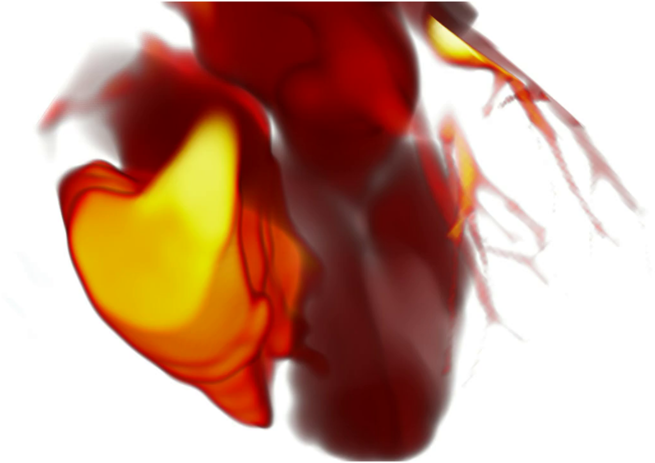

A stable scheme for simulation of incompressible flows in time-dependent domains and hemodynamic applications

Yuri Vassilevski[1,2,3]

Maxim Olshanskii[4]   Alexander Danilov[1,2,3]

Alexander Lozovskiy[1]   Victoria Salamatova[1,2,3]

[1]Institute of Numerical Mathematics RAS
[2]Moscow Institute of Physics and Technology
[3]Sechenov University
[4]University of Houston

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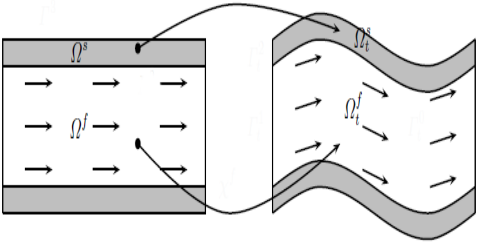
Fluid-Structure Interaction

Fluid-Structure Interaction problem

Prerequisites for FSI



- reference subdomains Ω_f, Ω_s
- transformation $\boldsymbol{\xi}$ maps Ω_f, Ω_s to $\Omega_f(t)$, $\Omega_s(t)$
- $\mathbf{v}$ and $\mathbf{u}$ denote velocities and displacements in $\widehat{\Omega} := \Omega_f \cup \Omega_s$
- $\boldsymbol{\xi}(\mathbf{x}) := \mathbf{x} + \mathbf{u}(\mathbf{x})$, $\mathbf{F} := \nabla\boldsymbol{\xi} = \mathbf{I} + \nabla\mathbf{u}$, $J := \det(\mathbf{F})$
- Cauchy stress tensors $\boldsymbol{\sigma}_f$, $\boldsymbol{\sigma}_s$
- pressures p_f,p_s
- density ρ_f is constant

Fluid-Structure Interaction problem

Universal equations in reference subdomains

Dynamic equations

$$\frac{\partial \mathbf{v}}{\partial t} = \begin{cases} \rho_s^{-1} \operatorname{div}\left(J \boldsymbol{\sigma}_s \mathbf{F}^{-T}\right) & \text{in } \Omega_s, \\ (J\rho_f)^{-1} \operatorname{div}\left(J \boldsymbol{\sigma}_f \mathbf{F}^{-T}\right) - \nabla \mathbf{v} \left(\mathbf{F}^{-1} \left(\mathbf{v} - \frac{\partial \mathbf{u}}{\partial t} \right) \right) & \text{in } \Omega_f \end{cases}$$

Fluid-Structure Interaction problem

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Kinematic equation

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{v} \quad \text{in } \Omega_s$$

Fluid-Structure Interaction problem

Universal equations in reference subdomains

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Kinematic equation

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{v} \quad \text{in } \Omega_s$$

Fluid incompressibility

$$\mathrm{div}\left(J\mathbf{F}^{-1}\mathbf{v}\right) = 0 \quad \text{in } \Omega_f \quad \text{or} \quad J\nabla \mathbf{v} : \mathbf{F}^{-T} = 0 \quad \text{in } \Omega_f$$

Fluid-Structure Interaction problem

Universal equations in reference subdomains

Dynamic equations

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Fluid incompressibility

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Constitutive relation for the fluid stress tensor

$$\boldsymbol{\sigma}_f = -p_f \mathbf{I} + \mu_f((\nabla \mathbf{v})\mathbf{F}^{-1} + \mathbf{F}^{-T}(\nabla \mathbf{v})^T) \quad \text{in } \Omega_f$$

FSI problem

User-dependent equations in reference subdomains

Constitutive relation for the solid stress tensor

$$\boldsymbol{\sigma}_s = \boldsymbol{\sigma}_s(J, \mathbf{F}, p_s, \lambda_s, \mu_s, \dots) \quad \text{in } \Omega_s$$

[1] Michler et al (2004), Hubner et al (2004), Hron&Turek (2006),...

FSI problem

User-dependent equations in reference subdomains

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Monolithic approach[1]: Extension of the displacement field to the fluid domain

$$\begin{aligned} G(\mathbf{u}) &= 0 \quad \text{in } \Omega_f, \\ \mathbf{u} &= \mathbf{u}^* \quad \text{on } \partial\Omega_f \end{aligned}$$

[1] Michler et al (2004), Hubner et al (2004), Hron&Turek (2006),...

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for example, vector Laplace equation or elasticity equation

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FSI problem

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for example, vector Laplace equation or elasticity equation

$+$ Initial, boundary, interface conditions $(\boldsymbol{\sigma}_f \mathbf{F}^{-T}\mathbf{n} = \boldsymbol{\sigma}_s \mathbf{F}^{-T}\mathbf{n})$

[1] Michler et al (2004), Hubner et al (2004), Hron&Turek (2006),...

Numerical scheme

- Conformal triangular or tetrahedral mesh Ω_h in $\widehat{\Omega}$
- LBB-stable pair for velocity and pressure P_2/P_1, P_2 for displacements
- Fortran open source software `Ani2D`, `Ani3D` (Advanced numerical instruments 2D/3D, K.Lipnikov, Yu.Vassilevski et al.)
  `http://sf.net/p/ani2d/` `http://sf.net/p/ani3d/`:
  - mesh generation
  - FEM systems
  - algebraic solvers

Numerical scheme

Find $\{\mathbf{u}^{k+1}, \mathbf{v}^{k+1}, p^{k+1}\} \in \mathbb{V}_h^0 \times \mathbb{V}_h \times \mathbb{Q}_h$ s.t.

$$\mathbf{v}^{k+1} = \mathbf{g}_h(\cdot, (k+1)\Delta t) \text{ on } \Gamma_{f0}, \quad \left[\frac{\partial \mathbf{u}}{\partial t}\right]_{k+1} = \mathbf{v}^{k+1} \text{ on } \Gamma_{fs}$$

Numerical scheme

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where

$$\mathbb{V}_h \subset H^1(\widehat{\Omega})^3, \mathbb{Q}_h \subset L^2(\widehat{\Omega}), \mathbb{V}_h^0 = \{\mathbf{v} \in \mathbb{V}_h \,:\, \mathbf{v}|_{\Gamma_{s0} \cup \Gamma_{f0}} = \mathbf{0}\}, \mathbb{V}_h^{00} = \{\mathbf{v} \in \mathbb{V}_h^0 \,:\, \mathbf{v}|_{\Gamma_{fs}} = \mathbf{0}\}$$

$$\left[\frac{\partial \mathbf{f}}{\partial t}\right]_{k+1} := \frac{3\mathbf{f}^{k+1} - 4\mathbf{f}^k + \mathbf{f}^{k-1}}{2\Delta t}$$

Numerical scheme

$$
\int_{\Omega_s} \rho_s \left[\frac{\partial \mathbf{v}}{\partial t}\right]_{k+1} \psi \, \mathrm{d}\Omega + \int_{\Omega_s} J_k \mathbf{F}(\widetilde{\mathbf{u}}^k)\mathbf{S}(\mathbf{u}^{k+1}, \widetilde{\mathbf{u}}^k) : \nabla \psi \, \mathrm{d}\Omega +
$$

$$
\int_{\Omega_f} \rho_f J_k \left[\frac{\partial \mathbf{v}}{\partial t}\right]_{k+1} \psi \, \mathrm{d}\Omega + \int_{\Omega_f} \rho_f J_k \nabla \mathbf{v}^{k+1} \mathbf{F}^{-1}(\widetilde{\mathbf{u}}^k) \left(\widetilde{\mathbf{v}}^k - \widetilde{\left[\frac{\partial \mathbf{u}}{\partial t}\right]_k} \right) \psi \, \mathrm{d}\Omega +
$$

$$
\int_{\Omega_f} 2\mu_f J_k \mathbf{D}_{\widetilde{\mathbf{u}}^k} \mathbf{v}^{k+1} : \mathbf{D}_{\widetilde{\mathbf{u}}^k} \psi \, \mathrm{d}\Omega - \int_{\Omega} p^{k+1} J_k \mathbf{F}^{-T}(\widetilde{\mathbf{u}}^k) : \nabla \psi \, \mathrm{d}\Omega = 0 \quad \forall \psi \in \mathbb{V}_h^0
$$

$$
J_k := J(\widetilde{\mathbf{u}}^k), \quad \widetilde{\mathbf{f}}^k := 2\mathbf{f}^k - \mathbf{f}^{k-1}, \quad \mathbf{D}_{\mathbf{u}} \mathbf{v} := \{\nabla \mathbf{v} \mathbf{F}^{-1}(\mathbf{u})\}_s, \quad \{\mathbf{A}\}_s := \frac{1}{2}(\mathbf{A} + \mathbf{A}^T)
$$

Numerical scheme

$$\int_{\Omega_s} \rho_s \left[\frac{\partial \mathbf{v}}{\partial t}\right]_{k+1} \psi \, \mathrm{d}\Omega + \int_{\Omega_s} J_k \mathbf{F}(\widetilde{\mathbf{u}}^k)\mathbf{S}(\mathbf{u}^{k+1}, \widetilde{\mathbf{u}}^k) : \nabla \psi \, \mathrm{d}\Omega +$$
$$\int_{\Omega_f} \rho_f J_k \left[\frac{\partial \mathbf{v}}{\partial t}\right]_{k+1} \psi \, \mathrm{d}\Omega + \int_{\Omega_f} \rho_f J_k \nabla \mathbf{v}^{k+1} \mathbf{F}^{-1}(\widetilde{\mathbf{u}}^k) \left(\widetilde{\mathbf{v}}^k - \widetilde{\left[\frac{\partial \mathbf{u}}{\partial t}\right]}_k\right) \psi \, \mathrm{d}\Omega +$$
$$\int_{\Omega_f} 2\mu_f J_k \mathbf{D}_{\widetilde{\mathbf{u}}^k} \mathbf{v}^{k+1} : \mathbf{D}_{\widetilde{\mathbf{u}}^k} \psi \, \mathrm{d}\Omega - \int_{\Omega} p^{k+1} J_k \mathbf{F}^{-T}(\widetilde{\mathbf{u}}^k) : \nabla \psi \, \mathrm{d}\Omega = 0 \quad \forall \psi \in \mathbb{V}_h^0$$

$$\int_{\Omega_s} \left[\frac{\partial \mathbf{u}}{\partial t}\right]_{k+1} \phi \, \mathrm{d}\Omega - \int_{\Omega_s} \mathbf{v}^{k+1} \phi \, \mathrm{d}\Omega + \int_{\Omega_f} G(\mathbf{u}^{k+1}) \phi \, \mathrm{d}\Omega = 0 \quad \forall \phi \in \mathbb{V}_h^{00}$$

$$J_k := J(\widetilde{\mathbf{u}}^k), \quad \widetilde{\mathbf{f}}^k := 2\mathbf{f}^k - \mathbf{f}^{k-1}, \quad \mathbf{D}_{\mathbf{u}}\mathbf{v} := \{\nabla \mathbf{v} \mathbf{F}^{-1}(\mathbf{u})\}_s, \quad \{\mathbf{A}\}_s := \frac{1}{2}(\mathbf{A} + \mathbf{A}^T)$$

Numerical scheme

$$\int_{\Omega_s} \rho_s \left[\frac{\partial \mathbf{v}}{\partial t}\right]_{k+1} \psi \, \mathrm{d}\Omega + \int_{\Omega_s} J_k \mathbf{F}(\widetilde{\mathbf{u}}^k)\mathbf{S}(\mathbf{u}^{k+1}, \widetilde{\mathbf{u}}^k) : \nabla\psi \, \mathrm{d}\Omega +$$
$$\int_{\Omega_f} \rho_f J_k \left[\frac{\partial \mathbf{v}}{\partial t}\right]_{k+1} \psi \, \mathrm{d}\Omega + \int_{\Omega_f} \rho_f J_k \nabla\mathbf{v}^{k+1}\mathbf{F}^{-1}(\widetilde{\mathbf{u}}^k)\left(\widetilde{\mathbf{v}}^k - \widetilde{\left[\frac{\partial \mathbf{u}}{\partial t}\right]}_k\right)\psi \, \mathrm{d}\Omega +$$
$$\int_{\Omega_f} 2\mu_f J_k \mathbf{D}_{\widetilde{\mathbf{u}}^k}\mathbf{v}^{k+1} : \mathbf{D}_{\widetilde{\mathbf{u}}^k}\psi \, \mathrm{d}\Omega - \int_{\Omega} p^{k+1} J_k \mathbf{F}^{-T}(\widetilde{\mathbf{u}}^k) : \nabla\psi \, \mathrm{d}\Omega = 0 \quad \forall\psi \in \mathbb{V}_h^0$$

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$$\int_{\Omega_f} J_k \nabla\mathbf{v}^{k+1} : \mathbf{F}^{-T}(\widetilde{\mathbf{u}}^k) q \, \mathrm{d}\Omega = 0 \quad \forall\ q \in \mathbb{Q}_h$$

$$J_k := J(\widetilde{\mathbf{u}}^k), \quad \widetilde{\mathbf{f}}^k := 2\mathbf{f}^k - \mathbf{f}^{k-1}, \quad \mathbf{D}_{\mathbf{u}}\mathbf{v} := \{\nabla\mathbf{v}\mathbf{F}^{-1}(\mathbf{u})\}_s, \quad \{\mathbf{A}\}_s := \frac{1}{2}(\mathbf{A} + \mathbf{A}^T)$$

Numerical scheme

$$\ldots + \int_{\Omega_s} J_k \mathbf{F}(\widetilde{\mathbf{u}}^k)\mathbf{S}(\mathbf{u}^{k+1}, \widetilde{\mathbf{u}}^k) : \nabla \psi \, \mathrm{d}\Omega + \ldots$$

- St. Venant–Kirchhoff model (geometrically nonlinear):

$$\mathbf{S}(\mathbf{u}_1, \mathbf{u}_2) = \lambda_s \mathtt{tr}(\mathbf{E}(\mathbf{u}_1, \mathbf{u}_2))\mathbf{I} + 2\mu_s \mathbf{E}(\mathbf{u}_1, \mathbf{u}_2);$$
$$\mathbf{E}(\mathbf{u}_1, \mathbf{u}_2) = \{\mathbf{F}(\mathbf{u}_1)^T \mathbf{F}(\mathbf{u}_2) - \mathbf{I}\}_s$$

- inc. Blatz–Ko model:

$$\mathbf{S}(\mathbf{u}_1, \mathbf{u}_2) = \mu_s(\mathtt{tr}(\{\mathbf{F}(\mathbf{u}_1)^T \mathbf{F}(\mathbf{u}_2)\}_s)\mathbf{I} - \{\mathbf{F}(\mathbf{u}_1)^T \mathbf{F}(\mathbf{u}_2)\}_s)$$

- inc. Neo-Hookean model:

$$\mathbf{S}(\mathbf{u}_1, \mathbf{u}_2) = \mu_s \mathbf{I}; \;\; \mathbf{F}(\widetilde{\mathbf{u}}^k) \to \mathbf{F}(\mathbf{u}^{k+1})$$

$$\{\mathbf{A}\}_s := \frac{1}{2}(\mathbf{A} + \mathbf{A}^T)$$

Numerical scheme

The scheme

- provides strong coupling on interface
- semi-implicit
- produces one linear system per time step
- second order in time

Numerical scheme

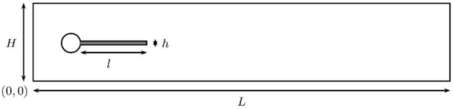
The scheme

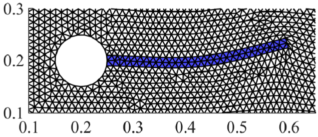
- provides strong coupling on interface
- semi-implicit
- produces one linear system per time step
- second order in time
- unconditionally stable (no CFL restriction), proved with assumptions:
  - 1st order in time
  - St. Venant–Kirchhoff inc./comp. (experiment: Neo-Hookean inc./comp.)
  - extension of $\mathbf{u}$ to Ω_f guarantees $J_k > 0$
  - Δt is not large

A.Lozovskiy, M.Olshanskii, V.Salamatova, Yu.Vassilevski. An unconditionally stable semi-implicit FSI finite element method. *Comput.Methods Appl.Mech.Engrg., 297, 2015*

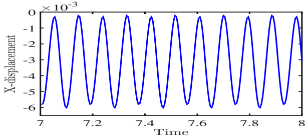
Validation in 2D: FSI3 benchmark problem

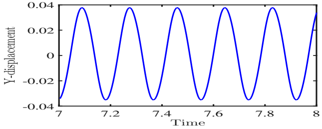
S. Turek and J. Hron. Proposal for numerical benchmarking of fluid-structure interaction between an elastic object and laminar incompressible flow. In: *Fluid-structure interaction*, Springer Berlin Heidelberg, 371–385, 2006.





- fluid: 2D transient Navier-Stokes, $\rho_f = 1000$, $\mu_f = 1$
- stick: SVK constitutive relation, $\rho_s = 1000$, $\lambda_s = 4\mu_s = 8 \cdot 10^6$
- inflow: parabolic velocity profile
- outflow: "do-nothing"
- rigid walls: no-slip condition
- $\Delta t = 10^{-3}$ until $T = 8$





Displacement extension in fluid domain: linear elasticity with $\mu_m = 20\mu_s$ and $\lambda_m = 20\lambda_s$ for adjacent to the beam elements

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	# of cells in Ω_f	# of cells in Ω_s	# of DOFs
Mesh 1	8652	162	76557
Mesh 2	17540	334	154242
Mesh 3	35545	658	310997

Mesh/method	$u_x \cdot 10^3$	$u_y \cdot 10^3$	F_D	F_L
1	-2.8 ± 2.6	1.5 ± 34.3	432.9 ± 22.3	0.98 ± 152.1
2	-3.0 ± 2.8	1.4 ± 35.9	453.8 ± 26.8	2.6 ± 154.0
3	$\mathbf{-3.0 \pm 2.9}$	$\mathbf{1.4 \pm 36.1}$	$\mathbf{458.0 \pm 27.6}$	$\mathbf{3.0 \pm 154.5}$
Turek, S. et al	$[-3.04, -2.84]$ $\pm[2.67, 2.87]$	$[1.28, 1.55]$ $\pm[34.61, 46.63]$	$[452.4, 474.9]$ $\pm[26.19, 36.63]$	$[1.81, 3.86]$ $\pm[152.7, 165.9]$
Liu, J.	-2.91 ± 2.74	1.46 ± 35.2	460.3 ± 27.67	2.41 ± 157

computed statistics for FSI3 test for the time interval [7,8]

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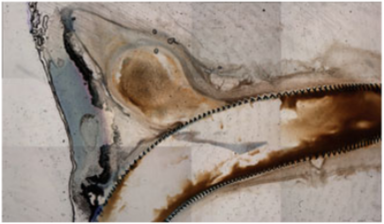
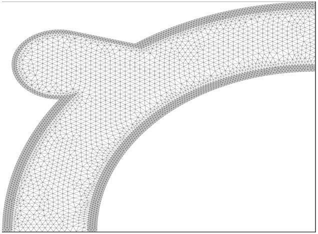
Displacement extension in fluid domain:

- Harmonic $\rightarrow$ mesh tangling
- Linear elasticity with $\mu_m = \mu_s$ and $\lambda_m = \lambda_s$ $\rightarrow$ mesh tangling
- Linear elasticity with $\mu_m = 20\mu_s$ and $\lambda_m = 20\lambda_s$ for adjacent to the beam elements $\rightarrow$ OK

2D test: blood vessel with aneurysm

S. Turek et al. Numerical simulation and benchmarking of a monolithic multigrid solver for fluid-structure interaction problems with application to hemodynamics. In: *Fluid Structure Interaction II*, Springer Berlin Heidelberg, 193–220, 2010.





- Investigating sensitivity to compressibility of the vessel material: measuring wall shear stress (WSS) since it serves as a good indicator for the risk of aneurysm rupture
- Showing reliability of the semi-implicit scheme for hemodynamic applications

2D test: blood vessel with aneurysm

S. Turek et al. Numerical simulation and benchmarking of a monolithic multigrid solver for fluid-structure interaction problems with application to hemodynamics. In: *Fluid Structure Interaction II*, Springer Berlin Heidelberg, 193–220, 2010.

- Material properties:

ρ_s	μ_s	ρ_f	μ_f
$1.12 \cdot 10^3$ kg/m^3	270000 Pa	$1.035 \cdot 10^3$ kg/m^3	$3.4983 \cdot 10^{-3}$ Pa $\cdot$ s

- Weakly compressible neo-Hookean model (μ_s for dog's artery):

$$\boldsymbol{\sigma}_s = \frac{\mu_s}{J^2}\left(\mathbf{F}\mathbf{F}^T - \frac{1}{2}\mathrm{tr}\,(\mathbf{F}\mathbf{F}^T)\mathbf{I}\right) + \left(\lambda_s + \frac{2\mu_s}{3}\right)(J-1)\mathbf{I}, \quad \lambda_s \to \infty$$

  Extrapolation is used in the model to retain semi-implicitness

- Pulsatile parabolic inflow profile:

$$v_1(0, y, t) = -50(8-y)(y-6)(1+0.75\sin(2\pi t)), \quad 6 \le y \le 8.$$

- λ_s takes values 10^4, 10^6, 10^8 kPa, i.e. Poisson's ratio $\nu \to 0.5$.
- Time step $\Delta t = 10^{-3}$ s until $T = 3$ s.
- Elasticity based displacement extension with $\mu_m = \mu_s$, $\lambda_m = 4\lambda_s$.

2D test: blood vessel with aneurysm

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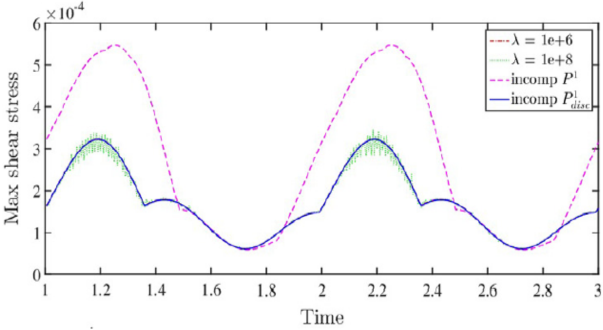
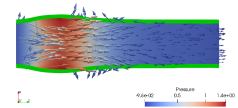


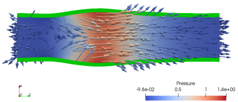
WSS for weakly incompressible and fully incompressible cases, with continuous and discontinuous pressure at the interface

Best choices (area of wall, WSS): Neo-Hookean compressible with moderate λ_s and incompressible with discontinuous pressures.

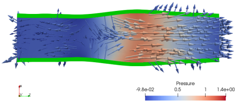
3D: pressure wave in flexible tube



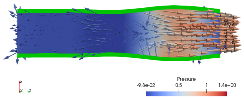
$t = 0.004s$



$t = 0.006s$



$t = 0.008s$

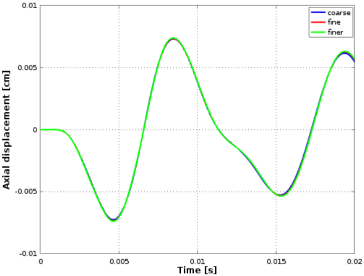


$t = 0.01s$

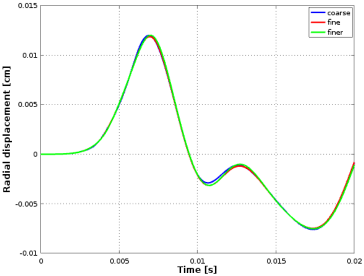
Pressure wave: middle cross-section velocity field, pressure distribution, velocity vectors and 10× enlarged structure displacement for several time instances.

- The tube (fixed at both ends) is 50mm long, it has inner diameter of 10mm and the wall (SVK) is 1mm thick.
- Left end: external pressure p_{ext} is set to $1.333 \cdot 10^3$Pa for $t \in (0, 3 \cdot 10^{-3})$s and zero afterwards, $\boldsymbol{\sigma}_f \mathbf{F}^{-T}\mathbf{n} = p_{ext}\mathbf{n}$. Right end: open boundary
- Simulation was run with $\Delta t = 10^{-4}$ s
- $\# Tets(\Omega_s) = 6336/11904/38016$, $\# Tets(\Omega_f) = 13200/29202/89232$

3D: pressure wave in flexible tube



axial component



radial component

Pressure wave: The radial and axial components of displacement of the inner tube wall at half the length of the pipe. Solutions are shown for three mesh sequentially refined meshes. The plots are almost indistinguishable.

3D: pressure wave in flexible tube

displacement extension in Ω_f

M.Landajuela et al. Coupling schemes for the FSI forward prediction challenge: comparative study and validation. *Int. J. for Numer. Meth. in Biomed. Engng.*, 33, 2017.

- Linear elasticity model is used for the **update** of the displacement extension in Ω_f

$$
-\operatorname{div}\left[J\left(\lambda_m \mathrm{tr}\left(\nabla\left[\frac{\partial \mathbf{u}}{\partial t}\right]^k \mathbf{F}^{-1}\right)\mathbf{I}\right.\right.
$$

$$
\left.\left.+\mu_m\left(\nabla\left[\frac{\partial \mathbf{u}}{\partial t}\right]^k \mathbf{F}^{-1}+\left(\nabla\left[\frac{\partial \mathbf{u}}{\partial t}\right]^k \mathbf{F}^{-1}\right)^T\right)\right)\mathbf{F}^{-T}\right]=0 \quad \text{in } \Omega_f,
$$

- the Lame parameters are element-volume dependent:

$$
\lambda_m = 16\mu_m = 16\frac{\mu_s}{v_e^{1.2}}
$$

3D: silicone filament in glycerol

Benchmark challenge for CMBE 2015, Paris

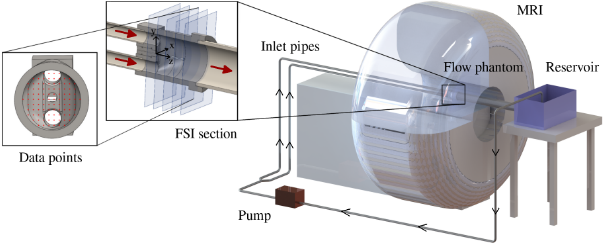
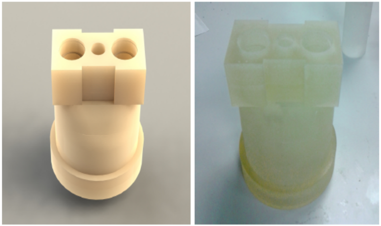
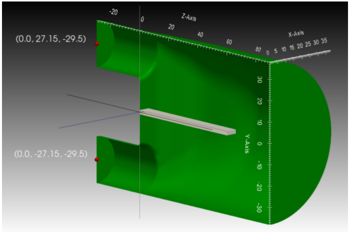


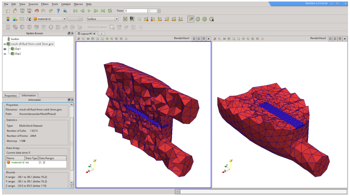
Image from A. Hessenthaler et al. Experiment for validation of fluid-structure interaction models and algorithms. *Int. J. for Numer. Meth. Biomed. Engng.*, 2017

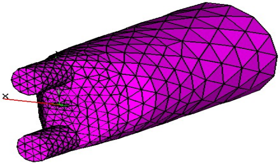
3D: silicone filament in glycerol





Meshed volume: original and extended domains.



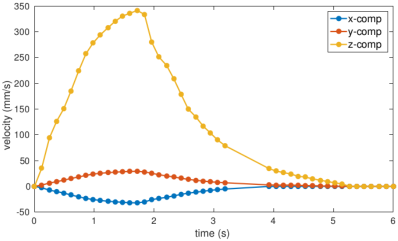


3D: silicone filament in glycerol

- Steady and pulsatile flow regimes

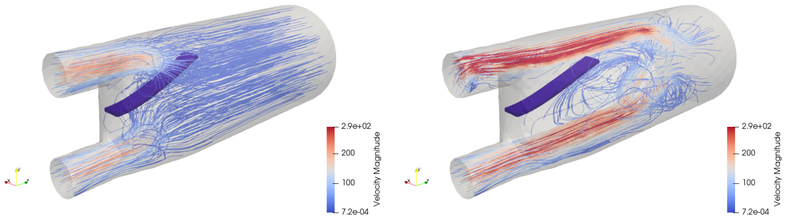
	Phase I	Phase II
velocity	stationary	pulsatile
ρ_f	$1.1633 \cdot 10^{-3}$ g mm^3	$1.164 \cdot 10^{-3}$ g mm^{-3}
μ_f	$12.5 \cdot 10^{-3}$ g mm^{-1}s^{-1}	$13.37 \cdot 10^{-3}$ g mm^{-1}s^{-1}

- Inflow velocities for one cycle of frequency 1/6 Hz for phase II:

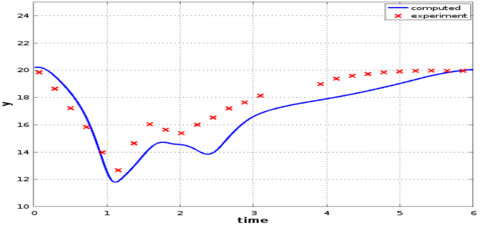


- Simulation was run with $\Delta t = 10^{-2}$ s, $t \in [0, 12]$
- $\#Tets(\Omega_s) = 733$, $\#Tets(\Omega_f) = 28712$, $\#unknowns = 254439$
- The filament (SVK) is lighter than the fluid and deflects upward
- Linear elasticity model is used for the **update** of the displacement extension in Ω_f, the Lame parameters are element-volume dependent

3D: silicone filament in glycerol



Streamlines colored by the velocity magnitude $t = 0.721s$ (left), $t = 2.017s$ (right)



Track of the computed y-displacement of the point in the structure with coordinate $z \approx 53$, $x = 0$ for $t \in [0, 6]$ and recorded experimental data

A.Lozovskiy, M.Olshanskii, Yu.Vassilevski. Analysis and assessment of a monolithic FSI finite element method.

Submitted to Computers & Fluids

Conclusions for Part I

- We proposed unconditionally stable semi-implicit ALE FE scheme for FSI
- Only one linear system is solved per time step
- The scheme can incorporate diverse elasticity models
- Works robustly in 2D and 3D and handles various time-discretizations
- Drawback: the scheme may suffer from mesh tangling for large deformations, and the cure is ad-hoc.

Incompressible fluid flow in a time-dependent domain

Navier-Stokes equations in a time-dependent domain

Prerequisites

- reference domain Ω_0
- transformation $\boldsymbol{\xi}$ maps Ω_0 to $\Omega(t)$
- $\mathbf{v}$ and $\mathbf{u}$ denote velocities and displacements in Ω_0
- $\boldsymbol{\xi}(\mathbf{x}) := \mathbf{x} + \mathbf{u}(\mathbf{x})$, $\mathbf{F} := \nabla \boldsymbol{\xi} = \mathbf{I} + \nabla \mathbf{u}$, $J := \det(\mathbf{F})$
- Cauchy stress tensor $\boldsymbol{\sigma}$
- pressure p
- density ρ is constant

Incompressible fluid flow in a moving domain

Navier-Stokes equations in reference domain Ω_0

Let $\boldsymbol{\xi}$ mapping Ω_0 to $\Omega(t)$, $\mathbf{F} = \nabla\boldsymbol{\xi} = \mathbf{I} + \nabla\mathbf{u}$, $J = \det(\mathbf{F})$ be given

Incompressible fluid flow in a moving domain

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Dynamic equations

$$\frac{\partial \mathbf{v}}{\partial t} = (J\rho_f)^{-1}\mathrm{div}\left(J\boldsymbol{\sigma}_f\mathbf{F}^{-T}\right) - \nabla\mathbf{v}\left(\mathbf{F}^{-1}\left(\mathbf{v} - \frac{\partial \mathbf{u}}{\partial t}\right)\right) \quad \text{in } \Omega_0$$

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Fluid incompressibility

$$\mathrm{div}\left(J\mathbf{F}^{-1}\mathbf{v}\right) = 0 \quad \text{in } \Omega_0 \quad \text{or} \quad J\nabla\mathbf{v} : \mathbf{F}^{-T} = 0 \quad \text{in } \Omega_0$$

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Constitutive relation for the fluid stress tensor

$$\boldsymbol{\sigma}_f = -p_f \mathbf{I} + \mu_f ((\nabla \mathbf{v})\mathbf{F}^{-1} + \mathbf{F}^{-T} (\nabla \mathbf{v})^T) \quad \text{in } \Omega_0$$

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Mapping $\boldsymbol{\xi}$ does not define material trajectories $\rightarrow$ quasi-Lagrangian formulation

Finite element scheme

Let $\mathbb{V}_h, \mathbb{Q}_h$ be Taylor-Hood P_2/P_1 finite element spaces.
Find $\{\mathbf{v}_h^k, p_h^k\} \in \mathbb{V}_h \times \mathbb{Q}_h$ satisfying b.c.
(”do nothing” $\boldsymbol{\sigma}\mathbf{F}^{-T}\mathbf{n} = 0$ or no-penetration no-slip $\mathbf{v}^k = (\boldsymbol{\xi}^k - \boldsymbol{\xi}^{k-1})/\Delta t$)

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("do nothing" $\boldsymbol{\sigma}\mathbf{F}^{-T}\mathbf{n} = 0$ or no-penetration no-slip $\mathbf{v}^k = (\boldsymbol{\xi}^k - \boldsymbol{\xi}^{k-1})/\Delta t$)

$$\int_{\Omega_0} J_k \frac{\mathbf{v}_h^k - \mathbf{v}_h^{k-1}}{\Delta t} \cdot \boldsymbol{\psi} \, \mathrm{d}\mathbf{x} + \int_{\Omega_0} J_k \nabla \mathbf{v}_h^k \mathbf{F}_k^{-1} \left(\mathbf{v}_h^{k-1} - \frac{\boldsymbol{\xi}^k - \boldsymbol{\xi}^{k-1}}{\Delta t} \right) \cdot \boldsymbol{\psi} \, \mathrm{d}\mathbf{x} -$$

$$\int_{\Omega_0} J_k p_h^k \mathbf{F}_k^{-T} : \nabla \boldsymbol{\psi} \, \mathrm{d}\mathbf{x} + \int_{\Omega_0} J_k q \mathbf{F}_k^{-T} : \nabla \mathbf{v}_h^k \, \mathrm{d}\mathbf{x} +$$

$$\int_{\Omega_0} \nu J_k (\nabla \mathbf{v}_h^k \mathbf{F}_k^{-1} \mathbf{F}_k^{-T} + \mathbf{F}_k^{-T} (\nabla \mathbf{v}_h^k)^T \mathbf{F}_k^{-T}) : \nabla \boldsymbol{\psi} \, \mathrm{d}\mathbf{x} = 0$$

$$\int_{\Omega_0} J_k \nabla \mathbf{v}^k : \mathbf{F}_k^{-T} q \, \mathrm{d}\Omega = 0$$

for all $\boldsymbol{\psi}$ and q from the appropriate FE spaces

Finite element scheme

The scheme

- semi-implicit
- produces one linear system per time step
- first order in time (may be generalized to the second order)

Finite element scheme

The scheme

- semi-implicit
- produces one linear system per time step
- first order in time (may be generalized to the second order)
- unconditionally stable (no CFL restriction) and 2nd order accurate, proved with assumptions:
  - $\inf_Q J \geq c_J > 0, \quad \sup_Q(\|\mathbf{F}\|_F + \|\mathbf{F}^{-1}\|_F) \leq C_F$
  - LBB-stable pairs (e.g. P_2/P_1)
  - Δt is not large

A.Danilov, A.Lozovskiy, M.Olshanskii, Yu.Vassilevski. A finite element method for the Navier-Stokes equations in moving domain with application to hemodynamics of the left ventricle. *Russian J. Numer. Anal. Math. Modelling, 32, 2017*

A.Lozovskiy, M.Olshanskii, Yu.Vassilevski. A quasi-Lagrangian finite element method for the Navier-Stokes equations in a time-dependent domain. Comput. Methods Appl. Mech. Engrg. 333, 2018

Energy equality for the weak solution

Let $\partial\Omega(t) = \partial\Omega^{ns}(t)$ and $\boldsymbol{\xi}_t$ be given on $\partial\Omega^{ns}(t)$. Then there exists $\mathbf{v}_1 \in C^1(Q)^d$, $\mathbf{v}_1 = \boldsymbol{\xi}_t$, $\mathrm{div}\,(J\mathbf{F}^{-1}\mathbf{v}_1) = 0$ [Miyakawa1982]

and we can decompose the solution $\mathbf{v} = \mathbf{w} + \mathbf{v}_1$, $\mathbf{w} = 0$ on $\partial\Omega^{ns}$

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Energy balance for w:

$$\underbrace{\frac{1}{2}\frac{d}{dt}\|J^{\frac{1}{2}}\mathbf{w}\|^2}_{\substack{\text{variation of}\\ \text{kinetic energy}}} \quad \underbrace{+2\nu\|J^{\frac{1}{2}}\mathbf{D}_\xi(\mathbf{w})\|^2}_{\substack{\text{energy of}\\ \text{viscous dissipation}}} \quad \underbrace{+(J(\nabla\mathbf{v}_1\mathbf{F}^{-1}\mathbf{w}),\mathbf{w})}_{\substack{\text{intensification}\\ \text{due to b.c.}}} \quad \underbrace{=(\widetilde{\mathbf{f}},\mathbf{w})}_{\substack{\text{work of}\\ \text{ext. forces}}}$$

$$\mathbf{D}_\xi(\mathbf{v}) = \tfrac{1}{2}(\nabla\mathbf{v}\mathbf{F}^{-1} + \mathbf{F}^{-T}(\nabla\mathbf{v})^T)$$

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Stability estimate for the FE solution

Let $\partial\Omega(t) = \partial\Omega^{ns}(t)$ and $\boldsymbol{\xi}_t$ be given on $\partial\Omega^{ns}(t)$. Then there exists $\mathbf{v}_1 \in C^1(Q)^d$, $\mathbf{v}_1 = \boldsymbol{\xi}_t$, $\mathrm{div}\,(J\mathbf{F}^{-1}\mathbf{v}_1) = 0$ [Miyakawa1982]

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Stability estimate for $\mathbf{w}_h^k$ FE approximation of $\mathbf{w}^k$:

$$\underbrace{\frac{1}{2\Delta t}\left(\|J_k^{\frac{1}{2}}\mathbf{w}_h^k\|^2 - \|J_{k-1}^{\frac{1}{2}}\mathbf{w}_h^{k-1}\|^2\right)}_{\substack{\text{variation of}\\ \text{kinetic energy}}} \underbrace{+2\nu\left\|J_k^{\frac{1}{2}}\mathbf{D}_k(\mathbf{w}_h^k)\right\|^2}_{\substack{\text{energy of}\\ \text{viscous dissipation}}} \underbrace{+\frac{(\Delta t)}{2}\left\|J_{k-1}^{\frac{1}{2}}[\mathbf{w}_h]_t^k\right\|^2}_{\substack{O(\Delta t)\text{ dissipative}\\ \text{term}}}$$

$$\underbrace{+(J_k(\nabla\mathbf{v}_1^k\mathbf{F}_k^{-1})\mathbf{w}_h^k, \mathbf{w}_h^k)}_{\substack{\text{intensification}\\ \text{due to b.c.}}} = \underbrace{(\widetilde{\mathbf{f}}^k, \mathbf{w}_h^k)}_{\substack{\text{work of}\\ \text{ext. forces}}}$$

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Stability estimate for the FE solution

Stability estimate for $\mathbf{w}_h^n$ FE approximation of $\mathbf{w}^n$:

$C_1 \|\nabla \mathbf{v}_1^k\| \leq \nu/2$:

$$\frac{1}{2}\|\mathbf{w}_h^n\|_n^2 + \nu \sum_{k=1}^{n} \Delta t \|\mathbf{D}_k(\mathbf{w}_h^k)\|_k^2 \leq \frac{1}{2}\|\mathbf{w}_0\|_0^2 + C \sum_{k=1}^{n} \Delta t \|\widetilde{\mathbf{f}}^k\|^2$$

$$\mathbf{D}_k(\mathbf{v}) := \tfrac{1}{2}(\nabla \mathbf{v} \mathbf{F}_k^{-1} + \mathbf{F}_k^{-T} (\nabla \mathbf{v})^T)$$

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$$\mathbf{D}_k(\mathbf{v}) := \tfrac{1}{2}(\nabla \mathbf{v} \mathbf{F}_k^{-1} + \mathbf{F}_k^{-T}(\nabla \mathbf{v})^T)$$

$C_1\|\nabla \mathbf{v}_1^k\| > \nu/2$:

$$\frac{1}{2}\|\mathbf{w}_h^n\|_n^2 + \nu \sum_{k=1}^{n} \Delta t \|\mathbf{D}_k(\mathbf{w}_h^k)\|_k^2 \le e^{\frac{2C_2}{\alpha}T} \left(\frac{1}{2}\|\mathbf{w}_0\|_0^2 + C \sum_{k=1}^{n} \Delta t \|\widetilde{\mathbf{f}}^k\|^2 \right),$$

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$$\mathbf{D}_k(\mathbf{v}) := \tfrac{1}{2}(\nabla \mathbf{v} \mathbf{F}_k^{-1} + \mathbf{F}_k^{-T}(\nabla \mathbf{v})^T)$$

$C_1\|\nabla \mathbf{v}_1^k\| > \nu/2$:

$$\frac{1}{2}\|\mathbf{w}_h^n\|_n^2 + \nu \sum_{k=1}^{n} \Delta t \|\mathbf{D}_k(\mathbf{w}_h^k)\|_k^2 \le e^{\frac{2C_2}{\alpha}T} \left(\frac{1}{2}\|\mathbf{w}_0\|_0^2 + C \sum_{k=1}^{n} \Delta t \|\widetilde{\mathbf{f}}^k\|^2 \right),$$

$$\text{if } (1 - 2C_2\Delta t) = \alpha > 0$$

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Convergence of the FE solution

Assume

1. LBB stable FE pair P_{m+1}-P_m;
2. Ω_0 is a convex polyhedron;
3. $\mathbf{u}_{tt} \in L^\infty(\Omega_0)$, $\mathbf{u}(t) \in H^{m+\frac{5}{2}}(\Omega_0)$, $p(t) \in H^{m+1}(\Omega_0)$ for all $t \in [0, T]$;
4. $c\Delta t \geq h^{2m+4}$ with some c independent of h, Δt;
5. either Δt is small enough s.t. $\frac{1}{2} - \tilde{C}\Delta t > 0$ or $\nu \geq \tilde{C}\, C_K$

Then

$$\max_{1\leq k\leq N} \|\mathbf{e}^k\|_k^2 + 2\nu\Delta t \sum_{k=1}^{N} \|\mathbf{D}_k(\mathbf{e}^k)\|_k^2 \leq C\left(h^{2(m+1)} + (\Delta t)^2 + (\Delta t)^{-1} h^{2(m+2)}\right).$$

In particular, for Taylor-Hood pair, $m = 1$:

$$\max_{1\leq k\leq N} \|\mathbf{e}^k\|_k^2 + 2\nu\Delta t \sum_{k=1}^{N} \|\mathbf{D}_k(\mathbf{e}^k)\|_k^2 \leq C \max\{h^2; \Delta t\} \text{ if } h^2 \leq c\Delta t.$$

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3D: left ventricle of a human heart

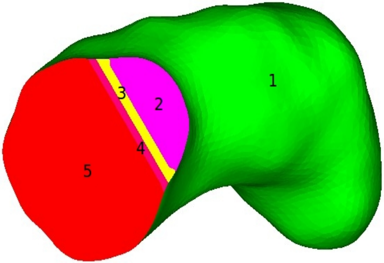


Figure: Left ventricle

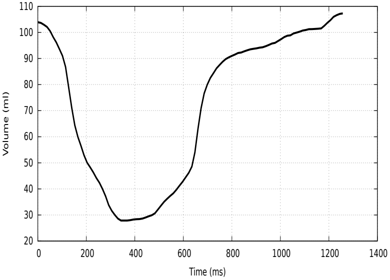
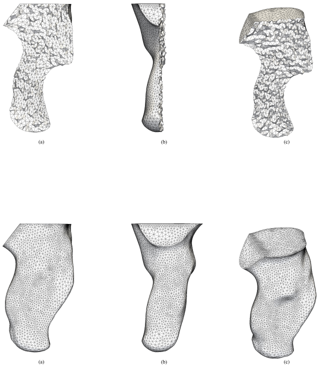


Figure: Ventricle volume

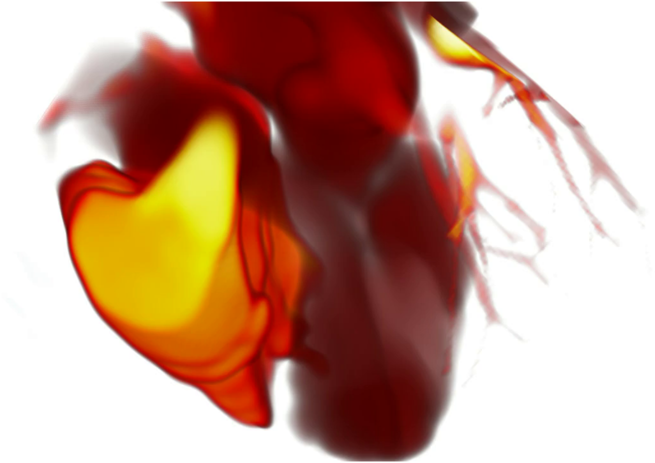
The law of motion for the ventricle walls is known thanks to ceCT scans $\to$ 100 mesh files with time gap 0.0127 s $\to$ $\mathbf{u}$ given as input $\to$ FSI reduced to NSE in a moving domain

- 2 - aortic valve (outflow)
- 5 - mitral valve (inflow)

3D: left ventricle of a human heart



- Quasi-uniform mesh: 14033 vertices, 69257 elements, 88150 edges.
- Boundary conditions: Dirichlet $\mathbf{v} = \frac{\partial \mathbf{u}}{\partial t}$ except:
  - Do-nothing on aortal valve during systole
  - Do-nothing on mitral valve during diastole
- Time step 0.0127 s is too large! $\Longrightarrow$ refined to $\Delta t = 0.0127/20$ s $\Longrightarrow$ Cubic-splined **u**.
- Blood parameters: $\rho_f = 10^3$ kg/m^3, $\mu_f = 4 \cdot 10^{-3}$ Pa $\cdot$s.



Conclusions for Part II

- We proposed unconditionally stable semi-implicit FE scheme for NS eqs in moving domain
- The scheme is proven to be second order accurate in space
- Only one linear system is solved per time step
- The scheme was applied to blood flow simulation in a geometrical dynamic model of the left ventricle

Stabilization in space for flow in the left ventricle

DNS resulted in convective instability during sharp deformation phases. We use a simple Smagorinsky dissipation model:

$$\mathbf{z}^{k-1} := \mathbf{v}^{k-1} - \frac{\mathbf{u}^k - \mathbf{u}^{k-1}}{\Delta t},$$

$$\int_{\Omega(t^{k-1})} \frac{\mathbf{v}^k - \mathbf{v}^{k-1}}{\Delta t} \cdot \boldsymbol{\psi}\, \mathrm{d}\mathbf{x} + \int_{\Omega(t^{k-1})} \nabla \mathbf{v}^k \mathbf{z}^{k-1} \cdot \boldsymbol{\psi}\, \mathrm{d}\mathbf{x}$$
$$- \int_{\Omega(t^{k-1})} s^k \mathrm{div}\, \boldsymbol{\psi}\, \mathrm{d}\mathbf{x} + \int_{\Omega(t^{k-1})} q \mathrm{div}\, \mathbf{v}^k\, \mathrm{d}\mathbf{x} + \int_{\Omega(t^{k-1})} 2\nu \{\nabla \mathbf{v}^k\}_s : \nabla \boldsymbol{\psi}\, \mathrm{d}\mathbf{x} +$$
$$\sum_e \int_{\Omega_e(t^{k-1})} 2\nu_T^{k-1} \{\nabla \mathbf{v}^k\}_s : \nabla \boldsymbol{\psi}\, \mathrm{d}\mathbf{x} = 0,$$

where

$$\nu_T^{k-1} = 0.04 h_e^2 \sqrt{2\{\nabla \mathbf{z}^{k-1}\}_s : \nabla \mathbf{z}^{k-1}}.$$

Worked for the entire cardiac cycle with the original viscosity and mesh!