Exact Smooth Piecewise Polynomial Sequences on Alfeld Splits

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Numerical Analysis of Coupled and Multi-Physics Problems with Dynamic Interfaces

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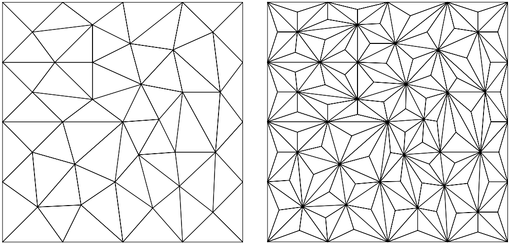
Outline

Alfeld Splits

- Certain triangulations have advantages in computational PDEs (structure-preserving, low-order,...)
- An Alfeld split/refinement of a simplicial triangulation is obtained by connecting the barycenter of each n-simplex with its vertices. This is also known as a barycenter refinement.

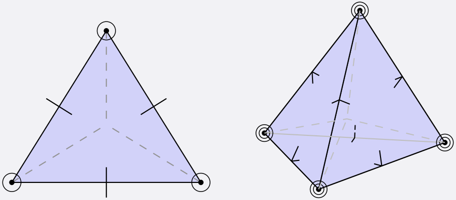


Simplicial triangulation (left) and resulting Alfeld split (right).

Alfeld Splits

- Alfeld splits of a simplicial triangulation are useful in several areas of computational mathematics. For example, it is possible to construct

H^2-conforming elements (Clough-Tocher ('65), Alfeld ('84))



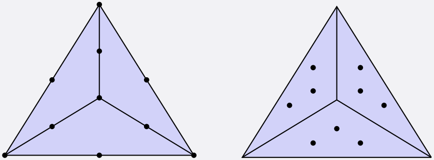
Cubic Clough-Tocher element (left) and quintic Alfeld element (right).

- Applications: Fourth-order problems (biharmonic, Cahn-Hilliard)
- Advantages: Relatively low-order

Alfeld Splits

- Alfeld splits of a simplicial triangulation are useful in several areas of computational mathematics. For example, it is possible to construct

Inf-Sup stable pairs (Arnold-Qin ('92), Zhang ('04))



Quadratic Lagrange velocity element (left) and discontinuous linear pressure space (right) form a stable finite element pair on Alfeld splits.

- Applications: Incompressible flow (Stokes/NSE)
- Advantages: Strongly imposes the divergence–free constraint, enhanced stability properties.

Alfeld Splits

Goal

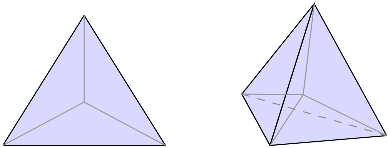
- We show that these spaces are connected via an exact sequences of finite element spaces.
- The sequences are a de Rham complexes, but where the finite element spaces have extra smoothness compared to the canonical Whitney-Nédélec spaces.
- We prove results in arbitrary dimension $n \geq 2$, and adopt a finite element exterior calculus (FEEC) framework.
- E.g., we identity H^2 functions as 0-forms, and the velocity and pressure functions as $(n-1)$-forms and n-forms, respectively.

Previous Work

Christiansen and Hu ('18) have recently studied discrete smooth de Rham (Stokes) complexes in any dimension. Their triangulations have different splits, and they consider the lowest (polynomial) degree case.

Notation

- Let $T = [x_0, x_1, \dots, x_n]$ be an n-dimensional simplex.
- Let $z = \frac{1}{n+1}\sum_{i=0}^{n} x_i$ be the barycenter of T.
- The Alfeld split of T is obtained by subdividing T into $(n+1)$ n-simplices by adjoining the vertices of T with z.
- Set $T_i = [z, x_0, \dots, \hat{x}_i, \dots, x_n]$.
- Set $T^{\mathrm{z}} = \{T_0, \dots, T_n\}$ to be the mesh of the sub-division.



The local mesh in two (left) and three (right) dimensions.

Notation

- Let $\{\lambda_i\}_{i=0}^n \subset \mathcal{P}_1(T)$ be the barycentric coordinates satisfying $\lambda_i(x_j) = \delta_{i,j}$.
- The differential $d\lambda_i : \mathbb{R}^n \to \mathbb{R}$ is given by $d\lambda_i(r) = \operatorname{grad} \lambda_i \cdot r$.
- For integer $k \in [1, n]$, and $0 \leq \sigma(1) < \sigma(2) < \cdots < \sigma(k) \leq n$, we use the notation
$$d\lambda_\sigma := d\lambda_{\sigma(1)} \wedge d\lambda_{\sigma(2)} \wedge \cdots \wedge d\lambda_{\sigma(k)}.$$
- Define the spaces of differential forms with polynomial coefficients
$$\mathcal{P}_r\Lambda^k(T) = \Big\{ \sum_{\sigma \in \Sigma(k,n)} a_\sigma d\lambda_\sigma : \ a_\sigma \in \mathcal{P}_r(T) \Big\},$$
where $\Sigma(k, n)$ is the set of increasing maps $\{1, 2, \ldots, n\} \to \{1, 2, \ldots, k\}$.
- The analogous piecewise-defined space on the Alfeld split is
$$\mathcal{P}_r\Lambda^k(T^{\mathrm{z}}) = \prod_{i=0}^{n} \mathcal{P}_r\Lambda^k(T^i).$$

Canonical Finite Element Spaces

One family of canonical (local) finite elements spaces defined on the split is defined as

$$V_r^k(T^{\mathrm{z}}) = \mathcal{P}_r\Lambda^k(T^{\mathrm{z}}) \cap H\Lambda^k(T) = \{\omega \in \mathcal{P}_r\Lambda^k(T^{\mathrm{z}}) :\ d\omega \in L^2\Lambda^{k+1}(T)\},$$

$$\mathring{V}_r^k(T^{\mathrm{z}}) = \mathcal{P}_r\Lambda(T^{\mathrm{z}}) \cap \mathring{H}\Lambda^k(T) = \{\omega \in V_r^k(T^{\mathrm{z}}) :\ \mathrm{tr}_{\partial T}\omega = 0\}\ (0 \le k \le n-1),$$

$$\mathring{V}_r^n(T^{\mathrm{z}}) = \{\omega \in V_r^n(T^{\mathrm{z}}) :\ \int_T \omega = 0\}.$$

Remark (Proxies)

In three dimensions, we may identity forms in $V_r^k(T^{\mathrm{z}})$ by scalar/vector proxies.

$$k=0:\quad d \simeq \mathrm{grad}\,,\quad V_r^0(T^{\mathrm{z}}) \ \simeq\ H^1 \textit{ Lagrange finite element space},$$

$$k=1:\quad d \simeq \mathrm{curl}\,,\quad V_r^1(T^{\mathrm{z}}) \ \simeq\ H(\mathrm{curl}) \textit{ Nedelec finite element space},$$

$$k=2:\quad d \simeq \mathrm{div}\,,\quad V_r^2(T^{\mathrm{z}}) \ \simeq\ H(\mathrm{div}) \textit{ Nedelec finite element space},$$

$$k=3:\quad V_r^3(T^{\mathrm{z}}) \ \simeq\ \textit{Discontinuous finite element space}.$$

A Well-Known Result (Arnold, Falk, Winther)

An Exact Sequence

$$0 \longrightarrow \mathring{V}_r^0(T^z) \xrightarrow{d} \mathring{V}_{r-1}^1(T^z) \xrightarrow{d} \cdots \xrightarrow{d} \mathring{V}_{r-n+1}^{n-1}(T^z) \xrightarrow{d} \mathring{V}_{r-n}^n(T^z) \longrightarrow 0.$$

- Exactness means:

$$\ker \mathring{V}_r^k(T^z) = \operatorname{range} \mathring{V}_{r+1}^{k-1}(T^z),$$

i.e., if $\omega \in \mathring{V}_r^k(T^z)$ with $d\omega = 0$, then $\omega = d\rho$ for some $\rho \in \mathring{V}_{r+1}^{k-1}(T^z)$.

Remark (Translation)

Let $n = 3$. Then the exactness of the sequence implies (with an abuse of notation)

- *If* $\omega \in \mathring{V}_r^0(T^z)$ *with* $\operatorname{grad} \omega = 0$, *then* $\omega = 0$.
- *If* $\omega \in \mathring{V}_{r-1}^1(T^z)$ *with* $\operatorname{curl} \omega = 0$, *then* $\omega = \operatorname{grad} \rho$ *for some* $\rho \in \mathring{V}_r^0(T^z)$.
- *If* $\omega \in \mathring{V}_{r-2}^2(T^z)$ *with* $\operatorname{div} \omega = 0$, *then* $\omega = \operatorname{curl} \rho$ *for some* $\rho \in \mathring{V}_{r-1}^1(T^z)$.
- *If* $\omega \in \mathring{V}_{r-3}^3(T^z)$, *then* $\omega = \operatorname{div} \rho$ *for some* $\rho \in \mathring{V}_{r-2}^2(T^z)$.

Lagrange Finite Element Spaces

We define the (Lagrange) finite element spaces

$$L_r^k(T^{\mathrm{z}}) = \mathcal{P}_r\Lambda^k(T^{\mathrm{z}}) \cap C^0\Lambda^k(T),$$
$$\mathring{L}_r^k(T^{\mathrm{z}}) = \{\omega \in L_r^k(T^{\mathrm{z}}) :\ \omega|_{\partial T} = 0\} \cap \mathring{V}_r^k(T^{\mathrm{z}}).$$

Remark (Proxies)

In three dimensions, we may identity forms in $L_r^k(T^{\mathrm{z}})$ by scalar/vector proxies.

$$k = 0 :\ L_r^0(T^{\mathrm{z}}) \simeq H^1 \text{ Lagrange finite element space},$$
$$k = 1 :\ L_r^1(T^{\mathrm{z}}) \simeq \text{vector-valued } H^1 \text{ Lagrange finite element space},$$
$$k = 2 :\ L_r^2(T^{\mathrm{z}}) \simeq \text{vector-valued } H^1 \text{ Lagrange finite element space},$$
$$k = 3 :\ L_r^3(T^{\mathrm{z}}) \simeq H^1 \text{ Lagrange finite element space}.$$

Smooth Finite Element Spaces

We define the smooth finite element spaces

$$S_r^k(T^{\mathrm{z}}) = \{\omega \in L_r^k(T^{\mathrm{z}}) : \ d\omega \in C^0\Lambda^{k+1}(T)\},$$

$$\mathring{S}_r^k(T^{\mathrm{z}}) = \{\omega \in S_r^k(T^{\mathrm{z}}) : \ \omega|_{\partial T} = 0, \ d\omega|_{\partial T} = 0\} \cap \mathring{V}_r^k(T^{\mathrm{z}}).$$

Remark (Proxies)

In three dimensions, we may identity forms in $S_r^k(T^{\mathrm{z}})$ by scalar/vector proxies.

$$k = 0: \ S_r^0(T^{\mathrm{z}}) \ \simeq \ H^2 \textit{ finite element space},$$

$$k = 1: \ S_r^1(T^{\mathrm{z}}) \ \simeq \ H^1(\mathrm{curl}) \textit{ finite element space},$$

$$k = 2: \ S_r^2(T^{\mathrm{z}}) \ \simeq \ H^1(\mathrm{div}) \textit{ finite element space},$$

$$k = 3: \ S_r^3(T^{\mathrm{z}}) \ \simeq \ H^1 \textit{ Lagrange finite element space}.$$

Local Results

Theorem

Suppose that $\omega \in \mathring{V}_r^k(T^{\mathrm{z}})$ $(r \geq 0)$ *and* $d\omega = 0$. *Then there exists a* $\rho \in \mathring{L}_{r+1}^{k-1}(T^{\mathrm{z}})$ *such that* $\omega = d\rho$*; that is,*

$$\ker \mathring{V}_r^k(T^{\mathrm{z}}) = \operatorname{range} \mathring{L}_{r+1}^{k-1}(T^{\mathrm{z}}).$$

Theorem (w/o boundary conditions)

Suppose that $\omega \in V_r^k(T^{\mathrm{z}})$ $(r \geq 0)$ *and* $d\omega = 0$. *Then there exists a* $\rho \in L_{r+1}^{k-1}(T^{\mathrm{z}})$ *such that* $\omega = d\rho$*; that is,*

$$\ker V_r^k(T^{\mathrm{z}}) = \operatorname{range} L_{r+1}^{k-1}(T^{\mathrm{z}}).$$

Remark

Recall that $\ker \mathring{V}_r^k(T^{\mathrm{z}}) = \operatorname{range} \mathring{V}_{r+1}^{k-1}(T^{\mathrm{z}})$. *Therefore,*

$$\operatorname{range} \mathring{V}_{r+1}^{k-1}(T^{\mathrm{z}}) = \operatorname{range} \mathring{L}_{r+1}^{k-1}(T^{\mathrm{z}}).$$

Local Results

The next two results are immediate due to the inclusion $L_r^k(T^{\mathrm{z}}) \subset V_r^k(T^{\mathrm{z}})$.

Corollary

Suppose that $\omega \in \mathring{L}_r^k(T^{\mathrm{z}})$ *and* $d\omega = 0$. *Then there exists a* $\rho \in \mathring{S}_{r+1}^{k-1}(T^{\mathrm{z}})$ *such that* $\omega = d\rho$; *that is,*

$$\ker \mathring{L}_r^k(T^{\mathrm{z}}) = \mathrm{range}\, \mathring{S}_{r+1}^{k-1}(T^{\mathrm{z}}).$$

Corollary

Suppose that $\omega \in \mathring{S}_r^k(T^{\mathrm{z}})$ *and* $d\omega = 0$. *Then there exists a* $\rho \in \mathring{S}_{r+1}^{k-1}(T^{\mathrm{z}})$ *such that* $\omega = d\rho$; *that is,*

$$\ker \mathring{S}_r^k(T^{\mathrm{z}}) = \mathrm{range}\, \mathring{S}_{r+1}^{k-1}(T^{\mathrm{z}}).$$

Divergence-free methods for the Stokes/NSE problem

Taking $k = n$ in the theorem yields

Corollary

For any $\omega \in \mathring{V}_r^n(T^z)$ $(r \geq 0)$, *there exists* $\rho \in \mathring{L}_{r+1}^{n-1}(T^z)$ *such that* $\operatorname{div} \rho = \omega$.

Remark

This result states that the divergence operator acting on the Lagrange finite element space is surjective onto the space of discontinuous piecewise polynomials. This result has been obtained by Arnold-Qin ('92) in two dimensions and Zhang ('04) in three dimensions.

Remark

Using Stenberg's Macro argument, it can be shown that analogous global results hold, thus yielding stable and divergence-free yielding methods for Stokes/NSE.

Divergence-free methods for the Stokes/NSE problem

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For any $\omega \in \mathring{V}_r^n(T^z)$ $(r \geq 0)$, *there exists* $\rho \in \mathring{L}_{r+1}^{n-1}(T^z)$ *such that* $\operatorname{div} \rho = \omega$.

Remark

Using this local result, we have developed (global) conforming and divergence-free yielding finite element pairs for the Stokes/NSE problem in any dimension and for any polynomial degree (Guzmán & N. ('18)).

Exact Sequences

The theorems yield several new exact sequences on Alfeld splits. For example:

Corollary

Let $n = 3$. Then the following sequences are exact:

$$0 \longrightarrow \mathring{L}_r^0(T^z) \xrightarrow{\text{grad}} \mathring{V}_{r-1}^1(T^z) \xrightarrow{\text{curl}} \mathring{V}_{r-2}^2(T^z) \xrightarrow{\text{div}} \mathring{V}_{r-3}^3(T^z) \longrightarrow 0,$$

$$0 \longrightarrow \mathring{S}_r^0(T^z) \xrightarrow{\text{grad}} \mathring{L}_{r-1}^1(T^z) \xrightarrow{\text{curl}} \mathring{V}_{r-2}^2(T^z) \xrightarrow{\text{div}} \mathring{V}_{r-3}^3(T^z) \longrightarrow 0,$$

$$0 \longrightarrow \mathring{S}_r^0(T^z) \xrightarrow{\text{grad}} \mathring{S}_{r-1}^1(T^z) \xrightarrow{\text{curl}} \mathring{L}_{r-2}^2(T^z) \xrightarrow{\text{div}} \mathring{V}_{r-3}^3(T^z) \longrightarrow 0,$$

$$0 \longrightarrow \mathring{S}_r^0(T^z) \xrightarrow{\text{grad}} \mathring{S}_{r-1}^1(T^z) \xrightarrow{\text{curl}} \mathring{S}_{r-2}^2(T^z) \xrightarrow{\text{div}} \mathring{L}_{r-3}^3(T^z) \longrightarrow 0.$$

Dimension Counts

Corollary

The dimension of $S_r^k(T^{\mathrm{z}})$ *satisfies*

$$\dim S_r^k(T^{\mathrm{z}}) = \dim L_{r-1}^{k+1}(T^{\mathrm{z}}) + \dim L_r^k(T^{\mathrm{z}}) - \dim V_{r-1}^{k+1}(T^{\mathrm{z}}).$$

Dimension Counts

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Example

Taking $k = 0$ gives the dimension of local C^1 elements on Alfeld splits:

$$\dim S_r^0(T^{\mathrm{z}}) = \binom{r+n}{n} + n\binom{r-1}{n}.$$

This dimension count was first established by Kolenikov & Sorokina ('14) using different techniques.

Example

Taking $n = 3$ and $k = 1$ yields the local dimension of an $H^1(\mathrm{curl}; T)$ space

$$\dim S_r^1(T^{\mathrm{z}}) = r(2r^2 - 3r + 13).$$

Super-smoothness at vertices

Corollary

Let $n = 3$. Then any $\omega \in S_r^k(T^{\mathrm{z}})$ is C^{2-k} on the vertices of T^{z}.

Example

- $k = 0$: If ω is a C^1 piecewise polynomial on T^{z}, then ω is C^2 on the vertices of T^{z}.
- $k = 1$: If ω is a C^0 piecewise polynomial and if $\operatorname{curl} \omega \in C^0(T)$, then ω is C^1 on the vertices of T^{z}.

The Three Dimensional Case

Recall, that in three dimensions ($n = 3$), we have four local exact sequences with varying level of smoothness:

$$0 \longrightarrow \mathring{L}_r^0(T^{\mathrm{z}}) \xrightarrow{\text{grad}} \mathring{V}_{r-1}^1(T^{\mathrm{z}}) \xrightarrow{\text{curl}} \mathring{V}_{r-2}^2(T^{\mathrm{z}}) \xrightarrow{\text{div}} \mathring{V}_{r-3}^3(T^{\mathrm{z}}) \longrightarrow 0,$$

$$0 \longrightarrow \mathring{S}_r^0(T^{\mathrm{z}}) \xrightarrow{\text{grad}} \mathring{L}_{r-1}^1(T^{\mathrm{z}}) \xrightarrow{\text{curl}} \mathring{V}_{r-2}^2(T^{\mathrm{z}}) \xrightarrow{\text{div}} \mathring{V}_{r-3}^3(T^{\mathrm{z}}) \longrightarrow 0,$$

$$0 \longrightarrow \mathring{S}_r^0(T^{\mathrm{z}}) \xrightarrow{\text{grad}} \mathring{S}_{r-1}^1(T^{\mathrm{z}}) \xrightarrow{\text{curl}} \mathring{L}_{r-2}^2(T^{\mathrm{z}}) \xrightarrow{\text{div}} \mathring{V}_{r-3}^3(T^{\mathrm{z}}) \longrightarrow 0,$$

$$0 \longrightarrow \mathring{S}_r^0(T^{\mathrm{z}}) \xrightarrow{\text{grad}} \mathring{S}_{r-1}^1(T^{\mathrm{z}}) \xrightarrow{\text{curl}} \mathring{S}_{r-2}^2(T^{\mathrm{z}}) \xrightarrow{\text{div}} \mathring{L}_{r-3}^3(T^{\mathrm{z}}) \longrightarrow 0.$$

- We develop analogous global finite element spaces and sequences.

The Three Dimensional Case

Recall, that in three dimensions ($n = 3$), we have four local exact sequences with varying level of smoothness:

$$0 \longrightarrow \mathring{L}^0_r(T^z) \xrightarrow{\text{grad}} \mathring{V}^1_{r-1}(T^z) \xrightarrow{\text{curl}} \mathring{V}^2_{r-2}(T^z) \xrightarrow{\text{div}} \mathring{V}^3_{r-3}(T^z) \longrightarrow 0,$$

$$0 \longrightarrow \mathring{S}^0_r(T^z) \xrightarrow{\text{grad}} \mathring{L}^1_{r-1}(T^z) \xrightarrow{\text{curl}} \mathring{V}^2_{r-2}(T^z) \xrightarrow{\text{div}} \mathring{V}^3_{r-3}(T^z) \longrightarrow 0,$$

$$0 \longrightarrow \mathring{S}^0_r(T^z) \xrightarrow{\text{grad}} \mathring{S}^1_{r-1}(T^z) \xrightarrow{\text{curl}} \mathring{L}^2_{r-2}(T^z) \xrightarrow{\text{div}} \mathring{V}^3_{r-3}(T^z) \longrightarrow 0,$$

$$0 \longrightarrow \mathring{S}^0_r(T^z) \xrightarrow{\text{grad}} \mathring{S}^1_{r-1}(T^z) \xrightarrow{\text{curl}} \mathring{S}^2_{r-2}(T^z) \xrightarrow{\text{div}} \mathring{L}^3_{r-3}(T^z) \longrightarrow 0.$$

- We develop analogous global finite element spaces and sequences.
- We focus on the sequence with the highest level of smoothness.

Set up

- Let $\Omega \subset \mathbb{R}^3$ be a polyhedral domain.
- Let $\mathcal{T}_h$ be a regular, simplicial triangulation of Ω.
- Let $\mathcal{T}_h^z$ be the triangulation obtained by performing an Alfeld split to each $T \in \mathcal{T}_h$.
- For $T \in \mathcal{T}_h$, $\Delta_\ell(T)$ is the set of ℓ-dimensional simplices of T (vertices, edges, faces)

DOFs: C^1 element

Lemma

Let $r \geq 5$. Then, a function $\omega \in S_r^0(T^{\mathrm{z}})$ is uniquely determined by the following DOFs:

$$
\begin{aligned}
& D^\alpha \omega(a), && \forall |\alpha| \leq 2, && \forall a \in \Delta_0(T), \\
& \int_e \omega\sigma \, \mathrm{ds}, && \forall \sigma \in \mathcal{P}_{r-6}(e), && \forall e \in \Delta_1(T), \\
& \int_e \frac{\partial \omega}{\partial n_e^{\pm}} \, \sigma \, \mathrm{ds}, && \forall \sigma \in \mathcal{P}_{r-5}(e), && \forall e \in \Delta_1(T), \\
& \int_F \omega \, \sigma \, \mathrm{dA}, && \forall \sigma \in \mathcal{P}_{r-6}(F), && \forall F \in \Delta_2(T) \\
& \int_F \frac{\partial \omega}{\partial n_F} \, \sigma \, \mathrm{dA}, && \forall \sigma \in \mathcal{P}_{r-4}(F), && \forall F \in \Delta_2(T), \\
& \int_T \operatorname{grad} \omega \cdot \operatorname{grad} \sigma \, \mathrm{dx}, && \forall \sigma \in \mathring{S}_r^0(T^{\mathrm{z}}). &&
\end{aligned}
$$

Here, $n_{e_\pm}$ are two orthonormal normal vectors that are orthogonal to the edge e.

DOFs: $H^1(\mathrm{curl})$ element

Lemma

A function $\omega \in S^1_{r-1}(T^z)$ $(r \geq 5)$ *is uniquely determined by the values*

$$
\begin{array}{lll}
D^\alpha \omega(a) & \forall |\alpha| \leq 1, & \forall a \in \Delta_0(T), \\
\int_e \omega \cdot \kappa \, \mathrm{ds} & \forall \kappa \in [\mathcal{P}_{r-5}(e)]^3, & \forall e \in \Delta_1(T), \\
\int_e (\mathrm{curl}\, \omega) \cdot \kappa \, \mathrm{ds} & \forall \kappa \in [\mathcal{P}_{r-4}(e)]^3, & \forall e \in \Delta_1(T), \\
\int_f (\omega \cdot n_F) \kappa \, \mathrm{dA} & \forall \kappa \in \mathcal{P}_{r-4}(F), & \forall F \in \Delta_2(T), \\
\int_F (n_F \times \omega \times n_F) \cdot \kappa \, \mathrm{dA} & \forall \kappa \in D_{r-5}(F), & \forall F \in \Delta_2(T), \\
\int_F (\mathrm{curl}\, \omega \times n_F) \cdot \kappa \, \mathrm{dA} & \forall \kappa \in [\mathcal{P}_{r-5}(F)]^3, & \forall F \in \Delta_2(T), \\
\int_T \omega \cdot \kappa \, \mathrm{dx} & \forall \kappa \in \mathrm{grad}\, \mathring{S}^0_r(T^z), & \\
\int_T \mathrm{curl}\, \omega \cdot \kappa \, \mathrm{dx} & \forall \kappa \in \mathrm{curl}\, \mathring{S}^1_{r-1}(T^z), &
\end{array}
$$

where $D_{r-5}(F) = \mathcal{P}_{r-6}(F) + x_F \mathcal{P}_{r-6}(F)$ *is the local Raviart–Thomas space.*

DOFs: $H^1(\mathrm{div})$ element

Lemma

A function $\omega \in S^2_{r-2}(T^z)$ $(r \geq 5)$ *is uniquely determined by the values*

$$
\begin{aligned}
&\omega(a),\ \operatorname{div}\omega(a) && && \forall a \in \Delta_0(T),\\
&\int_e \omega \cdot \kappa \,\mathrm{ds} && \forall \kappa \in [\mathcal{P}_{r-4}(e)]^3, && \forall e \in \Delta_1(T),\\
&\int_e (\operatorname{div}\omega)\kappa \,\mathrm{ds} && \forall \kappa \in \mathcal{P}_{r-5}(e), && \forall e \in \Delta_1(T),\\
&\int_F \omega \cdot \kappa \,\mathrm{dA} && \forall \kappa \in [\mathcal{P}_{r-5}(F)]^3, && \forall F \in \Delta_2(T),\\
&\int_F (\operatorname{div}\omega)\kappa \,\mathrm{dA} && \forall \kappa \in \mathcal{P}_{r-6}(F), && \forall F \in \Delta_2(T),\\
&\int_T \omega \cdot \kappa \,\mathrm{dx} && \forall \kappa \in \operatorname{curl} \mathring{S}^1_{r-1}(T^z), && \\
&\int_T (\operatorname{div}\omega)\kappa \,\mathrm{dx} && \forall \kappa \in \mathring{S}_{r-3}(T^z). &&
\end{aligned}
$$

DOFs: H^1 element

Lemma

Any $\omega \in S^3_{r-3}(T^{\mathrm{z}}) = L^3_{r-3}(T^{\mathrm{z}})$ $(r \geq 5)$ *is uniquely determined by the degrees of freedom*

$$
\begin{aligned}
&\omega(a) && && \forall a \in \Delta_0(T),\\
&\int_e \omega\kappa\,\mathrm{ds} && \forall \kappa \in \mathcal{P}_{r-5}(e), && \forall e \in \Delta_1(T),\\
&\int_F \omega\kappa\,\mathrm{dA} && \forall \kappa \in \mathcal{P}_{r-6}(F), && \forall F \in \Delta_2(T),\\
&\int_T \omega\,\mathrm{dx}, && && \\
&\int_T \omega\kappa\,\mathrm{dx} && \forall \kappa \in \mathring{S}^3_{r-3}(T^{\mathrm{z}}). &&
\end{aligned}
$$

Global Finite Element Spaces

The degrees of freedom induce the spaces:

$$S_r^0(\mathcal{T}_h^{\mathrm{z}}) := \{\omega \in H^2(\Omega) : \ \omega|_T \in S_r^0(T^{\mathrm{z}}) \ \forall T \in \mathcal{T}_h, \ \omega \text{ is } C^2 \text{ at vertices}\},$$

$$S_{r-1}^1(\mathcal{T}_h^{\mathrm{z}}) := \{\omega \in H^1(\mathrm{curl};\Omega) : \ \omega|_T \in S_{r-1}^1(T^{\mathrm{z}}) \ \forall T \in \mathcal{T}_h, \ \omega \text{ is } C^1 \text{ at vertices}\},$$

$$S_{r-2}^2(\mathcal{T}_h^{\mathrm{z}}) := \{\omega \in H^1(\mathrm{div};\Omega) : \ \omega|_T \in S_{r-2}^2(T^{\mathrm{z}}) \ \forall T \in \mathcal{T}_h\},$$

$$S_{r-3}^3(\mathcal{T}_h^{\mathrm{z}}) := \{\omega \in H^1(\Omega) : \ \omega|_T \in S_{r-3}^3(T^{\mathrm{z}}) \ \forall T \in \mathcal{T}_h\},$$

and projections

$$\Pi_0 : C^\infty(\Omega) \to S_r^0(\mathcal{T}_h^{\mathrm{z}}),$$

$$\Pi_1 : [C^\infty(\Omega)]^3 \to S_{r-1}^1(\mathcal{T}_h^{\mathrm{z}}),$$

$$\Pi_2 : [C^\infty(\Omega)]^3 \to S_{r-2}^2(\mathcal{T}_h^{\mathrm{z}}),$$

$$\Pi_3 : C^\infty(\Omega) \to S_{r-3}^3(\mathcal{T}_h^{\mathrm{z}}).$$

Commuting Properties

Theorem

For $r \geq 5$, the following diagram commutes

$$
\begin{array}{ccccccccc}
C^\infty(T) & \xrightarrow{\text{grad}} & [C^\infty(T)]^3 & \xrightarrow{\text{curl}} & [C^\infty(T)]^3 & \xrightarrow{\text{div}} & C^\infty(T) & \longrightarrow & 0 \\
\downarrow \Pi_0 & & \downarrow \Pi_1 & & \downarrow \Pi_2 & & \downarrow \Pi_3 & & \\
S_r^0(\mathcal{T}_h^z) & \xrightarrow{\text{grad}} & S_{r-1}^1(\mathcal{T}_h^z) & \xrightarrow{\text{curl}} & S_{r-2}^2(\mathcal{T}_h^z) & \xrightarrow{\text{div}} & S_{r-3}^3(\mathcal{T}_h^z) & \longrightarrow & 0.
\end{array}
$$

Specifically, we have

$$
\begin{aligned}
\text{grad}\ \Pi_0 p &= \Pi_1 \,\text{grad}\, p, && \forall p \in C^\infty(T), \\
\text{curl}\ \Pi_1 p &= \Pi_2 \,\text{curl}\, p, && \forall p \in [C^\infty(T)]^3, \\
\text{div}\ \Pi_2 p &= \Pi_3 \,\text{div}\, p, && \forall p \in [C^\infty(T)]^3.
\end{aligned}
$$

Exactness of Global Spaces

Theorem

For $r \geq 5$, the following sequence is exact on contractible domains:

$$S_r^0(\mathcal{T}_h^z) \xrightarrow{\text{grad}} S_{r-1}^1(\mathcal{T}_h^z) \xrightarrow{\text{curl}} S_{r-2}^2(\mathcal{T}_h^z) \xrightarrow{\text{div}} S_{r-3}^3(\mathcal{T}_h^z) \longrightarrow 0.$$

Concluding Remarks

Summary

- We have developed several new local, discrete de Rham complexes with varying level of smoothness on Alfeld splits.
- The results lead to characterizations of discrete divergence-free subspaces for the Stokes problem and formulas for the dimensions of smooth polynomial spaces.
- We have constructed analogous global complexes in three dimensions and projections that commute with the differential operators.

Open Problems/Future Work

- Apply techniques to study different splits (e.g., Worsey-Farin, Powell-Sabin)
- Degrees of freedom for global spaces for general dimension $n \geq 2$.