Locking-free, three-field formulations for coupled elasticity-poroelasticity

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Numerical Analysis of Coupled and Multi-Physics Problems with Dynamic Interfaces

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Overview

Introduction

Three-field formulation for poroelasticity
- Model equations
- Solvability analysis
- Discrete problems
- Error estimate
- Numerical results

Three-field formulation for linear elasticity
- Rotation-based formulation
- Finite element discretisation
- Finite volume element discretisation
- Numerical results

Coupled elasticity-poroelasticity

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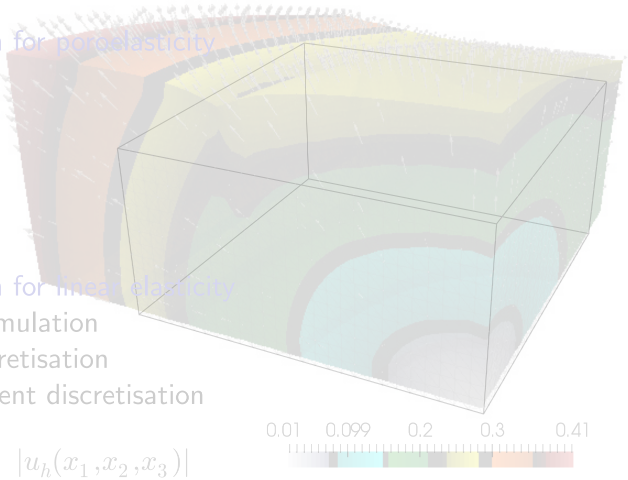


Coupled elasticity-poroelasticity

Motivating goal

Perfusion of cardiac tissue and contact with pericardial sac

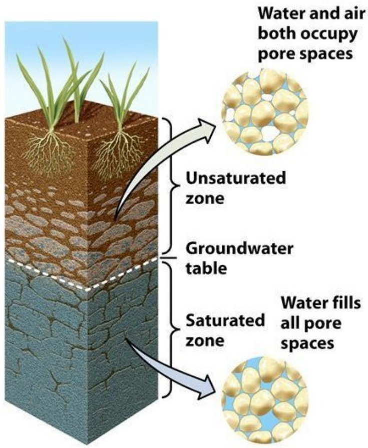
- Electro-chemical-poromechanical system with an interface
- Bidomain equations where conductivity is modified by porosity
- Equations of poromechanics with large deformations
- Hyperelasticity with fibre-oriented exponential constitutive equations

Numerical realisation (without the interface) already in place,
but no analysis!

⋮

Much simpler first step: linear poroelasticity

Model description



- Interconnected pore system uniformly saturated with fluid
- Only two phases: solid (Hooke's law for the skeleton deformation) and fluid (Darcy's law for fluid flow)
- Total volume of the pores $<<$ volume of the rock
- Rate (solid deformations) $<<$ rate (fluid flow)
- Total stress is distributed between fluid and solid particles

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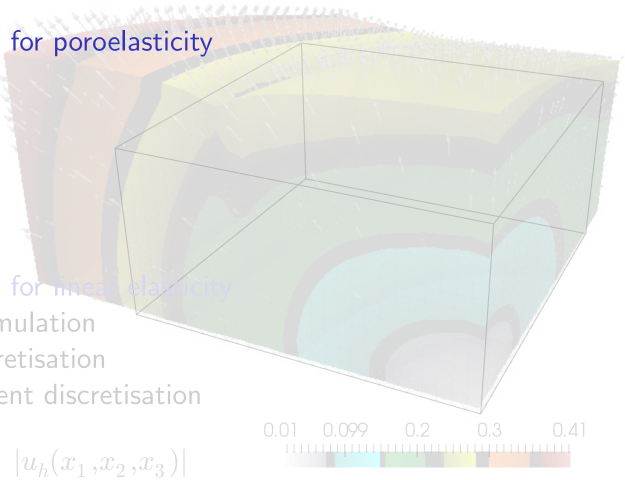

Coupled elasticity-poroelasticity

Governing equations

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Biot consolidation problem

For all $t > 0$, given a body force $\boldsymbol{f}(t) : \Omega \to \mathbb{R}^d$ and a volumetric fluid source (or sink) $s(t) : \Omega \to \mathbb{R}$, find the displacements of the porous skeleton, $\boldsymbol{u}(t) : \Omega \to \mathbb{R}^d$ and the pore pressure of the fluid, $p(t) : \Omega \to \mathbb{R}$, such that

$$
\begin{aligned}
\boldsymbol{\sigma} = \lambda(\operatorname{div} \boldsymbol{u})\mathbf{I} + 2\mu\varepsilon(\boldsymbol{u}) - p\mathbf{I} \qquad & \text{in } \Omega, \\
-\mathbf{div}\sigma = \boldsymbol{f} \qquad & \text{in } \Omega, \\
\partial_t\big(c_0 p + \alpha(\operatorname{div} \boldsymbol{u})\big) - \frac{1}{\eta}\operatorname{div}[\kappa(\nabla p - \rho \boldsymbol{g})] = s \qquad & \text{in } \Omega, \\
p = 0,\ \boldsymbol{\sigma}\boldsymbol{n} = \mathbf{0} \qquad & \text{on } \Gamma_p, \\
\boldsymbol{u} = \mathbf{0},\ (\kappa\nabla p)\cdot\boldsymbol{n} = 0 \qquad & \text{on } \Gamma_{\boldsymbol{u}}.
\end{aligned}
$$

Governing equations

- σ is the total Cauchy stress
- $\varepsilon(\boldsymbol{u}) = \frac{1}{2}(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T)$ is the infinitesimal strain tensor
- κ is the permeability of the porous solid $(0 < \kappa_{\text{inf}} \leq \kappa(\boldsymbol{x}) \leq \kappa_{\text{sup}} < \infty)$
- λ, μ are the Lamé constants of the solid
- $c_0 > 0$ is the constrained specific storage coefficient
- $\alpha > 0$ is the Biot-Willis parameter
- $\boldsymbol{g}$ is the gravity acceleration
- $\eta > 0, \rho > 0$ are the viscosity and density of the pore fluid
- $c_0 p + \alpha(\operatorname{div}\boldsymbol{u})$ represents the total fluid content

Governing equations

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Steady state problem

$$\boldsymbol{\sigma} = \lambda(\operatorname{div} \boldsymbol{u})\mathbf{I} + 2\mu\varepsilon(\boldsymbol{u}) - p\mathbf{I} \qquad \text{in } \Omega,$$

$$-\mathbf{div}\boldsymbol{\sigma} = \boldsymbol{f} \qquad \text{in } \Omega,$$

$$c_0 p + \alpha(\operatorname{div} \boldsymbol{u}) - \frac{1}{\eta}\operatorname{div}[\kappa(\nabla p - \rho\boldsymbol{g})] = s \qquad \text{in } \Omega,$$

$$p = 0,\ \boldsymbol{\sigma}\boldsymbol{n} = \mathbf{0} \qquad \text{on } \Gamma_p,$$

$$\boldsymbol{u} = \mathbf{0},\ (\kappa\nabla p)\cdot\boldsymbol{n} = 0 \qquad \text{on } \Gamma_{\boldsymbol{u}}.$$

Recall the linear elasticity case

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$$\boldsymbol{\sigma} = \lambda(\operatorname{div}\boldsymbol{u})\mathbf{I} + 2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}), \quad -\mathbf{div}\boldsymbol{\sigma} = \boldsymbol{f} \quad \text{in } \Omega, \quad \boldsymbol{u} = \mathbf{0} \text{ on } \partial\Omega.$$

Involving pressure (of the solid skeleton):

$$\hat{\phi} = -\lambda \operatorname{div}\boldsymbol{u}, \quad \boldsymbol{\sigma} = -\hat{\phi}\mathbf{I} + 2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}) \quad \text{in } \Omega,$$

$$-\mathbf{div}\boldsymbol{\sigma} = \boldsymbol{f} \quad \text{in } \Omega, \quad \boldsymbol{u} = \mathbf{0} \quad \text{on } \partial\Omega.$$

Mixed variational formulation: Find $\boldsymbol{u}, \hat{\phi}$ s.t.

$$\begin{aligned} 2\mu\int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}):\boldsymbol{\varepsilon}(\boldsymbol{v}) - \int_\Omega \hat{\phi}\operatorname{div}\boldsymbol{v} &= \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v} \quad \forall\, \boldsymbol{v} \in [\mathrm{H}^1_0(\Omega)]^d, \\ -\int_\Omega \psi\operatorname{div}\boldsymbol{u} - \frac{1}{\lambda}\int_\Omega \hat{\phi}\psi &= 0 \quad \forall\, \psi \in \mathrm{L}^2(\Omega). \end{aligned}$$

Any stable FE pair for Stokes $\Rightarrow$ Locking-free!

Back to our problem

$$\text{total pressure} \qquad \phi := p - \lambda \operatorname{div} \boldsymbol{u} \qquad \text{in } \Omega,$$

$$\sigma = \underbrace{2\mu\varepsilon(\boldsymbol{u}) - \phi\mathbf{I}}_{\lambda(\operatorname{div}\boldsymbol{u})\mathbf{I} + 2\mu\varepsilon(\boldsymbol{u}) - p\mathbf{I}} \qquad \text{in } \Omega,$$

$$-\mathbf{div}\sigma = \boldsymbol{f} \qquad \text{in } \Omega,$$

$$\underbrace{\left(c_0 + \frac{\alpha}{\lambda}\right)p - \frac{\alpha}{\lambda}\phi}_{c_0 p + \alpha(\operatorname{div}\boldsymbol{u})} - \frac{1}{\eta}\operatorname{div}[\kappa(\nabla p - \rho\boldsymbol{g})] = s \qquad \text{in } \Omega,$$

$$p = 0,\ \sigma\boldsymbol{n} = \mathbf{0} \qquad \text{on } \Gamma_p,$$

$$\boldsymbol{u} = \mathbf{0},\ (\kappa\nabla p)\cdot\boldsymbol{n} = 0 \qquad \text{on } \Gamma_{\boldsymbol{u}}.$$

Weak formulation

Find $\boldsymbol{u} \in \mathbf{H}^1_{\Gamma_{\boldsymbol{u}}}(\Omega)$, $p \in \mathrm{H}^1_{\Gamma_p}(\Omega)$ and $\phi \in \mathrm{L}^2(\Omega)$, such that

$$\begin{aligned} \int_\Omega \varepsilon(\boldsymbol{u}) : \varepsilon(\boldsymbol{v}) - \int_\Omega \phi \operatorname{div} \boldsymbol{v} &= F(\boldsymbol{v}) \\ \left(\frac{c_0}{\alpha} + \frac{1}{\lambda}\right) \int_\Omega pq + \frac{1}{\alpha\eta} \int_\Omega \kappa \nabla p \cdot \nabla q - \frac{1}{\lambda} \int_\Omega q\phi &= G(q) \\ - \int_\Omega \psi \operatorname{div} \boldsymbol{u} + \frac{1}{\lambda} \int_\Omega p\psi - \frac{1}{\lambda} \int_\Omega \phi\psi &= 0, \end{aligned}$$

for all $\boldsymbol{v} \in \mathbf{H}^1_{\Gamma_{\boldsymbol{u}}}(\Omega)$, $q \in \mathrm{H}^1_{\Gamma_p}(\Omega)$ and $\psi \in \mathrm{L}^2(\Omega)$. With

$$\begin{aligned} F(\boldsymbol{v}) &:= \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \\ G(q) &:= \frac{\rho}{\alpha\eta} \int_\Omega \kappa \boldsymbol{g} \cdot \nabla q - \frac{\rho}{\alpha\eta} \langle \kappa \boldsymbol{g} \cdot \boldsymbol{n}, q \rangle_{\Gamma_{\boldsymbol{u}}} + \frac{1}{\alpha} \int_\Omega sq. \end{aligned}$$

Weak formulation

Involved spaces

$$\mathbf{H} := \quad \mathbf{H}^1_{\Gamma_{\boldsymbol{u}}}(\Omega) = \{\boldsymbol{v} \in \mathbf{H}^1(\Omega) : \boldsymbol{v}|_{\Gamma_{\boldsymbol{u}}} = \mathbf{0}\}, \quad \mathrm{Z} := \mathrm{L}^2(\Omega),$$

$$\mathrm{Q} := \quad \mathrm{H}^1_{\Gamma_p}(\Omega) = \{q \in \mathrm{H}^1(\Omega) : q|_{\Gamma_p} = 0\}.$$

Bilinear forms

$$a_1(\boldsymbol{u}, \boldsymbol{v}) = 2\mu \int_\Omega \varepsilon(\boldsymbol{u}) : \varepsilon(\boldsymbol{v}),$$

$$a_2(p, q) = \left(\frac{c_0}{\alpha} + \frac{1}{\lambda}\right) \int_\Omega pq + \frac{1}{\alpha\eta} \int_\Omega \kappa \nabla p \cdot \nabla q,$$

$$b_1(\boldsymbol{v}, \psi) = -\int_\Omega \psi \operatorname{div} \boldsymbol{v}, \quad b_2(q, \psi) = \frac{1}{\lambda} \int_\Omega q\psi, \quad c(\phi, \psi) = \frac{1}{\lambda} \int_\Omega \phi\,\psi.$$

Weak formulation

Find $\boldsymbol{u} \in \mathbf{H}, p \in \mathrm{Q}, \phi \in \mathrm{Z}$ such that

$$
\begin{aligned}
a_1(\boldsymbol{u},\boldsymbol{v}) \qquad\qquad\qquad\qquad + b_1(\boldsymbol{v},\phi) &= F(\boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \mathbf{H},\\
a_2(p,q) - b_2(q,\phi) &= G(q) \qquad \forall q \in \mathrm{Q},\\
b_1(\boldsymbol{u},\psi) + b_2(p,\psi) - c(\phi,\psi) &= 0 \qquad \forall \psi \in \mathrm{Z}.
\end{aligned}
$$

Stability properties

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- Continuity:

$$|a_1(\boldsymbol{u},\boldsymbol{v})| \le 2\mu C_{k,2}\|\boldsymbol{u}\|_{1,\Omega}\|\boldsymbol{v}\|_{1,\Omega},$$

$$|a_2(p,q)| \le \max\left\{\frac{c_0}{\alpha}+\frac{1}{\lambda},\frac{\kappa_{\mathrm{sup}}}{\alpha\eta}\right\}\|p\|_{1,\Omega}\|q\|_{1,\Omega},$$

$$|b_1(\boldsymbol{v},\psi)| \le \sqrt{n}\|\boldsymbol{v}\|_{1,\Omega}\|\psi\|_{0,\Omega},$$

$$|b_2(q,\psi)| \le \lambda^{-1}\|q\|_{1,\Omega}\|\psi\|_{0,\Omega},$$

$$|c(\phi,\psi)| \le \lambda^{-1}\|\phi\|_{0,\Omega}\|\psi\|_{0,\Omega},$$

$$|F(\boldsymbol{v})| \le \|\boldsymbol{f}\|_{0,\Omega}\|\boldsymbol{v}\|_{1,\Omega},$$

$$|G(q)| \le \alpha^{-1}\left(\frac{\rho}{\eta}\kappa_{\mathrm{sup}}\|\boldsymbol{g}\|_{0,\Omega}+\frac{\rho}{\eta}\kappa_{\mathrm{sup}}C_\Gamma\|\boldsymbol{g}\cdot\boldsymbol{n}\|_{-1/2,\Gamma_{\boldsymbol{u}}}+\|s\|_{0,\Omega}\right)\|q\|_{1,\Omega}.$$

Stability properties

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- Positivity:

$$a_1(\boldsymbol{v},\boldsymbol{v}) \geq 2\mu C_{k,1}\|\boldsymbol{v}\|_{1,\Omega}^2, \quad \forall\, \boldsymbol{v} \in \mathbf{H},$$

$$a_2(q,q) \geq \alpha^{-1}\max\{c_0, \kappa_{\inf}\eta^{-1}\}\|q\|_{1,\Omega}^2 + \lambda^{-1}\|q\|_{0,\Omega}^2, \quad \forall\, q \in \mathrm{Q}$$

$$c(\psi,\psi) = \lambda^{-1}\|\psi\|_{0,\Omega}^2, \quad \forall\, \psi \in \mathrm{Z}.$$

- Inf-sup:

$$\sup_{\boldsymbol{v}\in\mathbf{H}\setminus\mathbf{0}} \frac{b_1(\boldsymbol{v},\psi)}{\|\boldsymbol{v}\|_{1,\Omega}} \geq \beta\|\psi\|_{0,\Omega} \quad \forall\, \psi \in \mathrm{Z}.$$

- Continuous dependence: If a solution exists, it satisfies

$$\|\boldsymbol{u}\|_{1,\Omega} + \|p\|_{1,\Omega} + \|\phi\|_{0,\Omega}$$

$$\leq \underbrace{C_{stab}}_{\text{indep. of } \lambda} \left(\|\boldsymbol{f}\|_{0,\Omega} + \|\boldsymbol{g}\|_{0,\Omega} + \|\boldsymbol{g}\cdot\boldsymbol{n}\|_{-1/2,\Gamma_{\boldsymbol{u}}} + \|s\|_{0,\Omega}\right).$$

No Babuška–Brezzi

Our problem

$$\begin{aligned} a_1(\boldsymbol{u},\boldsymbol{v}) \qquad\qquad &+ b_1(\boldsymbol{v},\phi) = F(\boldsymbol{v}) \qquad \forall \boldsymbol{v}\in \mathbf{H},\\ a_2(p,q) &- b_2(q,\phi) = G(q) \qquad \forall q \in \mathrm{Q},\\ b_1(\boldsymbol{u},\psi) + b_2(p,\psi) &- c(\phi,\psi) = 0 \qquad \forall \psi\in \mathrm{Z}. \end{aligned}$$

"Wrong signs" $\Rightarrow$ Babuška-Brezzi theory not applicable

But!!!!

$b_2(\cdot,\cdot)$ induces a compact operator. In fact

$$\langle \mathbb{B}_2(q),\psi\rangle_{0,\Omega} = b_2(q,\psi) = \frac{1}{\lambda}\int_\Omega q\psi = \big\langle(\lambda^{-1} I\circ i_c)(q),\psi\big\rangle_{0,\Omega},$$

$\forall\, q\in \mathrm{Q}, \forall\,\psi\in\mathrm{Z}$, where $i_c : \mathrm{H}^1(\Omega)\hookrightarrow \mathrm{L}^2(\Omega)$.

Solvability analysis

Decomposition of the problem

Find $\vec{\boldsymbol{u}} := (\boldsymbol{u}, p, \phi) \in \mathbb{V} := \mathbf{H} \times \mathrm{Q} \times \mathrm{Z}$, such that

$$(\mathscr{A} + \mathscr{K})\vec{\boldsymbol{u}} = \mathscr{F}_h,$$

where $\mathscr{A} : \mathbb{V} \to \mathbb{V}$, $\mathscr{K} : \mathbb{V} \to \mathbb{V}$ and $\mathscr{F}_h \in \mathbb{V}'$ are defined as:

$$\langle \mathscr{A}(\vec{\boldsymbol{u}}), \vec{\boldsymbol{v}} \rangle_{\mathbb{V}\times\mathbb{V}} := a_1(\boldsymbol{u}, \boldsymbol{v}) + b_1(\boldsymbol{v}, \phi) - b_1(\boldsymbol{u}, \psi) + c(\phi, \psi) + a_2(p, q)$$

$$\langle \mathscr{K}(\vec{\boldsymbol{u}}), \vec{\boldsymbol{v}} \rangle_{\mathbb{V}\times\mathbb{V}} := b_2(p, \psi) - b_2(q, \phi)$$

$$\langle \mathscr{F}_h, \vec{\boldsymbol{v}} \rangle_{\mathbb{V}\times\mathbb{V}} := F(\boldsymbol{v}) + G(q),$$

for all $\vec{\boldsymbol{u}} = (\boldsymbol{u}, p, \phi), \vec{\boldsymbol{v}} = (\boldsymbol{v}, q, \psi) \in \mathbb{V}$.

Lemma

$\mathscr{A}$ is invertible.

Proof: Proving the invertibility of $\mathscr{A}$, is equivalent to proving the unique solvability of the uncoupled problems:

- Find $(\boldsymbol{u},\phi) \in \mathbf{H} \times \mathrm{Z}$, such that

$$\begin{aligned} a_1(\boldsymbol{u},\boldsymbol{v}) + b_1(\boldsymbol{v},\phi) &= F_{\mathrm{H}}(\boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in \mathbf{H}, \\ b_1(\boldsymbol{u},\psi) - c(\phi,\psi) &= F_{\mathrm{Z}}(\psi) \quad \forall\, \psi \in \mathrm{Z}, \end{aligned}$$

- and: Find $p \in \mathrm{Q}$, such that

$$a_2(p,q) = F_{\mathrm{Q}}(q) \quad \forall\, q \in \mathrm{Q}.$$

Solvability analysis

Lemma

$(\mathscr{A}+\mathscr{K})$ *is one-to-one.*

Proof: It suffices to show that the unique solution to the homogeneous problem

$$\begin{aligned} a_1(\boldsymbol{u},\boldsymbol{v}) \qquad\qquad\qquad + b_1(\boldsymbol{v},\phi) &= 0 \qquad \forall \boldsymbol{v}\in\mathbf{H},\\ a_2(p,q) - b_2(q,\phi) &= 0 \qquad \forall q\in\mathrm{Q},\\ b_1(\boldsymbol{u},\psi)+b_2(p,\psi) - c(\phi,\psi) &= 0 \qquad \forall \psi\in\mathrm{Z}, \end{aligned}$$

is the null vector in $\mathbb{V}$.

Solvability analysis

Theorem

Given $\boldsymbol{f} \in \mathbf{L}^2(\Omega)$, $\boldsymbol{g} \in \mathbf{L}^2(\Omega)$ *and* $s \in L^2(\Omega)$, *there exists a unique solution* $(\boldsymbol{u}, p, \phi) \in \mathbf{H} \times \mathrm{Q} \times \mathrm{Z}$ *to the coupled problem. Moreover, there exists* $C_{stab} > 0$, *independent of* λ, *such that*

$$\|\boldsymbol{u}\|_{1,\Omega} + \|p\|_{1,\Omega} + \|\phi\|_{0,\Omega}$$

$$\leq C_{stab}\left(\|\boldsymbol{f}\|_{0,\Omega} + \|\boldsymbol{g}\|_{0,\Omega} + \|\boldsymbol{g}\cdot\boldsymbol{n}\|_{-1/2,\Gamma_{\boldsymbol{u}}} + \|s\|_{0,\Omega}\right).$$

Proof:

Inveribility of $\mathscr{A}$ + injectivity of $(\mathscr{A} + \mathscr{K})$ + compactness of $\mathscr{K}$ + Fredhlom alternative $\Rightarrow$ well-posedness.

A Galerkin method

Generic subspaces

$$\mathbf{H}_h \subseteq \mathbf{H}, \qquad \mathrm{Q}_h \subseteq \mathrm{Q}, \quad \text{and} \quad \mathrm{Z}_h \subseteq \mathrm{Z}.$$

Discrete problem

Find $\boldsymbol{u}_h \in \mathbf{H}_h$, $p_h \in \mathrm{Q}_h$ and $\phi_h \in \mathrm{Z}_h$, such that

$$\begin{aligned}
a_1(\boldsymbol{u}_h, \boldsymbol{v}_h) \qquad\qquad\qquad + b_1(\boldsymbol{v}_h, \phi_h) &= F(\boldsymbol{v}_h) && \forall \boldsymbol{v}_h \in \mathbf{H}_h, \\
a_2(p_h, q_h) - b_2(q_h, \phi_h) &= G(q_h) && \forall q_h \in \mathrm{Q}_h, \\
b_1(\boldsymbol{u}_h, \psi_h) + b_2(p_h, \psi_h) - c(\phi_h, \psi_h) &= 0 && \forall \psi_h \in \mathrm{Z}_h.
\end{aligned}$$

Solvability

Assumption

There exists $\hat{\beta} > 0$, independent of h, such that

$$\sup_{\boldsymbol{v}_h \in \mathbf{H}_h \setminus \mathbf{0}} \frac{b_1(\boldsymbol{v}_h, \psi_h)}{\|\boldsymbol{v}_h\|_{1,\Omega}} \geq \hat{\beta} \|\psi\|_{0,\Omega} \quad \forall\, \psi_h \in \mathrm{Z}_h.$$

Theorem

Given $\boldsymbol{f} \in \mathbf{L}^2(\Omega)$, $\boldsymbol{g} \in \mathbf{L}^2(\Omega)$ and $s \in L^2(\Omega)$, there exists a unique solution $(\boldsymbol{u}_h, p_h, \phi_h) \in \mathbf{H}_h \times \mathrm{Q}_h \times \mathrm{Z}_h$ to the discrete coupled problem. Moreover, there exists $\hat{C}_{stab} > 0$, independent of h and λ, s.t.

$$\|\boldsymbol{u}_h\|_{1,\Omega} + \|p_h\|_{1,\Omega} + \|\phi_h\|_{0,\Omega}$$

$$\leq \hat{C}_{stab} \left(\|\boldsymbol{f}\|_{0,\Omega} + \|\boldsymbol{g}\|_{0,\Omega} + \|\boldsymbol{g} \cdot \boldsymbol{n}\|_{-1/2,\Gamma_{\boldsymbol{u}}} + \|s\|_{0,\Omega} \right).$$

Error analysis I

Theorem: Céa's estimate

Let $(\boldsymbol{u}, p, \phi) \in \mathbf{H} \times \mathrm{Q} \times \mathrm{Z}$ and $(\boldsymbol{u}_h, p_h, \phi_h) \in \mathbf{H}_h \times \mathrm{Q}_h \times \mathrm{Z}_h$ be the unique solutions of the continuous and discrete coupled problems, respectively. Then, there exists $C_{\text{Céa}} > 0$, independent of h and λ, such that

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,\Omega} + \|p - p_h\|_{1,\Omega} + \|\phi - \phi_h\|_{0,\Omega}$$

$$\leq C_{\text{Céa}} \left(\operatorname{dist}(\boldsymbol{u}, \mathbf{H}_h) + \operatorname{dist}(p, \mathrm{Q}_h) + \operatorname{dist}(\phi, \mathrm{Z}_h) \right).$$

Proof. Inf-sup of b_1 + exploiting the kernel

$$\mathbf{K}_h := \left\{ \boldsymbol{v}_h \in \mathbf{H}_h : b_1(\boldsymbol{v}_h, \psi_h) = -b_2(p_h, \psi_h) + c(\phi_h, \psi_h), \quad \forall\, \psi_h \in \mathrm{Z}_h \right\}$$

$+$ error decomposition.

Error bounds

That was valid for any inf-sup stable approximation. Take e.g.

$$\mathbf{H}_h := \left\{ \boldsymbol{v}_h \in [C(\overline{\Omega})]^2 : \boldsymbol{v}_h\big|_K \in \mathbb{P}_{1,b}(K) \quad \forall\, K \in \mathscr{T}_h, \quad \boldsymbol{v}_h = 0 \text{ on } \Gamma_{\boldsymbol{u}} \right\}$$
$$\mathrm{Z}_h := \left\{ \psi_h \in C(\overline{\Omega}) : \psi_h\big|_K \in \mathbb{P}_1(K) \quad \forall\, K \in \mathscr{T}_h \right\}$$
$$\mathrm{Q}_h := \left\{ q_h \in C(\overline{\Omega}) : q_h\big|_K \in \mathbb{P}_1(K) \quad \forall\, K \in \mathscr{T}_h, \quad q_h = 0 \text{ on } \Gamma_p \right\}.$$

Theorem

Assume that $\boldsymbol{u} \in \mathbf{H}^2(\Omega)$, $p \in \mathrm{H}^2(\Omega)$ and $\phi \in \mathrm{H}^1(\Omega)$. Then, there exists $C > 0$, independent of h and λ, s.t.

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,\Omega} + \|p - p_h\|_{1,\Omega} + \|\phi - \phi_h\|_{0,\Omega}$$
$$\leq Ch\left\{ \|\boldsymbol{u}\|_{2,\Omega} + \|p\|_{2,\Omega} + \|\phi\|_{1,\Omega} \right\}.$$

What about without inf-sup?

Remark

- Without the inf-sup condition for b_1 it is still possible to prove that $\mathscr{A}$ is invertible and $\mathscr{A} + \mathscr{K}$ is injective.
- However, the continuous dependence and the Céa estimate involve constants depending on λ.
- Inf-sup unstable methods (e.g. the lowest order $[\mathbb{P}_1]^d \times \mathbb{P}_1 \times \mathbb{P}_0$) will fail for large λ.

Other possibilities, e.g. equal-order approximations

$$\mathbf{H}_h := \left\{ \boldsymbol{v}_h \in [C(\overline{\Omega})]^d : \boldsymbol{v}_h\big|_K \in [\mathbb{P}_k(K)]^d \ \forall K \in \mathscr{T}_h,\ \boldsymbol{v}_h = 0 \text{ on } \Gamma_{\boldsymbol{u}} \right\},$$
$$\mathrm{Z}_h := \left\{ \psi_h \in C(\overline{\Omega}) : \psi_h\big|_K \in \mathbb{P}_k(K) \ \forall K \in \mathscr{T}_h \right\},$$
$$\mathrm{Q}_h := \left\{ q_h \in C(\overline{\Omega}) : q_h\big|_K \in \mathbb{P}_k(K) \quad \forall K \in \mathscr{T}_h, \quad q_h = 0 \text{ on } \Gamma_p \right\}.$$

Take for instance the reflected GLS method for Stokes

$$\begin{aligned} a_1(\boldsymbol{u}_h, \boldsymbol{v}_h) \qquad\qquad &+ b_1(\boldsymbol{v}_h, \phi_h) = F(\boldsymbol{v}_h) && \forall \boldsymbol{v}_h \in \mathbf{H}_h, \\ a_2(p_h, q_h) &- b_2(q_h, \phi_h) = G(q_h) && \forall q_h \in \mathrm{Q}_h, \\ b_1(\boldsymbol{u}_h, \psi_h) + b_2(p_h, \psi_h) &- \tilde{c}(\phi_h, \psi_h) = \tilde{H}(\psi_h) && \forall \psi_h \in \mathrm{Z}_h, \end{aligned}$$

with

$$\tilde{c}(\phi_h, \psi_h) = \pm c(\phi_h, \psi_h) + \tau \sum_{K \in \mathscr{T}_h} h_K^2 (-2\mu \, \mathbf{div} \, \varepsilon(\boldsymbol{u}_h) + \nabla \phi_h, -2\mu \, \mathbf{div} \, \varepsilon(\boldsymbol{v}_h) \mp \nabla \psi_h)_{0,K}$$
$$\tilde{H}(\psi_h) = \tau \sum_{K \in \mathscr{T}_h} h_K^2 (\boldsymbol{f}, -2\mu \, \mathbf{div} \, \varepsilon(\boldsymbol{v}_h) \mp \nabla \psi_h)_{0,K}.$$

Ex I: Experimental convergence

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Manufactured solution in 2D

$$\boldsymbol{u} = a\begin{pmatrix}\sin^2(\pi x_1)\sin(\pi x_2)\cos(\pi x_2)\\ \sin^2(\pi x_2)\sin(\pi x_1)\cos(\pi x_1)\end{pmatrix}, \quad p = b\sin(\pi x_1)\sin(\pi x_2).$$

- Cantilever bracket with curved sides
- Scalings $a =$1e-4, $b = \pi$
- Young modulus $E = 1e4$, material permeability $\kappa =$ 1e-7, Biot-Willis coefficient $\alpha = 0.1$, constrained specific storage $c_0 =$ 1e-5,
- Boundary split into $\Gamma_{\boldsymbol{u}}$ and Γ_p

Ex I: Experimental convergence

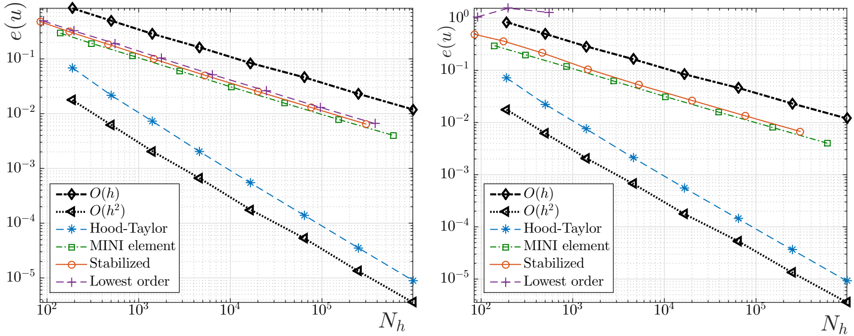


Figure: Velocity accuracy. Left: $\nu = 0.4$ ($\lambda = 14285.7$). Right: $\nu = 0.49999$ and $\lambda = 1.66\text{e}8$.

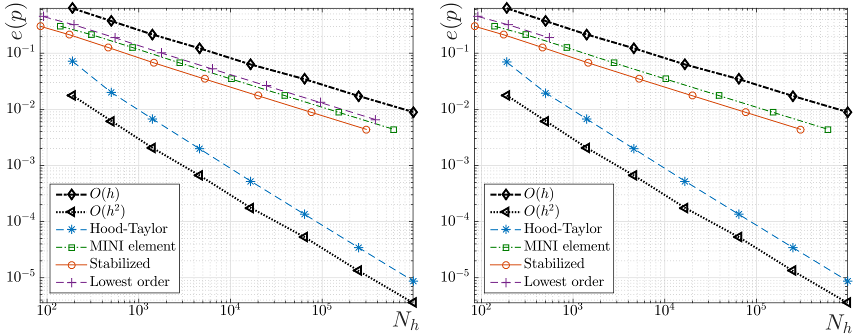


Figure: Pressure accuracy. Left: $\nu = 0.4$ ($\lambda = 14285.7$). Right: $\nu = 0.49999$ and $\lambda = 1.66\mathrm{e}8$.

Ex I: Experimental convergence

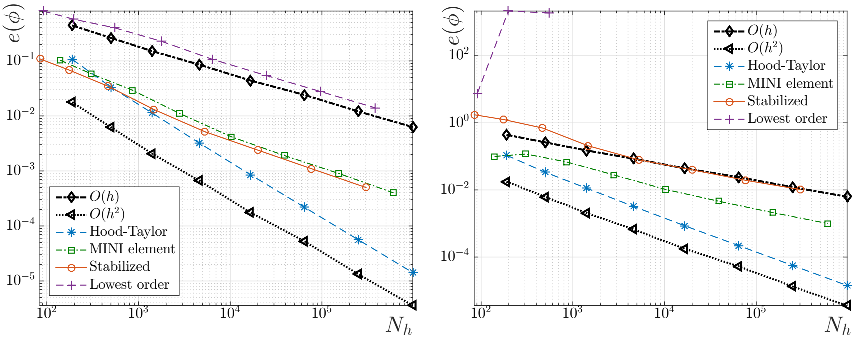
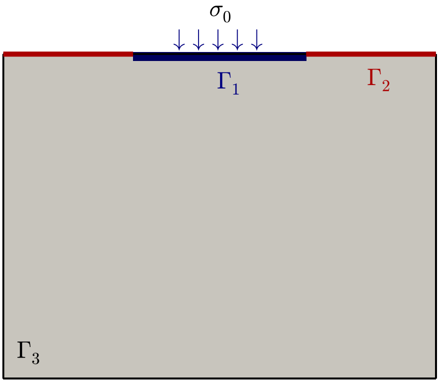


Figure: Total pressure accuracy. Left: $\nu = 0.4$ ($\lambda = 14285.7$). Right: $\nu = 0.49999$ and $\lambda = 1.66e8$.

Ex II: Footing problem



Undeformed domain and boundary splitting

- Block of porous soil undergoes a load of σ_0
- $\Omega = (-50, 50) \times (0, 75)$,
- E =3e4 N/m^2, κ =1e-4 m^2/Pa, σ_0 =1.5e4 N/m^2
- c_0 =1e-3, $\alpha = 0.1$,
- $\nu = 0.4995$
- $\boldsymbol{u} = \mathbf{0}$ on Γ_3
- $\boldsymbol{\sigma n} = m$ on $\Gamma_1 \cup \Gamma_2$
- $p = 0$ on $\partial\Omega$

Ex II: Footing problem

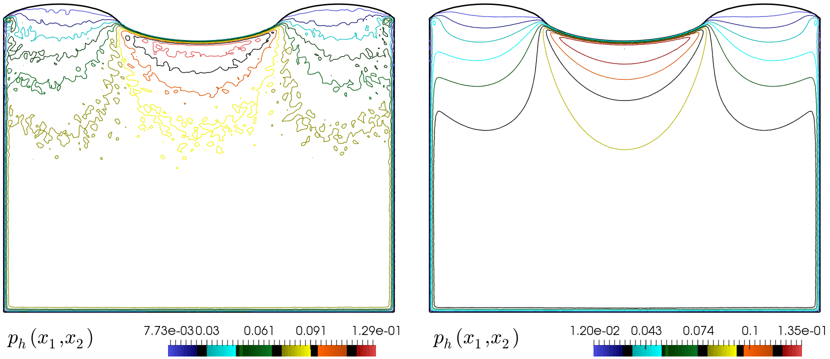


Figure: Pressure. Left: lowest order (inf-sup unstable). Right: MINI-element.

Ex II: Footing problem

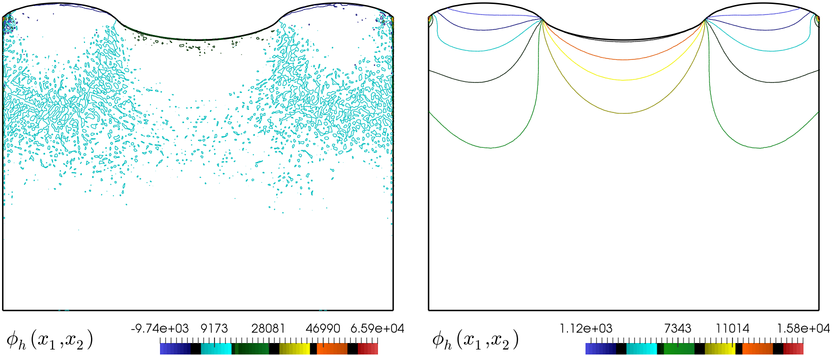
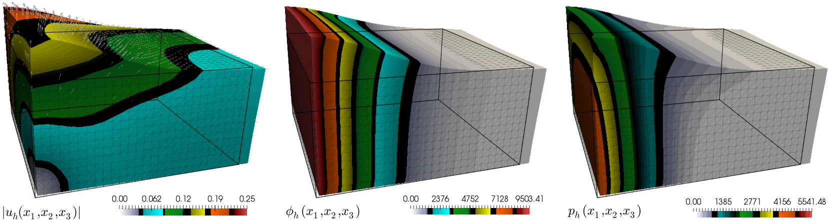


Figure: Total pressure. Left: lowest order (inf-sup unstable). Right: MINI-element.

Ex III: Swelling of a sponge

- Dirichlet pressure $x_1 = 0$ and $x_1 = 1$. Zero-flux pressure elsewhere.
- $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on $x_1 = 0$, $x_2 = 0$ and $x_3 = 0$, and zero normal stress elsewhere
- $E = 8000$, $\nu = 0.3$, $c_0 = 0.001$, $\kappa = 1\text{e-}5$, $\rho = \alpha = 1$, $\tau = 1/60$.
- No external or internal forces are considered, and neither fluid sources or sinks



Ex IV: Terzaghi's consolidation

- Transient consolidation of a thin porous column
- Top is pervious (zero pore pressure $p = 0$, constant mechanical load in the vertical direction $\boldsymbol{\sigma}\boldsymbol{n} = -\sigma_0 \boldsymbol{e}_3$, and free to drain)
- Bottom is impervious (zero pressure flux $\kappa \nabla p \cdot \boldsymbol{n} = 0$ and zero displacement $\boldsymbol{u} = \boldsymbol{0}$)
- Zero horizontal displacements on the walls
- Comparison against asymptotic 1D solution
- $\sigma_0 = 1\text{e}4$ [Pa], $E = 3\text{e}4$ [N/m^2], $\nu = 0.2$, $\kappa = 1\text{e-}10$ [m^2], $\eta = 1\text{e-}3$ [Pa s], $c_0 = 0$, $\alpha = 1$, $\rho = 1$, $T = 10$ [s], $\Delta t = 0.1$ [s]
- MINI-element + $\mathbb{P}_2$

Ex IV: Terzaghi's consolidation

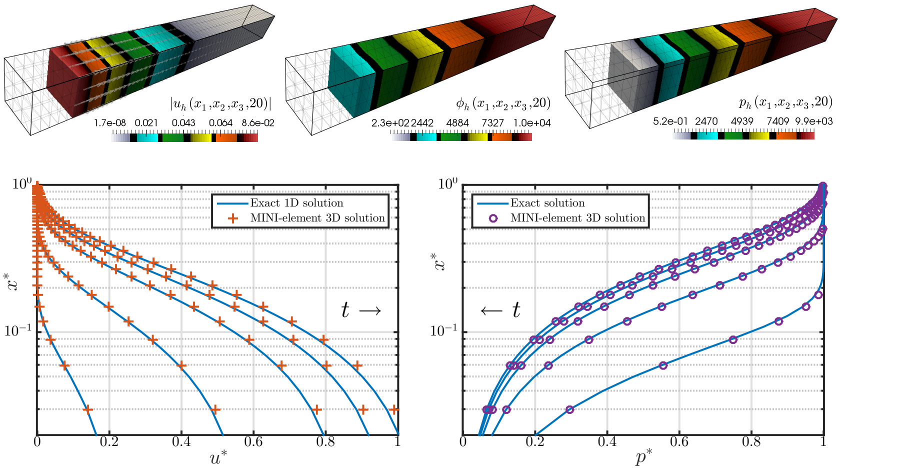


Figure: Pseudo-1D time-dependent consolidation benchmark.

Overview

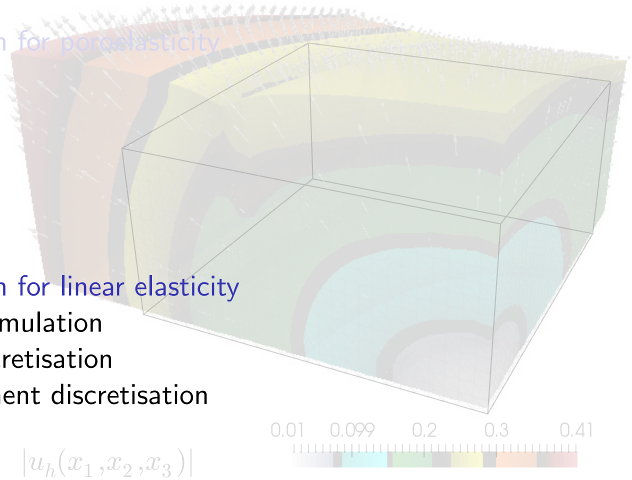
Introduction

Three-field formulation for poroelasticity

- Model equations
- Solvability analysis
- Discrete problems
- Error estimate
- Numerical results

Three-field formulation for linear elasticity

- Rotation-based formulation
- Finite element discretisation
- Finite volume element discretisation
- Numerical results



Coupled elasticity-poroelasticity

Rotation-based formulation

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Linear elasticity

Given a body force $\tilde{\boldsymbol{f}} : \Omega \to \mathbb{R}^d$ and a prescribed boundary motion $\boldsymbol{g}$, find the displacements $\boldsymbol{u} : \Omega \to \mathbb{R}^d$ s.t.

$$
\begin{aligned}
-\mathbf{div}[\lambda(\operatorname{div}\boldsymbol{u})\mathbf{I} + 2\mu\varepsilon(\boldsymbol{u})] &= \tilde{\boldsymbol{f}} && \text{in } \Omega,\\
\boldsymbol{u} &= \boldsymbol{g} && \text{on } \Gamma.
\end{aligned}
$$

Rotation-based formulation

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Displacement-rotation-pressure formulation

Introducing pressure $\underline{p := -\operatorname{div}\boldsymbol{u}}$ and rotations $\underline{\boldsymbol{\omega} := \sqrt{\eta}\,\mathbf{curl}\,\boldsymbol{u}}$:

$$
\begin{aligned}
\sqrt{\eta}\,\mathbf{curl}\,\omega + (1+\eta)\nabla p &= \boldsymbol{f} && \text{in } \Omega,\\
\omega - \sqrt{\eta}\,\mathbf{curl}\,\boldsymbol{u} &= 0 && \text{in } \Omega,\\
\operatorname{div}\boldsymbol{u} + p &= 0 && \text{in } \Omega,\\
\boldsymbol{u} &= \boldsymbol{g} && \text{on } \Gamma,
\end{aligned}
$$

where $\eta := \frac{\mu}{\lambda+\mu}$ and $\boldsymbol{f} = \frac{1}{\lambda+\mu}\tilde{\boldsymbol{f}}$.

(similarity with vorticity-based formulations for Stokes and Brinkman)

Weak formulation

Find ω, p and $\boldsymbol{u}$ s.t.

$$\int_\Omega \omega\cdot\theta - \sqrt{\eta}\int_\Omega \theta\cdot\mathbf{curl}\,\boldsymbol{u} = 0 \quad \forall\theta\in\mathbf{Z},$$

$$(1+\eta)\int_\Omega pq + (1+\eta)\int_\Omega q\,\mathrm{div}\,\boldsymbol{u} = 0 \quad \forall q\in\mathrm{Q},$$

$$-(1+\eta)\int_\Omega p\,\mathrm{div}\,\boldsymbol{v} + \sqrt{\eta}\int_\Omega \omega\cdot\mathbf{curl}\,\boldsymbol{v} = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v} \quad \forall\boldsymbol{v}\in\mathbf{H}.$$

Involved spaces:

$$\mathbf{H} := \mathrm{H}^1_0(\Omega)^d, \quad \mathbf{Z} := \mathrm{L}^2(\Omega)^d, \quad \text{and} \quad \mathrm{Q} := \mathrm{L}^2(\Omega).$$

Consider the η-dependent scaled norm (thanks to BCs!)

$$\|\boldsymbol{v}\|^2_{\mathbf{H}} := \eta\|\mathbf{curl}\,\boldsymbol{v}\|^2_{0,\Omega} + \|\mathrm{div}\,\boldsymbol{v}\|^2_{0,\Omega}.$$

Weak formulation

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Consider (ω, p) together in the product space $\mathbf{Z} \times \mathrm{Q}$ and introduce

$$a\big((\boldsymbol{\omega},p),(\boldsymbol{\theta},q)\big) := \int_\Omega \boldsymbol{\omega}\cdot\boldsymbol{\theta} + (1+\eta)\int_\Omega pq,$$

$$b\big((\boldsymbol{\theta},q),\boldsymbol{v}\big) := (1+\eta)\int_\Omega q\,\mathrm{div}\,\boldsymbol{v} - \sqrt{\eta}\int_\Omega \boldsymbol{\theta}\cdot\mathbf{curl}\,\boldsymbol{v},$$

$$F(\boldsymbol{v}) := -\int_\Omega \boldsymbol{f}\cdot\boldsymbol{v},$$

$\Rightarrow$ Find (ω, p) and $\boldsymbol{u}$ s.t.

$$a\big((\boldsymbol{\omega},p),(\boldsymbol{\theta},q)\big) + b\big((\boldsymbol{\theta},q),\boldsymbol{u}\big) = 0 \qquad \forall(\boldsymbol{\theta},q)\in\mathbf{Z}\times\mathrm{Q},$$

$$b\big((\boldsymbol{\omega},p),\boldsymbol{v}\big) = F(\boldsymbol{v}) \qquad \forall\boldsymbol{v}\in\mathbf{H}.$$

Well-posedness

- Coercivity:

$$a((\theta,q),(\theta,q)) \geq \alpha \|(\theta,q)\|^2_{\mathbf{Z}\times\mathrm{Q}} \qquad \forall (\theta,q) \in \mathbf{Z}\times\mathrm{Q}.$$

- Inf-sup:

$$\sup_{(\theta,q)\in\mathbf{Z}\times\mathrm{Q}} \frac{b((\theta,q),\boldsymbol{v})}{\|(\theta,q)\|_{\mathbf{Z}\times\mathrm{Q}}} \geq C\|\boldsymbol{v}\|_{\mathbf{H}} \qquad \forall \boldsymbol{v} \in \mathbf{H}.$$

$\Rightarrow$ There exists a unique solution to the continuous problem, and

$$\|\boldsymbol{u}\|_{\mathbf{H}} + \|(\omega,p)\|_{\mathbf{Z}\times\mathrm{Q}} \leq \underbrace{C_{\text{Stab.}}}_{\text{indep. of } \lambda} \|\boldsymbol{f}\|_{0,\Omega}.$$

FE discretisation

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Discrete functional spaces

Take a shape-regular family $\{\mathscr{T}_h(\Omega)\}_{h>0}$ of partitions and introduce

$$\mathbf{H}_h := \{\boldsymbol{v}_h \in \mathbf{H} : \boldsymbol{v}_h|_T \in \mathscr{P}_k(T)^d \quad \forall T \in \mathscr{T}_h(\Omega)\},$$
$$\mathbf{Z}_h := \{\theta_h \in \mathbf{Z} : \theta_h|_T \in \mathscr{P}_{k-1}(T)^d \quad \forall T \in \mathscr{T}_h(\Omega)\}, \quad k \geq 1,$$
$$\mathrm{Q}_h := \{q_h \in \mathrm{Q} : q_h|_T \in \mathscr{P}_{k-1}(T) \quad \forall T \in \mathscr{T}_h(\Omega)\}.$$

Galerkin scheme

Find (ω_h, p_h) and $\boldsymbol{u}_h$ s.t.

$$a\big((\omega_h, p_h), (\theta_h, q_h)\big) + b\big((\theta_h, q_h), \boldsymbol{u}_h\big) = 0 \qquad \forall (\theta_h, q_h) \in \mathbf{Z}_h \times \mathrm{Q}_h,$$
$$b\big((\omega_h, p_h), \boldsymbol{v}_h\big) = F(\boldsymbol{v}_h) \qquad \forall \boldsymbol{v}_h \in \mathbf{H}_h.$$

Well-posedness FE discretisation

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- Discrete inf-sup:

$$\sup_{(\theta_h,q_h)\in \mathbf{Z}_h\times \mathrm{Q}_h} \frac{b((\theta_h,q_h),\boldsymbol{v}_h)}{\|(\theta_h,q_h)\|_{\mathbf{Z}\times\mathrm{Q}}} \geq C\|\boldsymbol{v}_h\|_{\mathbf{H}} \qquad \forall \boldsymbol{v}_h \in \mathbf{H}_h.$$

- Well-posedness: There exists a unique solution that satisfies

$$\|\boldsymbol{u}_h\|_{\mathbf{H}} + \|(\omega_h,p_h)\|_{\mathbf{Z}\times\mathrm{Q}} \leq \underbrace{C_{\mathrm{Stab}}}_{\text{indep. of }\lambda} \|\boldsymbol{f}\|_{0,\Omega}.$$

- Quasi-optimality:

$$\|(\omega-\omega_h,p-p_h)\|_{\mathbf{Z}\times\mathrm{Q}} + \|\boldsymbol{u}-\boldsymbol{u}_h\|_{\mathbf{H}} \\ \leq \underbrace{C_{\text{Céa}}}_{\text{indep. of }\lambda} \inf_{((\theta_h,q_h),\boldsymbol{v}_h)\in(\mathbf{Z}_h\times\mathrm{Q}_h)\times\mathbf{H}_h} \|(\omega-\theta_h,p-q_h)\|_{\mathbf{Z}\times\mathrm{Q}} + \|\boldsymbol{u}-\boldsymbol{v}_h\|_{\mathbf{H}}.$$

Error analysis FE discretisation

- k-th order convergence in the energy norm:

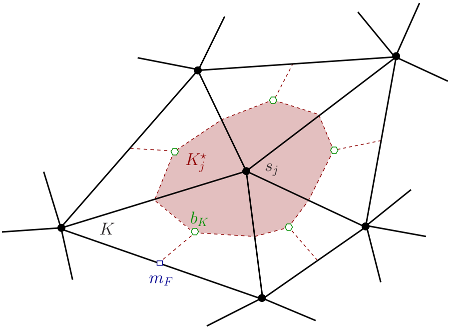
$$\|(\omega-\omega_h, p-p_h)\|_{\mathbf{Z}\times\mathrm{Q}} + \|\boldsymbol{u}-\boldsymbol{u}_h\|_{\mathrm{H}} \le Ch^k$$

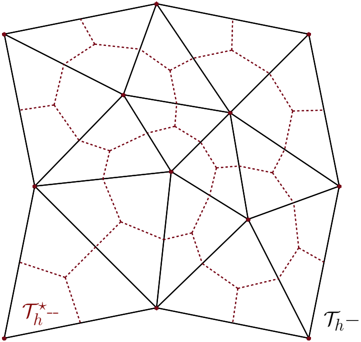
- L^2-convergence:

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{0,\Omega} \le Ch^{k+1}$$

Lowest-order FVE discretisation

Based on the primal mesh $\mathscr{T}_h$ we construct a dual mesh $\mathscr{T}_h^\star$ to ensure local conservativity.





Lowest-order FVE discretisation

Based on the primal and dual partitions $\mathscr{T}_h, \mathscr{T}_h^\star$, introduce

$$\mathbf{H}_h := \{\boldsymbol{v}_h \in \mathbf{H} : \boldsymbol{v}_h|_T \in \mathscr{P}_1(T)^d \quad \forall T \in \mathscr{T}_h\},$$
$$\mathbf{H}_h^\star := \{\boldsymbol{v}_h \in \mathrm{L}^2(\Omega)^d : \boldsymbol{v}_h|_{K_j^\star} \in \mathscr{P}_0(K_j^\star)^d \quad \forall K_j^\star \in \mathscr{T}_h^\star, \boldsymbol{v}|_{K_j^\star} = \mathbf{0} \text{ on } \partial\Omega\},$$
$$\mathbf{Z}_h := \{\theta_h \in \mathbf{Z} : \theta_h|_T \in \mathscr{P}_0(T)^d \quad \forall T \in \mathscr{T}_h\},$$
$$\mathrm{Q}_h := \{q_h \in \mathrm{Q} : q_h|_T \in \mathscr{P}_0(T) \quad \forall T \in \mathscr{T}_h\}.$$

Transfer operator $\mathscr{H}_h$ that relates the primal and dual meshes:

$$\boldsymbol{v}_h(\boldsymbol{x}) = \sum_j \boldsymbol{v}_h(s_j) \underbrace{\varphi_j(\boldsymbol{x})}_{\text{lin. nodal}} \mapsto \mathscr{H}_h \boldsymbol{v}_h(\boldsymbol{x}) = \sum_j \boldsymbol{v}_h(s_j) \underbrace{\chi_j(\boldsymbol{x})}_{\text{char. on CVs}}.$$

Lowest-order FVE discretisation

Find $(\hat{\omega}_h, \hat{p}_h)$ and $\hat{\boldsymbol{u}}_h$ s.t.

$$\begin{aligned} a\big((\hat{\omega}_h,\hat{p}_h),(\theta_h,q_h)\big)+b\big((\theta_h,q_h),\hat{\boldsymbol{u}}_h\big) &= 0,\\ B\big((\hat{\omega}_h,\hat{p}_h),\boldsymbol{v}_h\big) &= F(\mathscr{H}_h\boldsymbol{v}_h), \end{aligned}$$

for all $((\theta_h,q_h),\boldsymbol{v}_h)\in(\mathbf{Z}_h\times\mathrm{Q}_h)\times\mathbf{H}_h$, where

$$\begin{aligned} B((\theta_h,q_h),\boldsymbol{v}_h) := &-(1+\eta)\sum_{j=1}^{|\mathscr{N}_h|}\int_{\partial K_j^\star} q_h(\mathscr{H}_h\boldsymbol{v}_h\cdot\boldsymbol{n})\\ &-\sqrt{\eta}\sum_{j=1}^{|\mathscr{N}_h|}\int_{\partial K_j^\star}(\theta_h\times\boldsymbol{n})\cdot\mathscr{H}_h\boldsymbol{v}_h. \end{aligned}$$

- Galerkin scheme (instead of Petrov-Galerkin) thanks to the transfer operator!

Well-posedness FVE discretisation

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- Continuity + coercivity + Discrete inf-sup
- Unique solvability and continuous dependence on data
- Céa estimate
- Linear convergence:

$$\|(\omega-\hat{\omega}_h, p-\hat{p}_h)\|_{\mathbf{Z}\times\mathrm{Q}} + \|\boldsymbol{u}-\hat{\boldsymbol{u}}_h\|_{\mathrm{H}} \leq \underbrace{C_{\text{Conv.}}}_{\text{indep. of } \lambda} h$$

- L^2-convergence:

$$\|\boldsymbol{u}-\hat{\boldsymbol{u}}_h\|_{0,\Omega} \leq \underbrace{C_{\text{Conv.}}}_{\text{indep. of } \lambda} h^2$$

Example 1: Cantilevered beam

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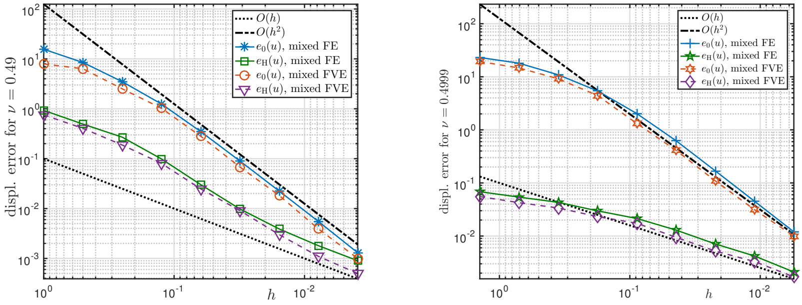
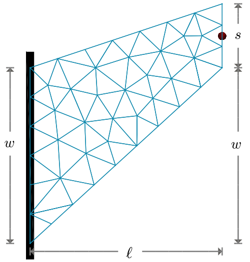
Convergence using the first order mixed FE and FVE schemes, for $\nu = 0.49$ and $\nu = 0.4999$, fixing $E = 1500$.

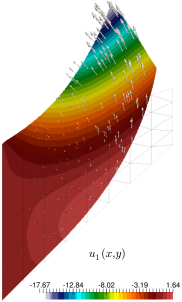


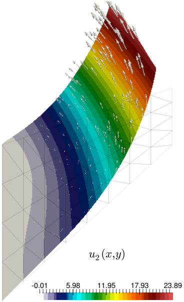
- Rectangular beam ($L = 10$, $l = 2$) subjected to a couple ($f = 300$)
- Zero horizontal displacement along the left edge
- Zero normal stress on the remainder of the boundary

Example 2: Cook's membrane

- Cook's membrane benchmark ($l = 48$, $w = 44$, $s = 16$)
- Clamped left edge $x = 0$, shearing load at $x = l$ of magnitude 1
- Zero body force $\boldsymbol{f} = 0$
- Traction free boundary condition on non-vertical edges
- $E = 1$, $\nu = 1/3$, s.t. $\eta = 1/3$







Example 2: Cook's membrane

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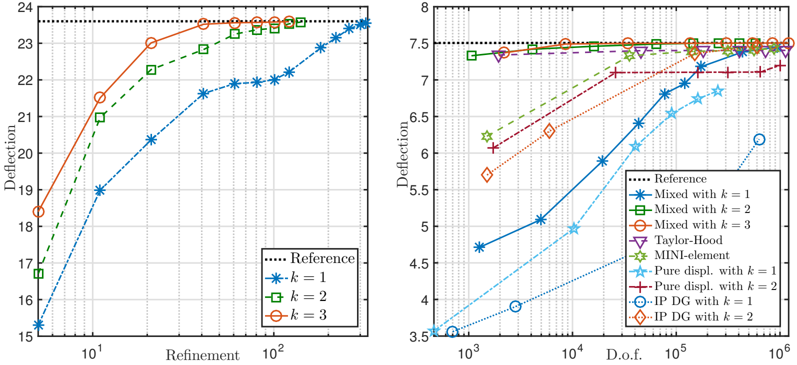
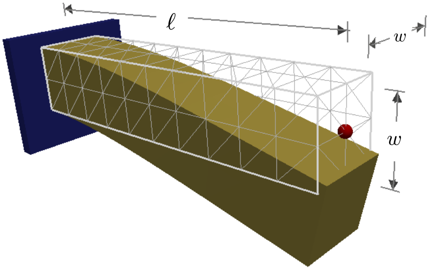
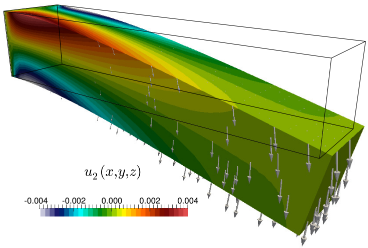
Comparison against other formulations for linear elasticity, for $\nu = 0.3$, $\nu = 0.49999$, fixing $E = 1500$.



Example 3: Clamped beam

Numerical solution using a second-order FE method.





k \ h	$(w/4)^3$	$(w/8)^3$	$(w/16)^3$	$(w/32)^3$	$(w/64)^3$
1	-0.4322	-0.4465	-0.4688	-0.4691	-0.4695
2	-0.4671	-0.4694	-0.4702	-0.4703	-0.4704
3	-0.4693	-0.4701	-0.4704	-0.4704	-0.4704

Max deflection at $(x_0, y_0, z_0) = (\ell, \frac{1}{2}w, \frac{1}{2}w)$ with $E = 1000$, $\nu = 0.3$. Expected $\delta = -0.47040$.

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- Finite volume element discretisation
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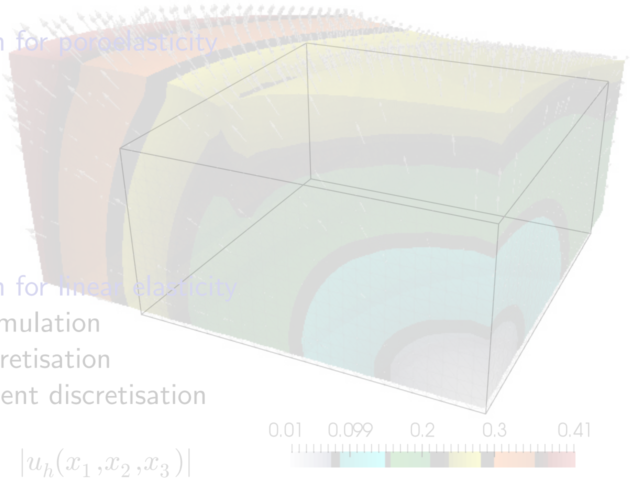
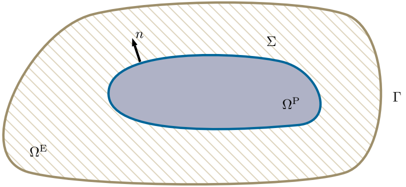


Coupled elasticity-poroelasticity



$$-\eta^{\mathrm{P}}\Delta \boldsymbol{u}^{\mathrm{P}} + \mathbf{div}(\phi \mathbf{I}) = \boldsymbol{f}^{\mathrm{P}} \quad \text{in } \Omega^{\mathrm{P}},$$
$$\phi - (\mu^{\mathrm{P}})^{-1}\eta^{\mathrm{P}} p^{\mathrm{P}} + \mathrm{div}(\boldsymbol{u}^{\mathrm{P}}) = 0 \quad \text{in } \Omega^{\mathrm{P}},$$
$$\left(c_0 + \alpha(\mu^{\mathrm{P}})^{-1}\eta^{\mathrm{P}}\right) p^{\mathrm{P}} - \alpha\phi - \frac{1}{\xi}\,\mathrm{div}\left[\kappa(\nabla p^{\mathrm{P}} - \rho \boldsymbol{g})\right] = s \quad \text{in } \Omega^{\mathrm{P}}.$$

$$\boldsymbol{u}^{\mathrm{P}} = \boldsymbol{u}^{\mathrm{E}}, \quad (\sigma^{\mathrm{E}} - \sigma^{\mathrm{P}})\boldsymbol{n} = \mathbf{0}, \quad \frac{\kappa}{\xi}(\nabla p^{\mathrm{P}} - \rho \boldsymbol{g}) \cdot \boldsymbol{n} = 0, \quad \text{on} \quad \Sigma$$

$$\sqrt{\eta^{\mathrm{E}}}\,\mathbf{curl}\,\omega + (1+\eta^{\mathrm{E}})\nabla p^{\mathrm{E}} = \boldsymbol{f}^{\mathrm{E}} \text{ in } \Omega^{\mathrm{E}},$$
$$\omega - \sqrt{\eta^{\mathrm{E}}}\,\mathbf{curl}\,\boldsymbol{u}^{\mathrm{E}} = 0 \quad \text{in } \Omega^{\mathrm{E}},$$
$$\mathrm{div}\,\boldsymbol{u}^{\mathrm{E}} + p^{\mathrm{E}} = 0 \quad \text{in } \Omega^{\mathrm{E}}.$$

$$\begin{pmatrix} \mathscr{A}_1 & \mathbf{0} & \mathscr{B}_1' & \mathscr{B}_3' \\ \mathbf{0} & \mathscr{A}_2 & -\mathscr{B}_2' & \mathbf{0} \\ \mathscr{B}_1 & \mathscr{B}_2 & -\mathscr{A}_3 & \mathbf{0} \\ \mathscr{B}_3 & \mathbf{0} & \mathbf{0} & \mathscr{A}_4 \end{pmatrix} \begin{pmatrix} \vec{\boldsymbol{u}} \\ \vec{p}^{\mathrm{P}} \\ \vec{\phi} \\ (\vec{\omega}, \vec{p}^{\mathrm{E}}) \end{pmatrix} = \begin{pmatrix} \mathscr{H} \\ \mathscr{G} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}$$

Ongoing extensions

Interface elasticity-poroelasticity problems

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- Analysis of the continuous formulation
  - Stability of all bilinear forms
  - Appropriate inf-sup conditions for the forms defining $\mathscr{B}_1$ and $\mathscr{B}_3$
  - Fredholm's alternative + two-fold saddle point theory
- Construction of a Galerkin method and derivation of error bounds
- Numerical validation and simulation of applicative problems
  - oil industry (reservoir and non-pay rock [Girault et al. 2011])
  - aircraft design (noise reduction [Rurkowska & Langer 2013])
  - dentistry (tooth and periodontal ligament [Favino et al. 2013])
  - cardiovascular models (blood cloth [Bukač 2016])
  - articular cartilage (structural response of joints [De Boer et al. 2017])
  - geotechnical structures (retaining walls, foundations [Zhang 2009])

Next steps

1. Poroelasticity applied to cardiac perfusion: modelling considerations and homogenisation framework for large strains; fixed-point analysis of mixed formulations and FE schemes
2. Porous-medium cardiac electromechanics: non-linear conductivities in the electrophysiology + geometric nonlinearities + stress-assisted conductivity + perfusion model from step 1
3. Interface conditions between myocardium and surrounding organs?

References

- R. Oyarzúa, R. Ruiz Baier, *Locking-free finite element methods for poroelasticity*. SINUM, **54** (2016) 2951–2973.
- V. Anaya, Z. De Wijn, D. Mora, R. Ruiz Baier, *Mixed displacement rotation pressure formulations for linear elasticity*. Submitted (2018).
- J.J. Lee, K.-A. Mardal, R. Winther, *Parameter-robust discretization and preconditioning of Biot's consolidation model*. SISC, **39** (2017) A1–A24.
- Z. De Wijn, R. Ruiz Baier, *A second order finite volume element method for three-field poroelasticity formulations*. Submitted (2018).
- S. Kumar, R. Oyarzúa, R. Ruiz-Baier, R. Sandilya, *Conservative mixed and discontinuous schemes for linear poroelasticity*. Submitted (2018).
- S.-Y. Yi, *Convergence analysis of a new mixed finite element method for Biot's consolidation model*. Numer. Methods PDEs, **30** (2014) 1189–1210.
- L. Berger, R. Bordas, D. Kay, S. Tavener, *Stabilized lowest-order approximation for linear three-field poroelasticity*. SISC, **37** (2015) A2222–A2245.

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