}^2\backslash\Gamma,$$

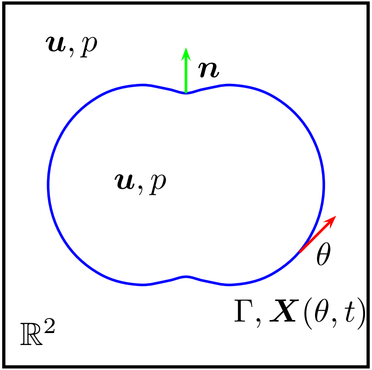
$$[\![\boldsymbol{u}]\!] = 0, \quad [\![\sigma\boldsymbol{n}]\!] = K\frac{\partial^2 \boldsymbol{X}}{\partial\theta^2}\left|\frac{\partial \boldsymbol{X}}{\partial\theta}\right|^{-1} \text{ on } \Gamma,$$

$$\frac{\partial \boldsymbol{X}}{\partial t}(\theta,t) = \boldsymbol{u}(\boldsymbol{X}(\theta,t),t).$$

$\boldsymbol{n}$: unit normal on Γ.

σ : stress tensor, $\sigma = \mu(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^{\mathrm{T}}) - pI$.

$[\![\cdot]\!]$: jump across Γ.



- Stokes equations satisfied in $\mathbb{R}^2\backslash\Gamma$ (with $\boldsymbol{u}\to 0$ as $|\boldsymbol{x}|\to\infty$, p bounded). Equal viscosity $\mu = 1$ in/out.
- No-slip and stress balance boundary conditions on Γ. Stress jump given by elastic filament force, elastic constant $K = 1$.
- Parametrization $\theta\in\mathbb{S}^1$ is material coordinate; moves with the fluid.

Immersed Boundary (IB) Formulation

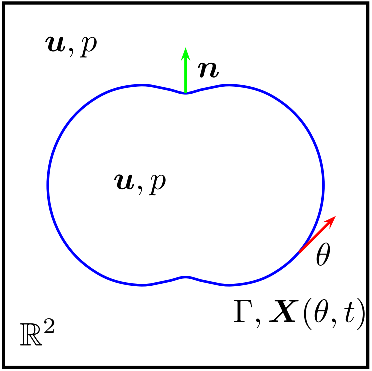
The *immersed boundary (IB) formulation* of the Peskin problem.

$$-\Delta \boldsymbol{u} + \nabla p = \boldsymbol{f}, \quad \nabla \cdot \boldsymbol{u} = 0 \text{ in } \mathbb{R}^2,$$

$$\boldsymbol{f} = \int_{\mathbb{S}^1} \frac{\partial^2 \boldsymbol{X}}{\partial \theta^2} \delta(\boldsymbol{x} - \boldsymbol{X}(\theta, t)) d\theta,$$

$$\frac{\partial \boldsymbol{X}}{\partial t}(\theta, t) = \boldsymbol{u}(\boldsymbol{X}(\theta, t), t).$$

δ : Dirac delta function.



- Stokes equation satisfied in a distributional sense.
- Interface condition replaced by distributional body force (surface measure) supported on Γ.

Boundary Integral (BI) Formulation

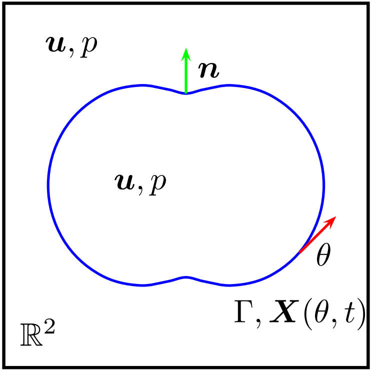
The *boundary integral (BI) formulation* of the Peskin problem:

$$\boldsymbol{u}(\boldsymbol{x},t) = \int_{\mathbb{S}^1} G(\boldsymbol{x} - \boldsymbol{X}(\theta',t)) \frac{\partial^2 \boldsymbol{X}}{\partial \theta^2}(\theta',t)\,d\theta',$$

$$G(\boldsymbol{x}) = \frac{1}{4\pi}\left(-\log|\boldsymbol{x}|\,I + \frac{\boldsymbol{x}\otimes\boldsymbol{x}}{|\boldsymbol{x}|^2}\right)$$

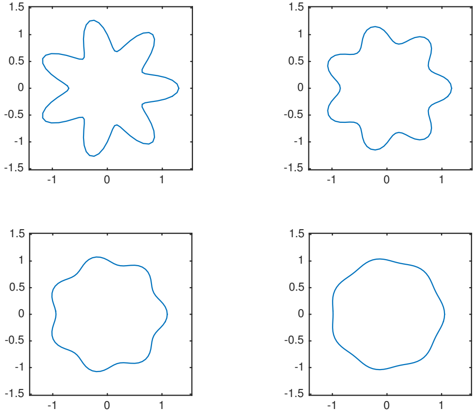
$$= \frac{1}{4\pi}\left(-\log|\boldsymbol{x}|\,I + \frac{1}{|\boldsymbol{x}|^2}\begin{pmatrix} x^2 & xy \\ xy & y^2 \end{pmatrix}\right),$$

$$\frac{\partial \boldsymbol{X}}{\partial t} = \boldsymbol{u}(\boldsymbol{X}(\theta,t),t).$$



- G is the Stokeslet tensor, the fundamental solution of Stokes equation ($\boldsymbol{x} = (x,y)^{\mathrm{T}}$).

Sample Simulation



- Approaches circle as $t \to \infty$.
- Computed using boundary integral method.

Significance and Goals

Siginificance of Peskin problem:

- Applied Analysis
  - Fluid structure interaction (FSI) problems are everywhere.
  - Arguably one of the simplest FSI problems.
- Numerical Analysis
  - Numerical analysis for fully dynamic FSI problems is non-existent. (Many interesting results for the stationary problem and some results for prescribed dynamic problems.)
  - Jump, IB and BI formulations basis for important FSI algorithms:

    Jump: immersed interface, cartesian embedded boundary, moving mesh methods (ALE methods).

    IB: immersed boundary, front-tracking, cut FEM (?), Lagrange multiplier methods (?).

    BI: boundary integral methods.
  - Peskin problem could serve as model numerical analysis problem for various FSI algorithms.

Goals:

- Well-posedness, regularity: Are all formulations equivalent? Equivalent if solution sufficiently smooth.
- Stability of equilibria, global behavior.

Related Problems/Previous Work

Related problems:

- Surface tension problem: Solonnikov, Dennisova, Tanaka, Shibata, Shimizu, Giga, Takahashi, Khöne, Prüss, Wilke, Escher, Günther, Prokert,. . .. Both Stokes/Navier Stokes fluids.
- Muskat/Hele Shaw problem: D'arcy flow, gravity and/or surface tension force at boundary. If no surface tension, the primary linearization is similar to Peskin problem considered here (Dirichlet-to-Neumann map): Ambrose, Cheng, Constantin, Cordoba, Escher, Gancedo, Shkoller, Siegel, Strain, . . .
- Water wave problem.

Fanghua Lin and Jiajun Tong (2017):

- Main results: local solution theory in $C([0,T];H^{5/2}(\mathbb{S}^1))$, local asymptotic (exponential) stability of circular equilibria.
- No regularity results; in particular, solution not classical. No results on global behavior.

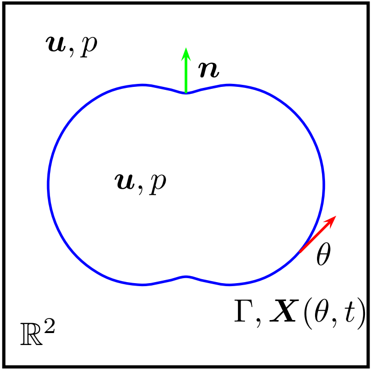
Reduction to Equation for X only

BI formulation of the Peskin problem:

$$\boldsymbol{u}(\boldsymbol{x},t) = \int_{\mathbb{S}^1} G(\boldsymbol{x} - \boldsymbol{X}(\theta',t)) \frac{\partial^2 \boldsymbol{X}}{\partial \theta^2}(\theta',t) d\theta'$$

$$G(\boldsymbol{x}) = \frac{1}{4\pi}\left(-\log|\boldsymbol{x}|\, I + \frac{\boldsymbol{x}\otimes\boldsymbol{x}}{|\boldsymbol{x}|^2}\right) = \frac{1}{4\pi}\left(-\log|\boldsymbol{x}|\, I + \frac{1}{|\boldsymbol{x}|^2}\begin{pmatrix} x^2 & xy \\ xy & y^2\end{pmatrix}\right),$$

$$\frac{\partial \boldsymbol{X}}{\partial t} = \boldsymbol{u}(\boldsymbol{X}(\theta,t),t).$$



Reduce the above to an equation for the evolution of X only:

$$\frac{\partial \boldsymbol{X}}{\partial t}(\theta,t) = \int_{\mathbb{S}^1} G(\boldsymbol{X}(\theta,t) - \boldsymbol{X}(\theta',t)) \frac{\partial^2 \boldsymbol{X}}{\partial \theta^2}(\theta',t) d\theta'.$$

Small Scale Decomposition I

Consider the BI formulation:

$$\partial_t \boldsymbol{X} = \int_{\mathbb{S}^1} G(\boldsymbol{X} - \boldsymbol{X}')\partial_{\theta'}^2 \boldsymbol{X}' d\theta'.$$

Integrate by parts in θ':

$$\partial_t \boldsymbol{X} = -\text{p.v.} \int_{\mathbb{S}^1} \partial_{\theta'} G(\boldsymbol{X} - \boldsymbol{X}')\partial_{\theta'} \boldsymbol{X}' d\theta',$$

$$-\partial_{\theta'} G(\boldsymbol{X} - \boldsymbol{X}') = -\frac{1}{4\pi}\left(\frac{\Delta \boldsymbol{X} \cdot \partial_{\theta'} \boldsymbol{X}'}{|\Delta \boldsymbol{X}|^2} I + \partial_{\theta'}\left(\frac{\Delta \boldsymbol{X} \otimes \Delta \boldsymbol{X}}{|\Delta \boldsymbol{X}|^2}\right)\right), \; \Delta \boldsymbol{X} = \boldsymbol{X} - \boldsymbol{X}'.$$

When $|\theta - \theta'| \ll 1$, $\Delta \boldsymbol{X} = \boldsymbol{X} - \boldsymbol{X}' \approx \partial_\theta \boldsymbol{X}(\theta - \theta')$, so:

$$\frac{\Delta \boldsymbol{X} \cdot \partial_{\theta'} \boldsymbol{X}'}{|\Delta \boldsymbol{X}|^2} \approx \frac{|\partial_\theta \boldsymbol{X}|^2 (\theta - \theta')}{|\partial_\theta \boldsymbol{X}|^2 (\theta - \theta')^2} = \frac{1}{\theta - \theta'}.$$

Thus, we may guess that:

$$\partial_t \boldsymbol{X} \approx -\frac{1}{4\pi}\text{p.v.} \int_{\mathbb{S}^1} \frac{1}{\theta - \theta'} \partial_\theta' \boldsymbol{X}' d\theta'.$$

Small Scale Decomposition II

Recall the Hilbert transform on circle:

$$(\mathcal{H}w)(\theta) = \frac{1}{2\pi}\text{p.v.}\int_{\mathbb{S}^1} \cot\left(\frac{\theta-\theta'}{2}\right) w(\theta')d\theta'.$$

We may write:

$$\partial_t \boldsymbol{X} = \Lambda \boldsymbol{X} + \mathcal{R}(\boldsymbol{X}),\ \ \Lambda \boldsymbol{X} = -\frac{1}{4}\mathcal{H}(\partial_\theta \boldsymbol{X}),$$

$$\mathcal{R}(\boldsymbol{X}) = -\frac{1}{4\pi}\int_{\mathbb{S}^1}\left(\left(\frac{\Delta\boldsymbol{X}\cdot\partial_{\theta'}\boldsymbol{X}'}{|\Delta\boldsymbol{X}|^2} - \frac{1}{2}\cot\left(\frac{\theta-\theta'}{2}\right)\right) I + \partial_{\theta'}\left(\frac{\Delta\boldsymbol{X}\otimes\Delta\boldsymbol{X}}{|\Delta\boldsymbol{X}|^2}\right)\right)\partial_{\theta'}\boldsymbol{X}'d\theta'.$$

- This is known as the *small scale decomposition* (SSD). Introduced by Hou, Lowengrub, Shelley ('94) for Hele-Shaw, water wave problems.
- In SSD, principal part ($\Lambda \boldsymbol{X}$ in above) treated implicitly to remove numerical stiffness.
- Hou and Shi (08) used SSD for IB method.

Integral Equation (Duhamel Formula)

$$\partial_t \boldsymbol{X} = \Lambda \boldsymbol{X} + \mathcal{R}(\boldsymbol{X}),\ \boldsymbol{X}(\theta, 0) = \boldsymbol{X}_0(\theta).$$

Use the Duhamel formula:

$$\boldsymbol{X}(t) = e^{t\Lambda}\boldsymbol{X}_0 + \int_0^t e^{(t-s)\Lambda}\mathcal{R}(\boldsymbol{X}(s))ds.$$

Strategy: Use fixed point argument to construct solution, viewing $\mathcal{R}$ as lower order perturbation.

- Standard technique for semilinear parabolic equations. c.f. For reaction diffusion equations:

$$\partial_t u = \Delta u + f(u),\ u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}),$$
$$u = e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta} f(u)ds,$$

  where $e^{t\Delta}$ is the heat kernel.
- Analysis depends critically on $\mathcal{R}$ being “lower order".
- We shall work in the Hölder spaces $C^{k,\gamma}(\mathbb{S}^1), k \in \{0\} \cup \mathbb{N}, 0 < \gamma < 1$.

Linear Semigroup Properties

The operator Λ can be written as:

$$\Lambda u = -\frac{1}{4}\mathcal{F}^{-1}\,|k|\,\mathcal{F}u,\ \mathcal{F}:\ \text{Fourier Transform (Series)}$$

Thus, Λ behaves like the square-root of the Laplacian and therefore is like taking one derivative. In fact:

$$e^{t\Lambda}u = \frac{1}{2\pi}\int_{\mathbb{S}^1} P(e^{-t/4}, \theta-\theta')u(\theta')d\theta',\ P(r,\theta) = \frac{1-r^2}{1-2r\cos(\theta)+r^2}.$$

where P is the Poisson kernel. We have:

$$\left\|e^{t\Lambda}u\right\|_{C^\beta} \leq \frac{C}{t^{\beta-\alpha}}\left\|u\right\|_{C^\alpha},\ 0<t\leq 1,\ 0\leq\alpha\leq\beta.$$

where, if $\alpha>0, \alpha\notin\mathbb{N},\ C^\alpha(\mathbb{S}^1) = C^{\lfloor\alpha\rfloor,\alpha-\lfloor\alpha\rfloor}(\mathbb{S}^1)$.
c.f. For the Laplacian:

$$\left\|e^{t\Delta}u\right\|_{C^\beta} \leq \frac{C}{t^{(\beta-\alpha)/2}}\left\|u\right\|_{C^\alpha}.$$

Estimates of $\mathcal{R}$

Recall:

$$\partial_t \boldsymbol{X} = \Lambda \boldsymbol{X} + \mathcal{R}(\boldsymbol{X}),$$

where

$$\mathcal{R}(\boldsymbol{X}) = -\frac{1}{4\pi}\int_{\mathbb{S}^1}\left(\left(\frac{\Delta\boldsymbol{X}\cdot\partial_{\theta'}\boldsymbol{X}'}{|\Delta\boldsymbol{X}|^2} - \frac{1}{2}\cot\left(\frac{\theta-\theta'}{2}\right)\right)I + \partial_{\theta'}\left(\frac{\Delta\boldsymbol{X}\otimes\Delta\boldsymbol{X}}{|\Delta\boldsymbol{X}|^2}\right)\right)\partial_{\theta'}\boldsymbol{X}'d\theta'.$$

Lemma

If $\boldsymbol{X}\in C^{1,\gamma}(\mathbb{S}^1)$, then $\mathcal{R}(\boldsymbol{X})\in C^{2\gamma}(\mathbb{S}^1)$.

- Proved by a careful estimation of difference quotients. Use "zero average" property of kernel.
- $\mathcal{R}$ has the effect of taking $1+\gamma-2\gamma = 1-\gamma$ derivatives. Thus, it is “lower order" than Λ.
- The above results come with appropriate estimates.

Local Existence/Uniqueness I

Duhamel formula:

$$X(t) = e^{t\Lambda}X_0 + \int_0^t e^{(t-s)\Lambda}\mathcal{R}(X(s))ds.$$

Define:

$$|X|_* = \inf_{\theta\neq\theta'} \frac{|X(\theta) - X(\theta')|}{|\theta - \theta'|}.$$

- $|X|_* > 0$ if and only if $|\partial_\theta X| > 0$ and no self-intersections of curve.

Definition (Mild Solution)

Let $T > 0$, $X(t) \in C([0,T]; C^{1,\gamma}(\mathbb{S}^1)), 0 < \gamma < 1$. Then, X is a mild solution if X satisfied the above Duhamel formula and $|X|_* > 0$ for $0 \leq t \leq T$ and $\lim_{t\to 0} X(t) = X_0$ in $C^{1,\gamma}(\mathbb{S}^1)$.

Local Existence/Uniqueness II

Let $h^{1,\gamma}(\mathbb{S}^1)$ (little Hölder space) be the completion of smooth functions in $C^{1,\gamma}(\mathbb{S}^1)$. Note that, for any $\alpha > \gamma$, $C^{1,\alpha}(\mathbb{S}^1) \subset h^{1,\gamma}(\mathbb{S}^1)$.

Theorem (M., Rodenberg, Spirn)

Suppose $\boldsymbol{X}_0 \in h^{1,\gamma}(\mathbb{S}^1)$ and $|\boldsymbol{X}_0|_* > 0$. Then, there is a $T > 0$ such that $\boldsymbol{X}(t)$ is a unique mild solution with initial value $\boldsymbol{X}_0$ up to $t = T$. Mild solution is continuous with respect to initial data in the $C^{1,\gamma}$ topology.

Proof.

- Use linear semigroup estimates with the fact that $\mathcal{R}$ is $1-\gamma$ order.
- Contraction mapping argument. Bounds as well as Lipschitz estimates on $\mathcal{R}$ needed (this is where all the work is).

□

- Local existence result (almost) optimal in that $\mathcal{R}$ only barely lower order with respect to Λ when γ small.

Regularity

Given the parabolic nature of our problem, it is natural to ask whether we have immediate smoothing for positive time.

Theorem (M., Rodenberg, Spirn)

A mild solution is in $C^1([\epsilon, T]; C^n(\mathbb{S}^1))$ *for any* $\epsilon > 0$ *and* $n \in \mathbb{N}$*.*

Proof.

- Need to obtain estimates on $\mathcal{R}$ for higher order Hölder spaces.
- This is obtained by commutator estimates on nonlinear kernels.

□

- Our regularity results immediately show that a classical solution exists and is unique.
- Furthermore, our regularity results establish the equivalence of the jump, IB and BI formulations of the problem.

Further Results

- The only equilibria are circles with uniformly-spaced material points.
- The circular equilibria are asymptotically stable, and is approached by exponential rate of $-1/4$.
- Define the γ-deformation ratio:

$$\varrho_\gamma(\boldsymbol{X}) = \frac{\|\partial_\theta \boldsymbol{X}\|_{C^\gamma}}{|\boldsymbol{X}|_*}.$$

  - Suppose solution ceases to exist at $t_* < \infty$. Then,

$$\lim_{t \to t_*} \varrho_\gamma(\boldsymbol{X}) \to \infty.$$

  - Suppose $\varrho_\gamma(\boldsymbol{X})$ remains bounded for all time. Then, solution is global and converges to a circle.
- Instead of $\boldsymbol{F} = \partial^2 \boldsymbol{X}/\partial\theta^2$, consider the more general elasticity law:

$$\boldsymbol{F}(\theta) = \partial_\theta \left(\mathcal{T}(|\partial_\theta \boldsymbol{X}|) \frac{\partial_\theta \boldsymbol{X}}{|\partial_\theta \boldsymbol{X}|} \right), \; \mathcal{T}(s) > 0, \; \frac{d\mathcal{T}}{ds} > 0.$$

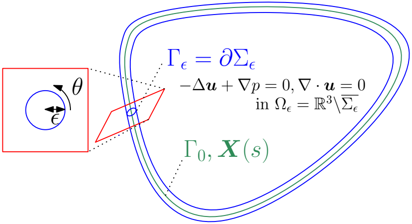
We can prove similar local-in-time well-posedness/regularity results.

Outline

Filament in 3D

Consider a (closed) filament $\Sigma_\epsilon \subset \mathbb{R}^3$. Center line Γ_0 given by $\boldsymbol{X}(s), 0 \leq s < 1$ (length normalized to 1) and of radius ϵ:

$$\Sigma_\epsilon = \{\boldsymbol{x} \in \mathbb{R}^3 | \text{dist}(\boldsymbol{x}, \Gamma_0) < \epsilon\}.$$



A Stokes fluid fills $\Omega_\epsilon = \mathbb{R}^3 \backslash \overline{\Sigma_\epsilon}$ (viscosity normalized to 1):

$$-\Delta \boldsymbol{u} + \nabla p = 0, \ \nabla \cdot \boldsymbol{u} = 0 \text{ for } \Omega_\epsilon.$$

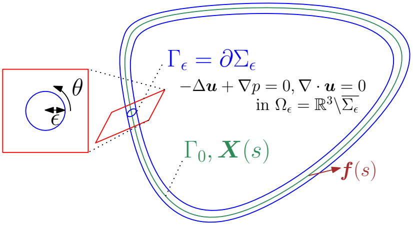
We want to understand the dynamics of this filament.

- Standard method: boundary integrals over the 2D surface $\Gamma_\epsilon = \partial\Sigma_\epsilon$. Too computationally expensive (especially if there are many filaments).
- We thus seek a 1D reduction.
- The real problem is dynamic (filament moves with time). Here we only consider stationary problem.

Slender Body Approximation: First Try

Suppose we are given a force density $\boldsymbol{f}(s), 0 \leq s < 1$ along the center line. A candidate velocity field $\widetilde{\boldsymbol{u}}$ is:

$$-\Delta \widetilde{\boldsymbol{u}} + \nabla \widetilde{p} = \int_0^L \boldsymbol{f}(s)\delta(\boldsymbol{x} - \boldsymbol{X}(s))ds,$$
$$\nabla \cdot \widetilde{\boldsymbol{u}} = 0.$$



Thus

$$\widetilde{\boldsymbol{u}}(x) = \int_0^1 \mathcal{S}(\boldsymbol{x} - \boldsymbol{X}(s))\boldsymbol{f}(s)ds, \; \mathcal{S}(\boldsymbol{x}) = \frac{1}{8\pi}\left(\frac{1}{|\boldsymbol{x}|}I + \frac{\boldsymbol{x}\boldsymbol{x}^{\mathrm{T}}}{|\boldsymbol{x}|^2}\right)$$

This, however, is problematic. There is a strong θ-dependence on the velocity field on Γ_ϵ.

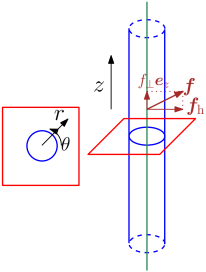
- If the non-slip boundary condition is to be satisfied, a strong θ dependence implies that the filament cross-section will deform very quickly, violating fiber integrity.

Slender Body Approximation on Straight Line

Suppose we have a straight filament of infinite extent along z axis, $\boldsymbol{f} = \text{const}$. Let

$$\boldsymbol{f} = f_z \boldsymbol{e}_z + \boldsymbol{f}_{\text{h}}, \ \widetilde{\boldsymbol{u}} = \widetilde{u}_z \boldsymbol{e}_z + \widetilde{\boldsymbol{u}}_{\text{h}}.$$

Let (r, θ, z) be the cylindrical coordinate system. Then, $\widetilde{u}_z = \widetilde{u}_z(r)$ and:



$$\widetilde{\boldsymbol{u}}_{\text{h}}(r, \theta) = \frac{1}{4\pi} \left(-\log|r| \boldsymbol{f}_{\text{h}} + \frac{1}{2} \begin{pmatrix} 1 + \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & 1 - \cos(2\theta) \end{pmatrix} \boldsymbol{f}_{\text{h}} \right)$$

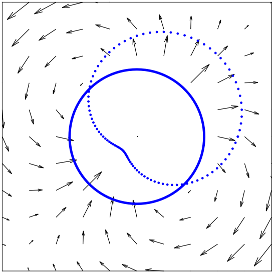
Note that there is a strong θ at $r = \epsilon$, the cylinder surface Γ_ϵ. To fix this, set:

$$\boldsymbol{u}_{\text{h}}^{\text{SB}} = \widetilde{\boldsymbol{u}}_{\text{h}} + \frac{\epsilon^2}{4} \Delta \widetilde{\boldsymbol{u}}_{\text{h}}.$$

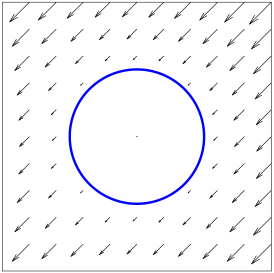
This has no θ dependence. Hence, in this case, a reasonable expression may be:

$$\boldsymbol{u}^{\text{SB}}(\boldsymbol{x}) = \int_{-\infty}^{\infty} \left(\mathcal{S} + \frac{\epsilon^2}{4} \Delta \mathcal{S} \right) (\boldsymbol{x} - s\boldsymbol{e}_z) \boldsymbol{f}(s) ds.$$

Slender Body Approximation on Straight Line



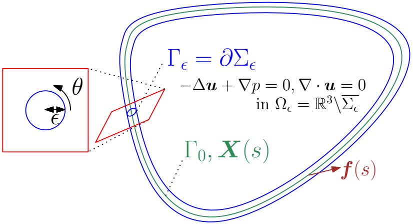
Velocity field $\widetilde{\boldsymbol{u}}_{\mathrm{h}}$ for straight line. Note θ dependence along circle Γ_ϵ $(r = \epsilon)$.



Velocity field $\boldsymbol{u}_{\mathrm{h}}^{\mathrm{SB}}$ for straight line. Note θ dependence on Γ_ϵ is absent.

Slender Body Approximation

In general, given a (closed) filament $X(s)$ with radius ϵ, the *Slender Body Approximation* is, for $x \in \Omega_\epsilon = \mathbb{R}^3 \backslash \overline{\Sigma_\epsilon}$:

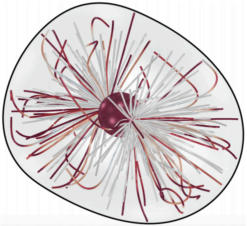


$$u^{\mathrm{SB}}(x) = \int_0^1 \left(\mathcal{S} + \frac{\epsilon^2}{2}\mathcal{D}\right)(x - X(s)) f(s) ds,$$

$$\mathcal{S} = \frac{1}{8\pi}\left(\frac{1}{|x|}I + \frac{xx^{\mathrm{T}}}{|x|^2}\right), \ \mathcal{D} = \frac{1}{2}\Delta\mathcal{S} = \frac{1}{8\pi}\left(\frac{1}{|x|^3}I - \frac{3xx^{\mathrm{T}}}{|x|^5}\right),$$

For $X(s)$ non-straight and $f(s)$ non-constant, u only approximately constant in θ on s cross-sections.

- Proposed in the 70's-80's by Lighthill, Keller, Rubinow, Johnson.
- Widely used in computation of filament dynamics: Shelley, Tornberg, Lauga, Fauci, Cortez, Zorin ...
- What is this an approximation to?



Nazockdast et.al. (2016)

Slender Body Problem I

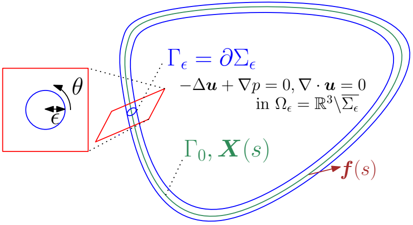
We define the *Slender Body Problem* to be:

$$-\Delta \boldsymbol{u} + \nabla p = 0,\ \nabla \cdot \boldsymbol{u} = 0 \text{ in } \Omega_\epsilon,$$

On Γ_ϵ:

$$\boldsymbol{u}(s,\theta) = \boldsymbol{u}(s),$$

$$-\int_0^{2\pi} \sigma \boldsymbol{n} \epsilon J_\epsilon(s,\theta) d\theta = \boldsymbol{f}(s).$$



where $\boldsymbol{n}$ is the outward unit normal on $\Gamma_\epsilon = \partial\Sigma_\epsilon$ and

$$\sigma \boldsymbol{n} = (\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^{\mathrm{T}}) - pI,\ J_\epsilon = 1 - \epsilon\kappa(s)\cos(\theta),\ \kappa : \text{curvature}.$$

- For every fixed s cross-section, $\boldsymbol{u}$ on Γ_ϵ is constant in θ. This is the *fiber integrity condition* (this condition of *Dirichlet* type).
- Total stress exerted on each cross section must be equal to the line force density $\boldsymbol{f}(s)$ (this condition is of *Neumann* type).
- $\boldsymbol{f}(s)$ (and center-line coordinates $\boldsymbol{X}(s)$) is the only given data.

Slender Body Problem II

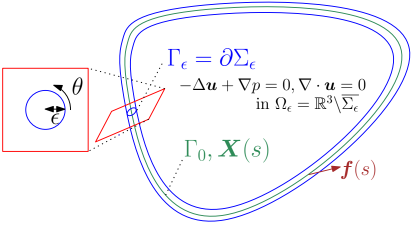
We define the *Slender Body Problem* to be:

$$-\Delta \boldsymbol{u} + \nabla p = 0,\ \nabla \cdot \boldsymbol{u} = 0 \text{ in } \mathbb{R}^3 \backslash \overline{\Sigma_\epsilon},$$

On $\Gamma_\epsilon = \partial\Sigma_\epsilon$:

$$\boldsymbol{u}(s,\theta) = \boldsymbol{u}(s),$$

$$-\int_0^{2\pi} \sigma \boldsymbol{n} \epsilon J_\epsilon(s,\theta)\, d\theta = \boldsymbol{f}(s).$$



where $\boldsymbol{n}$ is the outward unit normal on $\Gamma_\epsilon = \partial\Sigma_\epsilon$ and

$$\sigma = (\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^{\mathrm{T}}) - pI,\ J_\epsilon = 1 - \epsilon\kappa(s)\cos(\theta),\ \kappa : \text{curvature.}$$

- Does the Slender Body Problem have a solution?
- Does the Slender Body Approximation $\boldsymbol{u}^{\mathrm{SB}}$ provide a good approximation to the Slender Body Problem?

Weak Formulation I

Take a divergence-free test function $\boldsymbol{v}$ *that is constant along s cross-sections*, multiply to Stokes equation and integrate by parts:

$$\begin{aligned}
&\int_{\Omega_\epsilon} -(\nabla\cdot\sigma)\cdot\boldsymbol{v}d\boldsymbol{x} = \int_{\Gamma_\epsilon}(\sigma\boldsymbol{n})\cdot\boldsymbol{v}d\mu_{\Gamma_\epsilon} + \int_{\Omega_\epsilon}\sigma:\nabla\boldsymbol{v}d\boldsymbol{x}\\
&= \int_0^1\int_0^{2\pi}(\sigma\boldsymbol{n}\cdot\boldsymbol{v})\epsilon J_\epsilon d\theta ds + \int_{\Omega_\epsilon} 2\nabla_S\boldsymbol{u}:\nabla_S\boldsymbol{v}d\boldsymbol{x}\\
&= \int_0^1\left(\int_0^{2\pi}\sigma\boldsymbol{n}\epsilon J_\epsilon d\theta\right)\cdot\boldsymbol{v}(s)ds + \int_{\Omega_\epsilon} 2\nabla_S\boldsymbol{u}:\nabla_S\boldsymbol{v}d\boldsymbol{x}\\
&= -\int_0^1\boldsymbol{f}(s)\cdot\boldsymbol{v}(s)ds + \int_{\Omega_\epsilon} 2\nabla_S\boldsymbol{u}:\nabla_S\boldsymbol{v}d\boldsymbol{x},\quad \text{where } \nabla_S\boldsymbol{u} = \frac{1}{2}\left(\nabla\boldsymbol{u}+(\nabla\boldsymbol{u})^{\mathrm{T}}\right).
\end{aligned}$$

Note that, if $\boldsymbol{v}=\boldsymbol{u}$, we have:

$$\int_{\Omega_\epsilon} 2\left|\nabla_S\boldsymbol{u}\right|^2 d\boldsymbol{x} = \int_0^1 \boldsymbol{f}\cdot\boldsymbol{u}ds.$$

- This has a natural physical interpretation: power equals energy dissipation per unit time.

Weak Formulation II

Let:

$$\begin{aligned}
\dot{H}^1(\Omega_\epsilon) &= \{\boldsymbol{u} \in L^6(\Omega_\epsilon) \,|\, \|\nabla \boldsymbol{u}\|_{L^2(\Omega_\epsilon)} < \infty\}, \\
\mathcal{A}_\epsilon &= \{\boldsymbol{u} \in \dot{H}^1(\Omega_\epsilon) | \boldsymbol{u}(s,\theta) = \boldsymbol{u}(s) \text{ on } \Gamma_\epsilon\}, \\
\mathcal{A}_\epsilon^{\text{div}} &= \{\boldsymbol{u} \in \mathcal{A}_\epsilon | \nabla \cdot \boldsymbol{u} = 0\}.
\end{aligned}$$

- Fiber integrity condition (Dirichlet-like) is encoded in definition of function space (essential b.c.).

A velocity field $\boldsymbol{u} \in \mathcal{A}_\epsilon^{\text{div}}$ is a *weak solution* to the Slender Body Problem if

$$\int_{\Omega_\epsilon} 2\nabla_S \boldsymbol{u} : \nabla_S \boldsymbol{v} d\boldsymbol{x} = \int_0^1 \boldsymbol{f}(s)\boldsymbol{v}(s) ds, \text{ for all } \boldsymbol{v} \in \mathcal{A}_\epsilon^{\text{div}}.$$

Equivalently (requires proof), $\boldsymbol{u} \in \mathcal{A}_\epsilon^{\text{div}}, p \in L^2(\Omega_\epsilon)$ is a weak solution if,

$$\int_{\Omega_\epsilon} (2\nabla_S \boldsymbol{u} : \nabla_S \boldsymbol{v} - p\nabla \cdot \boldsymbol{v})\, d\boldsymbol{x} = \int_0^1 \boldsymbol{f}(s)\boldsymbol{v}(s) ds, \text{ for all } \boldsymbol{v} \in \mathcal{A}_\epsilon.$$

Existence/Uniqueness

Theorem (M., Ohm., Spirn)

Let $\boldsymbol{X}$ be a C^2 curve. Given $\boldsymbol{f} \in L^2(\mathbb{T}^1)$, $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$, there exists a unique weak solution $(\boldsymbol{u}, p) \in \mathcal{A}_\epsilon^{\text{div}} \times L^2(\Omega_\epsilon)$ with (C does not depend on ϵ):

$$\|\nabla \boldsymbol{u}\|_{L^2(\Omega_\epsilon)} + \|p\|_{L^2(\Omega)} \leq C \left|\log \epsilon\right|^{1/2} \|\boldsymbol{f}\|_{L^2(\mathbb{T}^1)}.$$

Proof.

$$\mathcal{B}[\boldsymbol{u}, \boldsymbol{v}] \equiv \int_{\Omega_\epsilon} 2\nabla_{\mathrm{S}}\boldsymbol{u} : \nabla_{\mathrm{S}}\boldsymbol{v} d\boldsymbol{x} = \int_0^1 \boldsymbol{f}(s)\boldsymbol{v}(s) ds \equiv \mathcal{F}[\boldsymbol{v}], \ \boldsymbol{u}, \boldsymbol{v} \in \mathcal{A}_\epsilon^{\text{div}}$$

- Coercivity of $\mathcal{B}$ on $\mathcal{A}_\epsilon \times \mathcal{A}_\epsilon$ follows from the Korn inequality:

$$\|\nabla \boldsymbol{v}\|_{L^2(\Omega_\epsilon)} \leq C_{\mathrm{K}} \|\nabla_{\mathrm{S}} \boldsymbol{v}\|_{L^2(\Omega_\epsilon)}.$$

- Continuity of $\mathcal{F}$ in A_ϵ follows from trace inequality:

$$\|\boldsymbol{v}\|_{L^2(\mathbb{T}^1)} \leq C_{\mathrm{T}} \|\nabla \boldsymbol{v}\|_{L^2(\Omega_\epsilon)}.$$

- ϵ dependence requires further work. For p, use inequality on right inverse of divergence operator.

□

PDE satisfied by Error

Recall that $\boldsymbol{u}^{\mathrm{SB}}$ was the Slender Body Approximation. We seek to estimate the error $\boldsymbol{u}^{\mathrm{e}} = \boldsymbol{u} - \boldsymbol{u}^{\mathrm{SB}}, p^{\mathrm{e}} = p - p^{\mathrm{SB}}$. We have:

$$
\begin{aligned}
&-\Delta \boldsymbol{u}^{\mathrm{e}} + \nabla p^{\mathrm{e}} = 0, \ \nabla \cdot \boldsymbol{u}^{\mathrm{e}} = 0 \text{ in } \Omega_\epsilon, \\
&\boldsymbol{u}^{\mathrm{e}} = -\boldsymbol{u}^{\mathrm{res}}(s,\theta) + \widetilde{\boldsymbol{u}}(s) \text{ on } \Gamma_\epsilon \text{ for some } \widetilde{\boldsymbol{u}}(s), \\
&-\int_0^{2\pi} (\sigma^{\mathrm{e}} \boldsymbol{n}) \epsilon J_\epsilon d\theta = \boldsymbol{f}^{\mathrm{res}}(s) \text{ on } \Gamma_\epsilon,
\end{aligned}
$$

where

$$
\begin{aligned}
\sigma^{\mathrm{e}} &= \sigma - \sigma^{\mathrm{SB}}, \ \sigma^{\mathrm{SB}} = 2\nabla_S \boldsymbol{u}^{\mathrm{SB}} - p^{SB} I, \\
\boldsymbol{u}^{\mathrm{res}}(s,\theta) &= \boldsymbol{u}^{\mathrm{SB}} - \frac{1}{2\pi}\int_0^{2\pi} \boldsymbol{u}^{\mathrm{SB}}(s,\theta)d\theta, \\
\boldsymbol{f}^{\mathrm{res}}(s) &= \boldsymbol{f} + \int_0^{2\pi} (\sigma^{\mathrm{SB}} \boldsymbol{n}) \epsilon J_\epsilon d\theta.
\end{aligned}
$$

- $\boldsymbol{u}^{\mathrm{res}}(s,\theta)$ is the “non-conforming" residual; $\boldsymbol{u}^{\mathrm{SB}} \notin \mathcal{A}_\epsilon^{\mathrm{div}}$.
- $\boldsymbol{f}^{\mathrm{res}}(s)$ is the “conforming residual".

Estimation of Residual

Lemma (M., Ohm, Spirn)

Suppose $\boldsymbol{f}$ is C^1, $\boldsymbol{X}$ is in $C^{2,\alpha}, 0 < \alpha < 1$. Then,

$$\|\boldsymbol{f}^{\mathrm{res}}\|_{L^\infty} \leq C\epsilon \|\boldsymbol{f}\|_{C^1(\mathbb{T}^1)}, \quad \|\boldsymbol{u}^{\mathrm{res}}\|_{L^\infty} \leq C\epsilon |\log \epsilon| \|\boldsymbol{f}\|_{C^1(\mathbb{T}^1)}$$

$$\left\| \frac{1}{\epsilon} \frac{\partial \boldsymbol{u}^{\mathrm{res}}}{\partial \theta} \right\|_{L^\infty} + \left\| \frac{\partial \boldsymbol{u}^{\mathrm{res}}}{\partial s} \right\|_{L^\infty} \leq C |\log \epsilon| \|\boldsymbol{f}\|_{C^1(\mathbb{T}^1)}$$

where C does not depend on ϵ.

Proof.

- When $\boldsymbol{X}(s)$ is a straight infinite line and $\boldsymbol{f}$ is constant, $\boldsymbol{f}^{\mathrm{res}} = \boldsymbol{u}^{\mathrm{res}} = 0$.
- $C^{2,\alpha}$ curve with C^1 force can be locally approximated by straight line/constant force as $\epsilon \to 0$.
- Estimate nearly singular integrals using above observation. Need to consider “far field" and "near field" residual contributions separately.

□

Error Estimate

Theorem (M., Ohm, Spirn)

Given $\boldsymbol{f}$ in C^1 and $\boldsymbol{X}$ in $C^{2,\alpha}, 0 < \alpha < 1$, the difference between $(\boldsymbol{u}, p)$ and its Slender Body Approximation $(\boldsymbol{u}^{\mathrm{SB}}, p^{\mathrm{SB}})$ satisfies:

$$\left\| \nabla (\boldsymbol{u} - \boldsymbol{u}^{\mathrm{SB}}) \right\|_{L^2(\Omega_\epsilon)} + \left\| p - p^{\mathrm{SB}} \right\|_{L^2(\Omega_\epsilon)} \leq C\epsilon \left| \log \epsilon \right| \left\| \boldsymbol{f} \right\|_{C^1(\mathbb{T}^1)} .$$

where the constant C does not depend on ϵ.

Proof.

- Proof essentially follows a Lax Equivalence principle type argument.
- *Consistency*: Residual estimated as in previous slide.
- *Stability with respect to $\epsilon \to 0$*: Consider the Korn and trace inequalities:

$$\left\| \nabla \boldsymbol{v} \right\|_{L^2(\Omega_\epsilon)} \leq C_{\mathrm{K}} \left\| \nabla_{\mathrm{S}} \boldsymbol{v} \right\|_{L^2(\Omega_\epsilon)} , \quad \left\| \boldsymbol{v} \right\|_{L^2(\mathbb{T}^1)} \leq C_{\mathrm{T}} \left\| \nabla \boldsymbol{v} \right\|_{L^2(\Omega_\epsilon)} .$$

We must study ϵ dependence of C_{K} and C_{T}. We can show C_{K} independent of ϵ, $C_{\mathrm{T}} = \mathcal{O}(\left|\log \epsilon\right|^{1/2})$. Similar independence of ϵ for the operator norm of the right inverse of divergence operator.

□

Future Directions/Acknowledgments/Funding

Future Directions:

- Peskin Problem:
  - Global well-posedness/singularity formation.
  - Variants of the Peskin problem: different viscosity, incompressible elasticity, 3D, etc.
  - Numerical analysis of IB and/or BI methods.
- Slender Body Theory:
  - Computational verification of optimality of error estimates.
  - Variants: open filaments, inextensible filaments, twisting filaments, etc.
  - Dynamic problems.

References:

- Y. Mori, A. Rodenberg and D. Spirn, *Well-posedness and global behavior of the Peskin problem of an immersed elastic filament in Stokes flow*, Communications on Pure and Applied Mathematics, to appear.
- Y. Mori, L. Ohm and D. Spirn, *Theoretical justification and error analysis for slender body theory*, submitted.

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