Extremal (in)dependence structures of copulas with multiplicative constructions

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Self-similarity, Long-range dependence, and Extremes
Casa Matématica Oaxaca, Mexico

Random scale constructions

$$(\mathbf{X_1}, \mathbf{X_2}) = \mathbf{R}(\mathbf{W_1}, \mathbf{W_2}), \qquad R \perp\!\!\!\perp (W_1, W_2)$$

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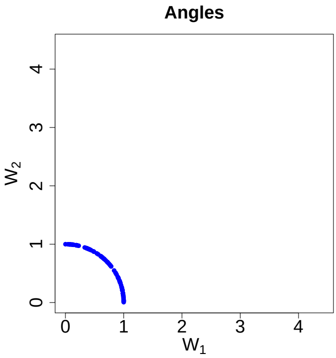

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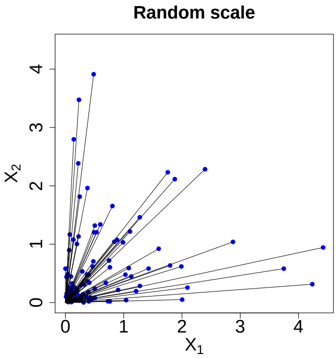

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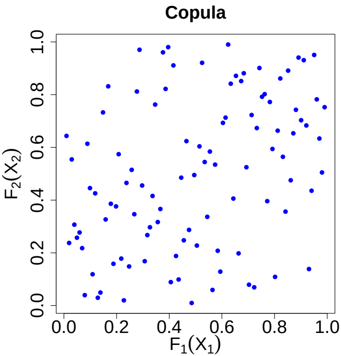

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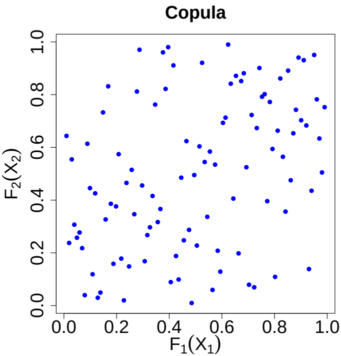
Why is this model important?

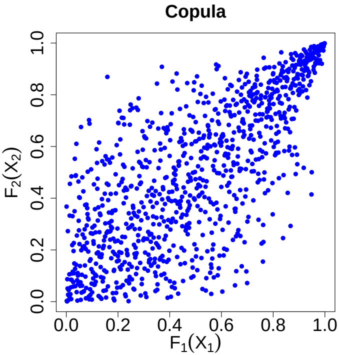
- Archimedean/Liouville copulas
- (Scale mixtures of) Gaussian copulas
- Student-t copulas
- Elliptical copulas
- Pareto copulas, includes all extreme value dependence structures
- etc.



Extremal properties: asymptotic dependence

Tail dependence coefficient $\chi_X \in [0,1]$

$$\chi_X = \lim_{q \to 1} \mathrm{P}\{F_1(X_1) > q \mid F_2(X_2) > q\}$$



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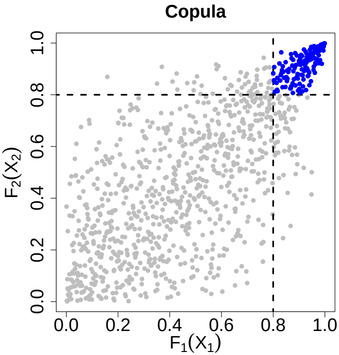

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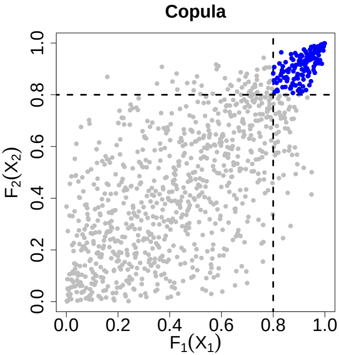

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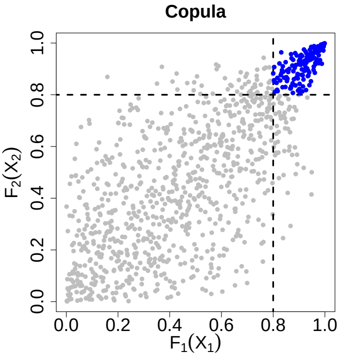


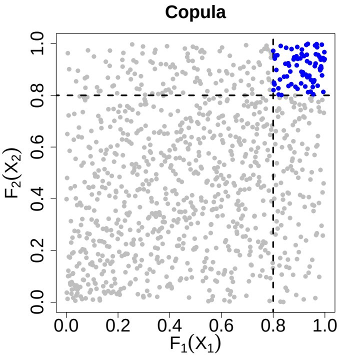
- $\chi_X > 0$: Asymptotic dependence (Pareto, student-t,...)
- $\chi_X = 0$: Asymptotic independence (Gaussian copula,...)

Extremal properties: asymptotic independence

Residual tail dependence coefficient $\eta_X \in [0,1]$

$$P\{F_1(X_1) > q, F_2(X_2) > q\} = \ell(1-q)P\{F_1(X_1) > q\}^{1/\eta_X}$$

where ℓ is slowly varying.



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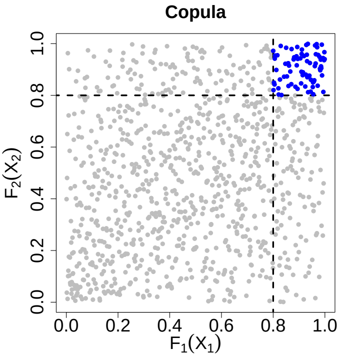

- $\eta_X \in (1/2, 1]$: Positive association.
- $\eta_X \in [0, 1/2)$: Negative association.

Asymp. dep.: $\eta_X = 1$
Independence: $\eta_X = 1/2$
Gaussian: $\eta_X = (1+\rho_X)/2$

Asymptotic dependence or independence?

Pre-asymptotic tail dependence coefficient

$$\chi_X(q) = \mathrm{P}\{F_1(X_1) > q, F_2(X_2) > q\}/(1-q)$$

for finite levels $q \in (0,1)$.

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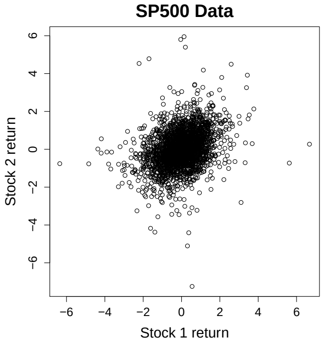

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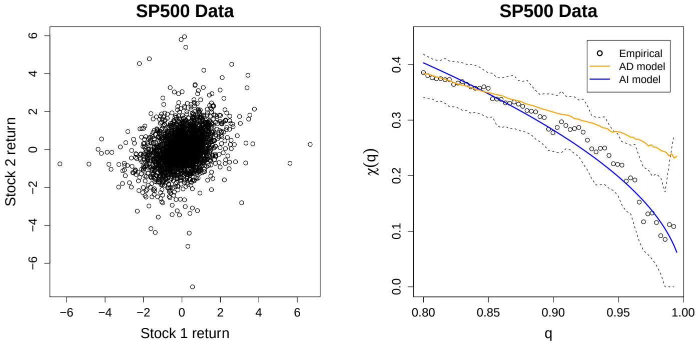
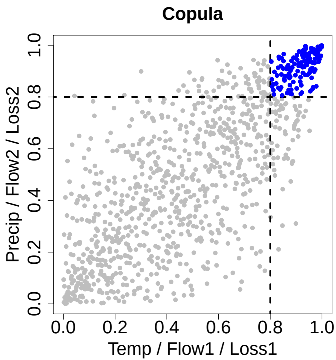


Extreme Value Theory and Statistics

- Rare events do happen!
- Impact on various risks (health, environment, economy,...)
- Often result of simultaneous events
- Joint exceedance estimates drastically differ between AD and AI models



The goals

$$(\mathbf{X_1}, \mathbf{X_2}) = \mathbf{R}(\mathbf{W_1}, \mathbf{W_2}), \qquad R \perp\!\!\!\perp (W_1, W_2)$$

In this project, we want to

- systematically characterize extremal dependence in (X_1, X_2), in terms of
  - the tail heaviness of R and (W_1, W_2);
  - the extremal dependence χ_W and η_W of (W_1, W_2);

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  - the extremal dependence χ_W and η_W of (W_1, W_2);
- compute the dependence coefficients χ_X and η_X;
- unify existing theory and models;
- build new statistical models with desirable properties.

Main assumptions on R and (W_1, W_2)

For $R \geq 0$: There exists $\xi \in \mathbb{R}$ and a function $b(t) > 0$ s.t.

$$\lim_{t \to r^*} \mathrm{P}(R > t + r/b(t) \mid R > t) = (1 + \xi r)_+^{-1/\xi}, \qquad r \geq 0,$$

where $r^* = \sup\{r : F_R(r) < 1\}$ is upper endpoint; cf. Embrechts et al. (1997).

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Tail heaviness of R increases with shape ξ:

1. $\xi < 0$: R has upper endpoint and is in negative Weibull MDA;
2. $\xi = 0$: R light tailed $\Rightarrow$ MDA of Gumbel distribution;
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For $(W_1, W_2) \in \mathbb{R}^2$: $W_1 \stackrel{d}{=} W_2 \stackrel{d}{=} W \geq 0$ and same range of tail decays as R.

Notation: χ_W and η_W are tail dependence and residual tail dependence coefficient of (W_1, W_2).

Some intuition: the "Independence Model"

$$(\mathbf{X_1}, \mathbf{X_2}) = \mathbf{R}\ (\mathbf{W_1}, \mathbf{W_2}), \qquad R \perp\!\!\!\perp W_1 \perp\!\!\!\perp W_2$$

- Simple model: R, W_1 and W_2 independent, i.e., $\chi_W = 0$, $\eta_W = 1/2$.

Some intuition: the "Independence Model"

$$(\mathbf{X_1}, \mathbf{X_2}) = \mathbf{R}_\xi(\mathbf{W_1}, \mathbf{W_2}), \qquad R \perp\!\!\!\perp W_1 \perp\!\!\!\perp W_2$$

- Simple model: R_ξ, W_1 and W_2 independent, i.e., $\chi_W = 0$, $\eta_W = 1/2$.
- Let $W_1, W_2 \sim \mathrm{Unif}[0, 1]$.
- Let $R_\xi = F_{R_\xi}^{-1}(U)$, with $U \sim \mathrm{Unif}[0, 1]$, shape $\xi \in \mathbb{R}$ and

$$F_{R_\xi}(r) = 1 - (1 + \xi r)_+^{-1/\xi}, \qquad r \geq 0.$$

Tail decays for R and (W_1, W_2)

Angle W / Radius R	Super-heavy	Reg. varying	Weibull	Neg. Weibull
Super-heavy				
Reg. varying				
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Superheavy-tails ($\xi = \infty$)

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1. *$R \in \mathrm{SHT}$ and $\overline{F}_W(x) \sim c\overline{F}_R(x)$, $c \in [0, \infty)$. Then $\eta_X = 1$ and*

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2. *$W \in \mathrm{SHT}$ and $\overline{F}_R = o(\overline{F}_W)$. Then $\chi_X = \chi_W$. If $\chi_W = 0$ and*
   - *$\overline{F}_R = O(\overline{F}_{\min(W_1, W_2)})$, then $\eta_X = \eta_W$;*
   - *$\overline{F}_{\min(W_1, W_2)} = o(\overline{F}_R)$, then, if the limit exists,*

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Example (R, W_1, W_2 independent)

$\overline{F}_{\min(W_1,W_2)} = (\overline{F}_W)^2$, $\chi_W = 0$, $\eta_W = 1/2$.

1. $R \in \mathrm{SHT}$: $\chi_X = 1/(1 + c)$
2. $W \in \mathrm{SHT}$, R lighter: $\chi_X = 0$ and $\eta_X = 1/2$

The "Independence model"

Angle W / Radius R	Super-heavy	Reg. varying	Weibull	Neg. Weibull
Super-heavy	$\chi_X = \frac{1}{1+c}$	$\chi_X = 1$	$\chi_X = 1$	$\chi_X = 1$
Reg. varying	*			
Weibull	*			
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Table : Values of χ_X and η_X for $(X_1, X_2) = R(W_1, W_2)$ with $W_1, W_2 \overset{d}{=} W$ independent. The *'s indicate $\chi_X = \chi_W = 0$ and $\eta_X = \eta_W = 1/2$.

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$Y \in \mathrm{RV}_{-\alpha}$: $\mathrm{P}(Y > x) \sim \ell(x)x^{-\alpha}$ with $\alpha > 0$, ℓ slowly varying

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$$\eta_X = \begin{cases} \alpha_W/\alpha_R, & \text{if } \alpha_R < \alpha_W/\eta_W, \\ \eta_W, & \text{if } \alpha_R > \alpha_W/\eta_W \text{ or } \alpha_R = +\infty. \end{cases}$$

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3. *$\alpha_W = \alpha_R$: More involved.*

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Example

1. All Pareto copulas, *t*-distributions, ...
2. Asymptotically independent model in Huser & Wadsworth (2018).

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Angle W / Radius R	Super-heavy	Reg. varying	Weibull	Neg. Weibull
Super-heavy	$\chi_X = \frac{1}{1+c}$	$\chi_X = 1$	$\chi_X = 1$	$\chi_X = 1$
Reg. varying	*	$\alpha_R < \alpha_W : \chi_X > 0$ $\alpha_W < \alpha_R < 2\alpha_W$: $\eta_X = \alpha_W/\alpha_R$ $\alpha_R > 2\alpha_W$: $\eta_X = 1/2$	$\chi_X > 0$	$\chi_X > 0$
Weibull	*	*		
Neg. Weibull	*	*		

Table : Values of χ_X and η_X for $(X_1, X_2) = R(W_1, W_2)$ with $W_1, W_2 \overset{d}{=} W$ independent. The *'s indicate $\chi_X = \chi_W = 0$ and $\eta_X = \eta_W = 1/2$.

Weibull-type tails ($\xi = 0$)

$Y \in \mathrm{W}(\theta)$: $\mathsf{P}(Y > x) \sim c x^{-\gamma} \exp(-\alpha x^{\beta})$, with $\theta = (c, \gamma, \alpha, \beta)$.

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3. $\beta_{\tilde{W}} = \beta_W$, $\alpha_{\tilde{W}} > \alpha_W$. *Then* $\chi_X = \chi_W = 0$ *and*

$$\eta_X = \eta_W^{\beta_R/(\beta_R+\beta_W)} = \left(\frac{\alpha_W}{\alpha_{\tilde{W}}}\right)^{\beta_R/(\beta_R+\beta_W)}.$$

4. $\beta_{\tilde{W}} > \beta_W$. *Then* $\chi_X = \chi_W = 0$. $\eta_X = \eta_W = 0$.

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$$\eta_X = \eta_W^{\beta_R/(\beta_R+\beta_W)} = \left(\frac{\alpha_W}{\alpha_{\tilde{W}}}\right)^{\beta_R/(\beta_R+\beta_W)} .$$

4. $\beta_{\tilde{W}} > \beta_W$. *Then* $\chi_X = \chi_W = 0$. $\eta_X = \eta_W = 0$.

Example

3. Independence model: $\alpha_{\tilde{W}} = 2\alpha_W$ and $\eta_X = 2^{-\beta_R/(\beta_R+\beta_W)}$.
3. Gaussian scale mixtures: $\eta_X = \{(1 + \rho_W)/2\}^{\beta_R/(\beta_R+2)}$.

The "Independence model"

Angle W / Radius R	Super-heavy	Reg. varying	Weibull	Neg. Weibull
Super-heavy	$\chi_X = \frac{1}{1+c}$	$\chi_X = 1$	$\chi_X = 1$	$\chi_X = 1$
Reg. varying	*	$\alpha_R < \alpha_W : \chi_X > 0$ $\alpha_W < \alpha_R < 2\alpha_W$: $\eta_X = \alpha_W/\alpha_R$ $\alpha_R > 2\alpha_W$: $\eta_X = 1/2$	$\chi_X > 0$	$\chi_X > 0$
Weibull	*	*	$\eta_X = 2^{-\beta_R/(\beta_R+\beta_W)}$	
Neg. Weibull	*	*		

Table : Values of χ_X and η_X for $(X_1, X_2) = R(W_1, W_2)$ with $W_1, W_2 \overset{d}{=} W$ independent. The *'s indicate $\chi_X = \chi_W = 0$ and $\eta_X = \eta_W = 1/2$.

Negative Weibull domain of attraction ($\xi < 0$)

$$Y \in \mathrm{NW}(\alpha):\ \mathsf{P}(Y > 1 - s) \sim \ell(1/s)s^{\alpha}, \quad s \to 0,\ \ell \in \mathsf{RV}_0, \alpha > 0$$

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Proposition

1. $R \in \mathrm{W}(\theta_R)$, $W \in \mathrm{NW}(\alpha_W)$, $\tilde{W} \in \mathrm{NW}(\alpha_{\tilde{W}})$. *Then* $\chi_X = \chi_W$ *and* $\eta_X = 1$.
2. $R \in \mathrm{NW}(\alpha_R)$, $W \in \mathrm{W}(\theta_W)$, $\tilde{W} \in \mathrm{NW}(\alpha_{\tilde{W}})$. *Then* $\chi_X = \chi_W$ *and* $\eta_X = \eta_W$.
3. $R \in \mathrm{NW}(\alpha_R)$, $W \in \mathrm{NW}(\alpha_W)$, $\tilde{W} \in \mathrm{NW}(\alpha_{\tilde{W}})$. *If* $\alpha_{\tilde{W}} = \alpha_W$ *then* $\chi_X = \chi_W$ *and* $\eta_X = 1$. *If* $\alpha_{\tilde{W}} > \alpha_W$ *then* $\chi_X = \chi_W = 0$ *and*

$$\eta_X = \frac{\alpha_W + \alpha_R}{\alpha_{\tilde{W}} + \alpha_R} > \frac{\alpha_W}{\alpha_{\tilde{W}}} = \eta_W.$$

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Example

1. Support of (W_1, W_2) on sphere: model of Wadsworth et al. (2017).
3. Independence model: $\eta_X = \frac{\alpha_W + \alpha_R}{2\alpha_W + \alpha_R} \in (1/2, 1)$.

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Weibull	*	*	$\eta_X = 2^{-\beta_R/(\beta_R+\beta_W)}$	$\chi_X = 0$ $\eta_X = 1$
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How do we use this in a statistical model?

Parametric model: Bridging between AD and AI

Let $\{C_{\xi,\alpha} : (\xi, \alpha) \in \mathbb{R} \times \mathbb{R}_+\}$ be the family of copulas corresponding to:

$$(X_1, X_2) = R(W_1, W_2), \qquad R \perp\!\!\!\perp W_1 \perp\!\!\!\perp W_2,$$

- $\mathrm{P}(R \leq r) = 1 - (1 + \xi r)_+^{-1/\xi}, \qquad r \geq 0;$
- $W_1, W_2 \sim \mathrm{Beta}(\alpha, \alpha)$, independent.

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Properties:

1. $\xi < 0$: AI ($\chi_X = 0$) with
$$\eta_X = \frac{\alpha + \xi^{-1}}{2\alpha + \xi^{-1}}.$$
2. $\xi = 0$: AI ($\chi_X = 0$) with $\eta_X = 1$.
3. $\xi > 0$: AD with
$$\chi_X = \mathrm{E}\left[\min\left\{\frac{W_1^{1/\xi}}{\mathrm{E}(W_1^{1/\xi})}, \frac{W_2^{1/\xi}}{\mathrm{E}(W_2^{1/\xi})}\right\}\right].$$

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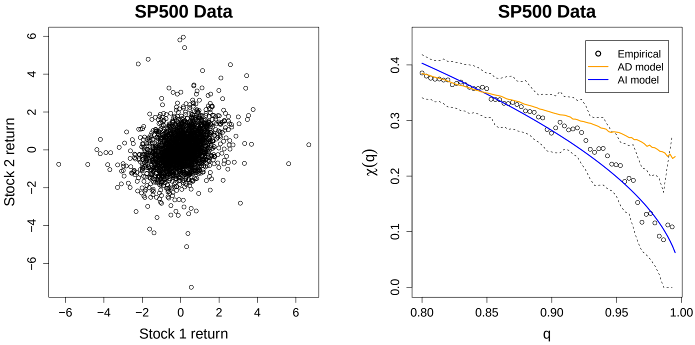
Estimation:

- Densities for ML estimation readily available.
- Marginal normalization requires one-dim. integration.

Parametric model: Bridging between AD and AI

Estimates: $\hat{\xi} = -1.18$ and $\hat{\alpha} = 0.85$

- Asymptotically independent model (blue curve in plot)



Conclusion

We unify theory and cover/extend existing examples:

- Archimedean/Liouville copulas: Larsson & Nešlehová (2011), Belzile & Nešlehová (2017)
- (Scale mixtures of) Gaussian copulas: Sibuya (1960), Huser et al. (2017)
- Student-t copulas Nikoloulopoulos et al. (2012)
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We build new models bridging between AD and AI:

- "Independence model"
- "Spiky norm model"
- etc.

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Thank you!

References

P. Embrechts, C. Klüppelberg, and T. Mikosch.
Modelling Extremal Events: for Insurance and Finance.
Springer, London, 1997.

S. Engelke, T. Opitz, and J. Wadsworth.
Extremal dependence of random scale constructions.
Available from `http://arxiv.org/abs/1803.04221`, 2018.

R. Huser, T. Opitz, and E. Thibaud.
Bridging asymptotic independence and dependence in spatial extremes using gaussian scale mixtures.
Spatial Statistics, 21:166 – 186, 2017.

R. Huser and J. Wadsworth.
Modeling spatial processes with unknown extremal dependence class.
J. Amer. Statist. Assoc., 2018.
to appear.

M. Larsson and J. Nešlehová.
Extremal behavior of archimedean copulas.
Advances in Applied Probability, 43(1):195–216, 2011.

A. K. Nikoloulopoulos, H. Joe, and H. Li.
Extreme value properties of multivariate t copulas.
Extremes, 12(2):129–148, 2009.

H. Rootzén and N. Tajvidi.
Multivariate generalized Pareto distributions.
Bernoulli, 12:917–930, 2006.

M. Sibuya.
Bivariate extreme statistics. I.
Ann. Inst. Statist. Math. Tokyo, 11:195–210, 1960.

J.L. Wadsworth, J.A. Tawn, A.C. Davison, and D.M. Elton.
Modelling across extremal dependence classes.
Journal of the Royal Statistical Society: Series B (Statistical Methodology), 79(1):149–175, 2017.