A Pseudo-Parabolic PDE for Compaction of a Sedimentary Basin

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A Two-phase Compaction Model

A **sedimentary basin** is a

- fully-saturated granular deforming porous medium
- consolidating under its own weight + overburden pressure.

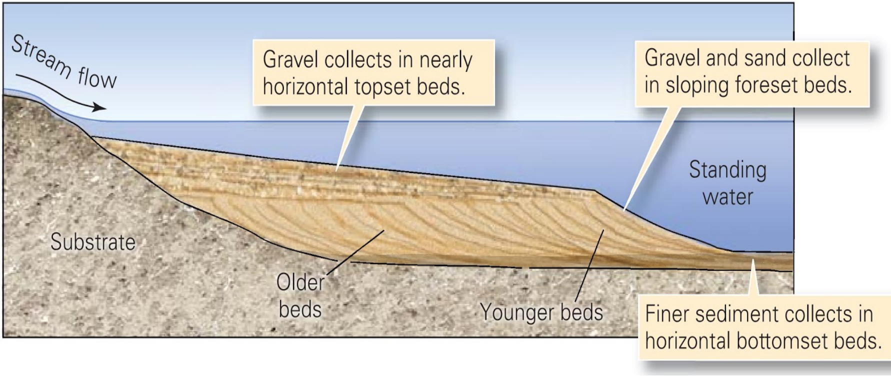
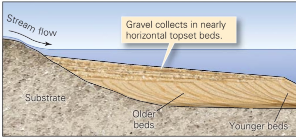


Figure: Wikipedia

The Spatial and Time Scales in Basin Modeling

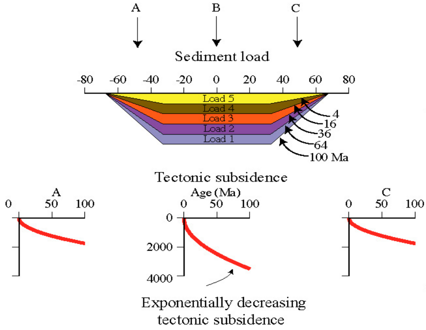


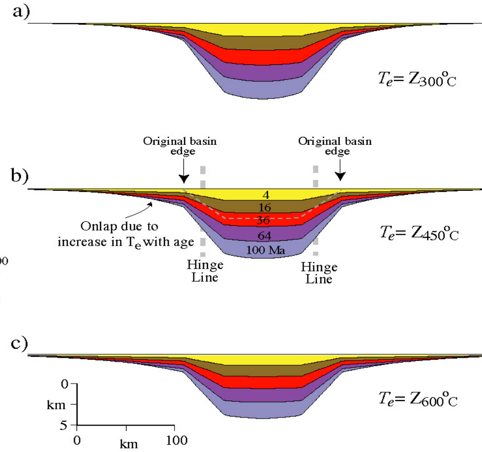
- Depths: kilometers
- Widths & lengths: 100s of kilometers
- Geological time scales: 50 Ma
- Pressures are huge!

Visco-elastic models needed for

- *viscous creep* ... shallow & intermediate depths,
- *pressure solution* ... about 500 meters and deeper,
- *tectonic subsidence* ... below 4 kilometers.

More on scales and dimensions





Most essential variation in the vertical direction

Laboratory Scales

Experiments on fluid driven deformation of

- soft open-cell polymer foams,
- visco-elastic granular medium of soft gel particles.

REFERENCES

- B. Sobac, M. Colombani, Y. Forterre, On the Dynamics of Poroelastic Foams, Mećanique & Industries **12** (2001), 231-238.
- C.W. MacMinn, E.R. Dufresne, J.S. Wettlaufer, Fluid-Driven Deformation of a Soft Granular Material, Physical Review X **5** (2015),

Experimental results

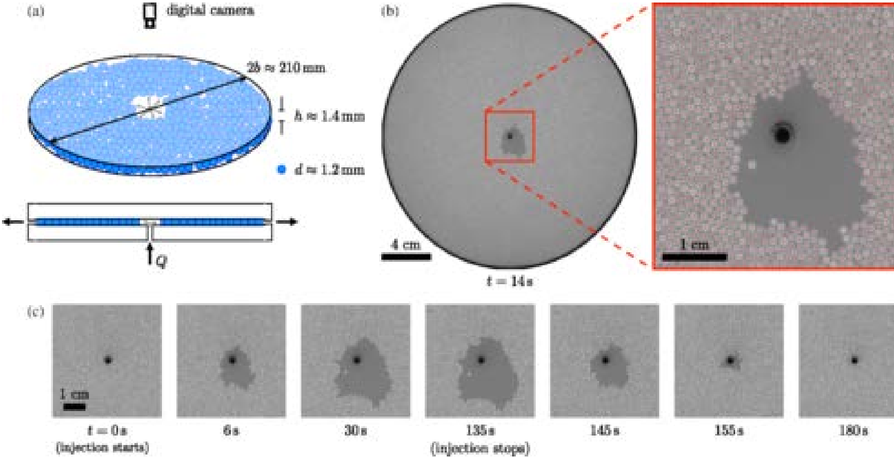


Figure: MacMinn et al (2015)

Basin model: variables and assumptions

The *variables* are

- **porosity** $\phi(x,t)$,
- **fluid pressure** $p(x,t)$.
- fluid velocity $\mathbf{v}_f(x,t)$ and solid velocity $\mathbf{v}_s(x,t)$,

The assumptions are

- **Shear stress** in the sediment matrix negligible:
  - **effective stress** is $\sigma(x,t)\delta = -p_s(x,t)\delta$
  - $p_s(x,t)$ is *effective pressure* of solid.
- **Overburden pressure** $P = p + p_s = p - \sigma$ is given.
- **Composite flow** $v_T = \phi\mathbf{v}_f + (1-\phi)\mathbf{v}_s$ is irrotational.

Model: conservation of mass

Mass conservation of **fluid** and **solid** (Audet & Fowler, 1992)

$$\frac{\partial \rho_f \phi}{\partial t} + \nabla \cdot (\rho_f \phi \mathbf{v}_f) = \rho_f F, \text{ and}$$
$$\frac{\partial \rho_s (1-\phi)}{\partial t} + \nabla \cdot (\rho_s (1-\phi) \mathbf{v}_s) = 0.$$

Divide out the (constant) densities and add the equations:

$$\nabla \cdot \mathbf{v}_T = \nabla \cdot (\phi \mathbf{v}_f + (1-\phi)\mathbf{v}_s) = F.$$

Note: $v_T \cdot n|_{boundary}$ to be used as a Neumann condition

Model: derive the porosity-pressure equation

Helmholtz theorem ($\nabla \times \mathbf{v}_T = 0$) implies

$$\mathbf{v}_T = \phi \mathbf{v}_f + (1-\phi)\mathbf{v}_s = \nabla w$$

To get w, we solve the Neumann problem

$$\nabla \cdot \nabla w = F, \quad \nabla w \cdot \mathbf{n}|_{boundary} = g, \quad (w \equiv \Delta^{-1} F)$$

Darcy's law for the fluid flow

$$\phi(\mathbf{v}_f - \mathbf{v}_s) = -\frac{k(\phi)}{\mu} \nabla p$$

Porosity-pressure equation

$$\frac{\partial \phi}{\partial t} - \nabla \cdot \left((1-\phi)\frac{k(\phi)}{\mu}\nabla p\right) = \nabla \cdot (1-\phi)\nabla \Delta^{-1} F$$

The Porosity-Stress Equation

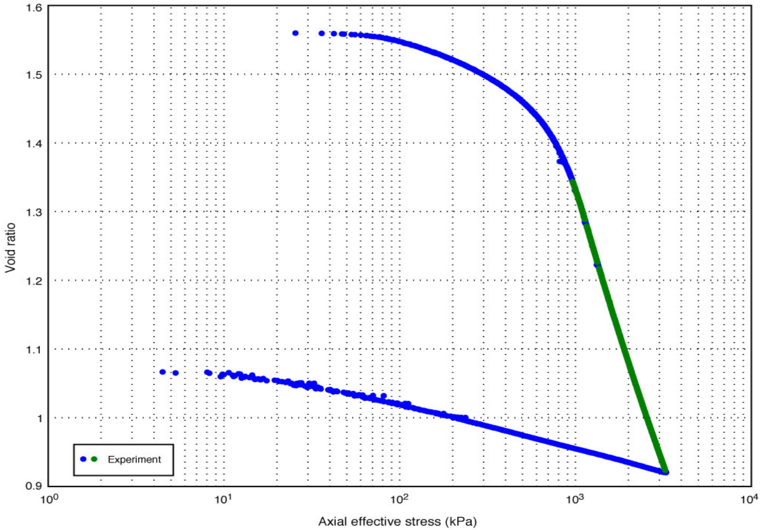
From the given overburden pressure $P = p - \sigma$, we obtain the **porosity-stress equation**

$$\frac{\partial \phi}{\partial t} - \nabla \cdot \left((1-\phi)\frac{k(\phi)}{\mu}\nabla(\sigma + P)\right) = \nabla \cdot (1-\phi)\nabla\Delta^{-1}F.$$

Typically $k(\phi) = \phi^m,\ m > 1$ (Carman-Kozeny)

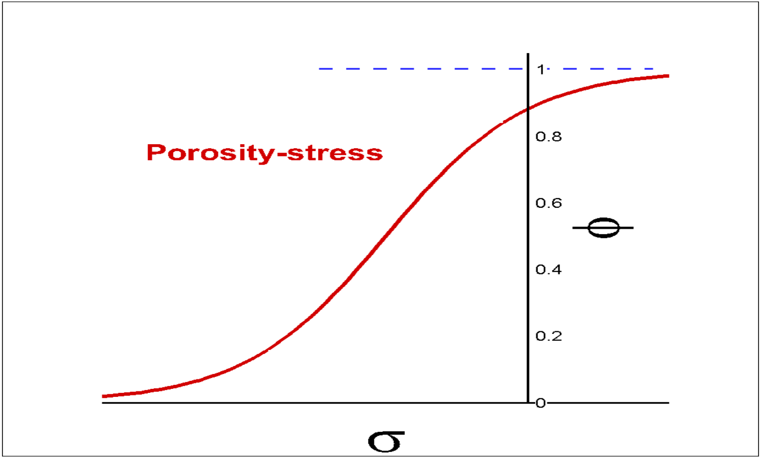
- It remains to relate the porosity ϕ and the stress σ to characterize the mechanical deformation of solid matrix ....

Void ratio $\frac{\phi}{1-\phi}$ & effective solid pressure 10^{p_s}



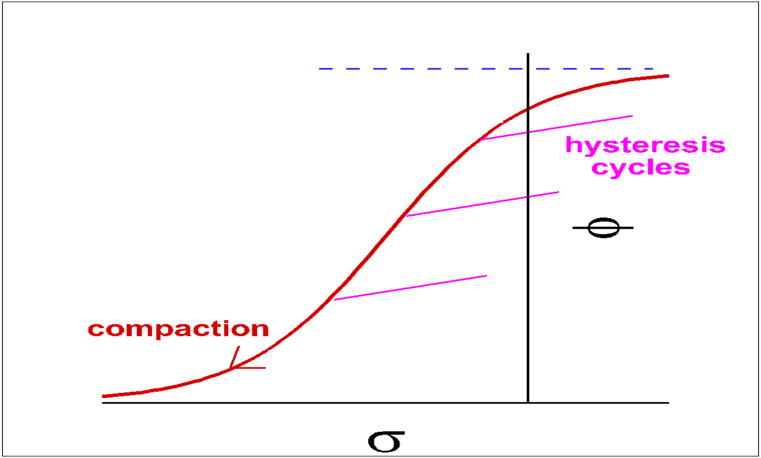
Porosity $\phi = \beta(\sigma)$ & effective stress $\sigma = -p_s$

$\phi = \beta(\sigma)$ ... nonlinear elastic compaction



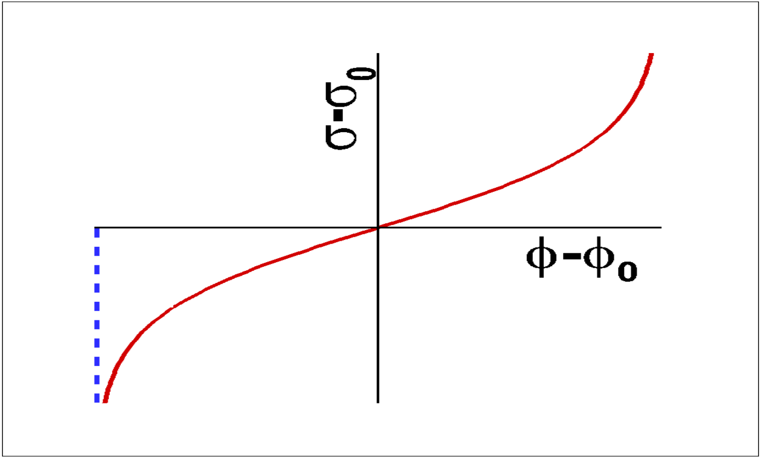
Hysteresis Cycles

$\phi = \beta(\sigma)$ ... unidirectional compaction



Stress $\sigma = \alpha(\phi)$ & porosity ϕ

$\alpha = \beta^{-1}$ ... the inverse of nonlinear compaction, translation



Constitutive assumption

Solid matrix is visco-elastic

$$\sigma = \alpha(\phi) + \eta \frac{\partial \phi}{\partial t}.$$

... **visco-elastic departures** from the compaction curves

- J.M. Sharp, Math. Geol. **8** (1976), 305-332.
- C.L. Angevine & D.L. Turcotte, Geol. Soc. Am. Bull. **94** (1983), 1129-1134.
- A.C. Fowler & X.S. Yang, SIAM Jour. Appl. Math. **59** (1998), 365-385.
- A. Rivel, Geophysical Research Letters **26** (1999), 255-258.
- A.C. Fowler & X.S. Yang, Jour Geophys Res., B **104** (1999), 989-997.
- X.S. Yang, Nonlinear Proc. Geophysics **7** (2000), 1-8.
- X.S. Yang, Geophysical Research Letters **29** (2002),

Model summary

The **pseudo-parabolic compaction equation**

$$\frac{\partial \phi}{\partial t} - \nabla\cdot (1-\phi)\frac{k(\phi)}{\mu}\nabla\left(\alpha(\phi) + \eta\frac{\partial \phi}{\partial t} + P\right) = \nabla\cdot (1-\phi)\nabla\Delta^{-1}F.$$

Assume

- $\alpha(\cdot)$ is continuous and affine bounded,

$$|\alpha(s)| \leq K_\alpha(|s|+1)$$

- $\alpha(\cdot) + KI$ is monotone for some $K \in \mathbb{R}$,
- $\eta > 0$.

What are pseudo-parabolic PDEs, and why ?

Simple linear pseudo-parabolic PDE

$$u_t - \eta \Delta u_t - \Delta u = 0$$

Nonlocal form

$$u_t - \Delta (I - \eta \Delta)^{-1} u = 0$$

Note: pseudo-parabolic ≈ parabolic

- $-\Delta (I - \eta \Delta)^{-1} = \frac{1}{\eta}(I - (I - \eta \Delta)^{-1})$
- bounded linear operator with norm $O(\frac{1}{\eta})$
- Yosida approximation of parabolic equation

$$u_t - \Delta u = 0 \qquad (\eta = 0)$$

Sobolev equation

$$u_{tt} - \eta \Delta u_{tt} - \Delta u = 0$$

rotating fluids, $\eta =$ inertia

- Sobolev (1954)
- A.E.H. Love (1940)
- Poincaré (1885)

General PDE of Sobolev type: **implicit evolution equations**

Pseudo-parabolic equation

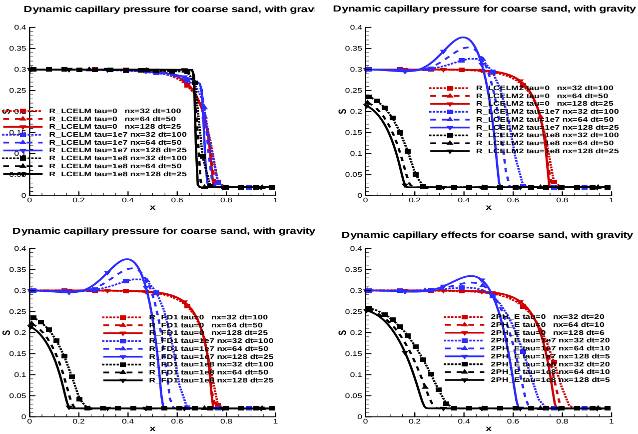
$$u_t - \eta \Delta u_t - \Delta u = 0$$

- Milne (1926) ... gas diffusion with delay
- Coleman-Noll (1960) ... 2^{nd}-order fluids
- Barenblatt (1960) ... dual-porosity porous media flow
- Chen-Gurtin (1968) ... two-temperature heat conduction
- Showalter-Ting (1970) ... existence, regularity, ...
- Carroll-Showalter, Singular & Degenerate Cauchy Problems (1976) ... 150 references !

Richards' equation with dynamic capillary pressure

- M. Hassanizadeh, Celia, Dahle (2002) ... experiments & model
- M. Peszynska - S. Yi (2008) ... numerical
- M. Peszynska - RES - S. Yi (2009) ... homogenization
- A. Mikelic (2010) ... existence - highly degenerate case

Numerical experiments: Peszynska-Yi (2008)



Instabilities arise from advection with some methods.

The Compaction Model: Theory

For each $\phi \in L^\infty(G),\ 0 < \delta \le \phi < 1$, define the elliptic operator

$$A_\phi = -\nabla \cdot (1-\phi)\frac{k(\phi)}{\mu}\nabla(\cdot).$$

The nonlinear pseudo-parabolic **compaction equation** takes the form

$$\frac{\partial \phi}{\partial t} + A_\phi\left(\alpha(\phi) + \eta\frac{\partial \phi}{\partial t} + P(t)\right) = f(\phi, t), \quad \phi(0) = \phi_0.$$

Rewrite in the function space $L^2(0, T; L^2(G))$

$$\frac{\partial \phi}{\partial t} + \frac{1}{\eta}(\alpha(\phi)+P) = (I+\eta A_\phi)^{-1}(\frac{1}{\eta}(\alpha(\phi)+P)+f(\phi)), \quad \phi(0) = \phi_0.$$

Main analysis results [SIAM Jour Math Anal 50 (2018)]

Theorem 1. Let $\phi_0 \in L^2(G)$. There exists a solution in $H^1(0, T; L^2(G))$.
It is obtained as a fixed point and satisfies

$$\eta\phi'(t) + \alpha(\phi(t)) + P(t) \in L^2(0, T; H_0^1(G)).$$

Theorem 2. Regularity (both local and global) of initial data is preserved up to $H^1(G)$. That is, for any subdomain $G_0 \subset G$, if $\phi_0 \in H^1(G_0)$, then $\phi(t) \in H^1(G_0)$ for $t \in [0, T]$.

Theorem 3. Any jump $[\phi(x, t)]$ in values along an interior submanifold Γ decays according to the ODE

$$\eta\frac{d[\phi(x,t)]}{dt} + \alpha([\phi(x, t)]) = 0, \ \ x \in \Gamma.$$

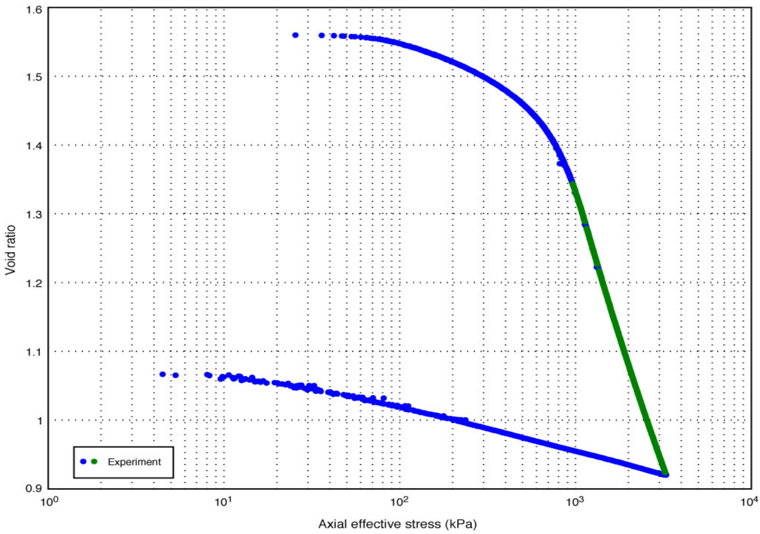
Challenge 1
The Hysteresis Problem

We model the **irreversible compaction** with the system

$$\frac{\partial\phi}{\partial t} - \nabla\cdot(1-\phi)\frac{k(\phi)}{\mu}\nabla(\sigma + P) = \nabla\cdot(1-\phi)\nabla\Delta^{-1}F,$$
$$\phi = \mathcal{H}(\sigma),$$

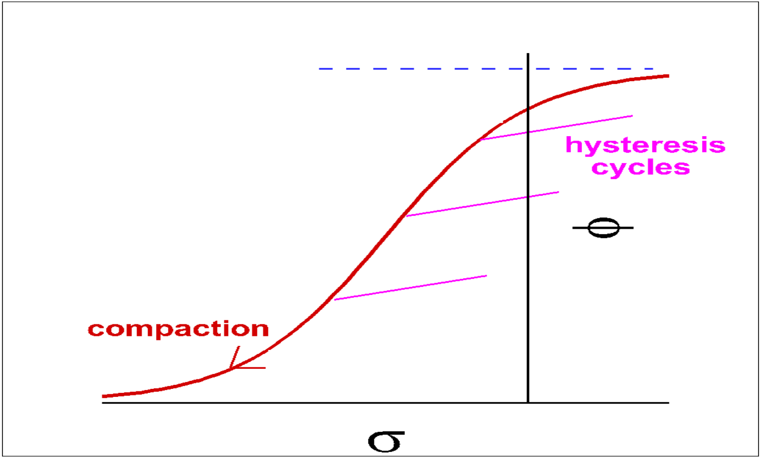
where $\mathcal{H}$ is a **hysteresis functional**.

Porosity & effective solid pressure



Hysteresis Cycles

$$\phi \in \mathcal{H}(\sigma)$$



Compaction model with hysteresis

The hysteresis functional is modeled by the singular ODE

$$\phi = \beta(\xi), \ \frac{\partial \phi}{\partial t} + c(\xi - \sigma) \ni 0,$$

where $c(\cdot)$ is the constraint graph, ξ: auxiliary variable

$$c(s) = -m \text{ if } s < 0, \quad c(0) = [-m, +\infty),$$

($m \geq 0$: slope of hysteresis cycles)

The model is now the system

$$\frac{\partial \phi}{\partial t} - \nabla \cdot \left[(1-\phi) \frac{k(\phi)}{\mu} \nabla (\sigma + P) \right] = \nabla \cdot (1-\phi) \nabla \Delta^{-1} F,$$

$$\phi = \beta(\xi), \ \frac{\partial \phi}{\partial t} + c(\xi - \sigma) \ni 0.$$

Challenge 2
A Free-Boundary Problem

At the basin surface, there is another problem.
The *regions* are *sediment* $G_T^s = \{(x,t) \in G_T : \sigma(x,t) < 0\}$, the pure *fluid* $G_T^0 = \{(x,t) \in G_T : \sigma(x,t) = 0\}$, and their *interface* $S = \overline{G_T^s} \cap G_T^0$.

A variational inequality describes the unknown surface S:

- Sediment can not support any tensile stress: $\sigma \leq 0$,
- The surface is given by $S = \{(x,t) : \sigma(x,t) = 0\}$, and
- $\mathbf{v}_s \cdot \mathbf{n} = V$ is the normal velocity of the surface.

The last condition determines the motion of S.

Summary

- Model
  Darcy flow & irrotational total fluid & viscoelastic or hysteretic constituve assumptions
- Analysis & recent references to pseudo-parabolic PDEs
  Eleanor Holland, RES,
  "Poro-visco-elastic Compaction in Sedimentary Basins",
  SIAM Jour Math Anal (2018)
- Current work on remaining challenges

Thank you!