On a characterization of spectral tail processes of stationary time series

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Overview

The (spectral) tail process

We can describe the extremal behavior of multivariate time series by looking at the tail process $(Y_t)_{t\in\mathbb{Z}}$

$$\lim_{u\to\infty} \mathcal{L}\left(\left(\frac{X_{-m}}{u},\dots,\frac{X_n}{u}\right)\Big|\, \|X_0\| > u\right) =: \mathcal{L}((Y_{-m},\dots,Y_n)), \quad m,n\in\mathbb{N}.$$

Decomposition of the tail process - Basrak & Segers (2009)

Existence of (non-degenerate) $(Y_t)_{t\in\mathbb{Z}}$ is equivalent to

1. $\|X_0\|$ is regularly varying with some index $\alpha > 0$
2. $\lim_{u\to\infty} \mathcal{L}\left(\left(\frac{X_{-m}}{\|X_0\|},\dots,\frac{X_n}{\|X_0\|}\right)\Big|\, \|X_0\| > u\right) =: \mathcal{L}((\Theta_{-m},\dots,\Theta_n))$

for spectral tail process $(\Theta_t)_{t\in\mathbb{Z}}$. Then

$$(Y_t)_{t\in\mathbb{Z}} \stackrel{d}{=} Y\cdot(\Theta_t)_{t\in\mathbb{Z}},$$

for $Y\sim \mathrm{Par}(\alpha)$, independent of $(\Theta_t)_{t\in\mathbb{Z}}$.

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The meaning of $(\Theta_t)_{t\in\mathbb{Z}}$

$\mathcal{L}(\Theta_0)$ describes extremal dependence between components:

$$\mathcal{L}(\Theta_0) = \lim_{u\to\infty} \mathcal{L}\left(\frac{X_0}{\|X_0\|} \,\middle|\, \|X_0\| > u\right).$$

$\mathcal{L}((\Theta_t)_{t\in\mathbb{Z}})$ the development of extremal events over time.

Applications of the (spectral) tail process

- Looking at $((\Theta_t)_{t\in\mathbb{Z}})$ is a way to distinguish between different models with regard to extremal properties
- Extremal characteristics like extremal index, extremal coefficient, etc. can be determined from $\mathcal{L}((\Theta_t)_{t\in\mathbb{Z}})$ and α
- $\mathcal{L}((\Theta_t)_{t\in\mathbb{Z}})$ determines asymptotic variance and dependence structure of extremal estimators (Drees and Rootzén (2010) and others)
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(Non-)Stationarity properties of the spectral tail process

If the underlying process $(X_t)_{t\in\mathbb{Z}}$ is stationary, the corresponding (spectral) tail process is in general not stationary (due to conditioning on event at time 0).

Instead, the following relation holds:

"Time change formula" (Basrak & Segers (2009))

Let $(\Theta_t)_{t\in\mathbb{Z}}$ be the spectral tail process of a stationary underlying process. Then, for each bounded, measurable $f : (\mathbb{R}^d)^{t-s+1} \to \mathbb{R}$ such that $f(x_s, \dots, x_t) = 0$, whenever $x_0 = 0$, we have

$$E(f(\Theta_{s-i}, \dots, \Theta_{t-i})) = E\left(f\left(\frac{\Theta_s}{\|\Theta_i\|}, \dots, \frac{\Theta_t}{\|\Theta_i\|}\right)\|\Theta_i\|^\alpha\right),$$

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Characterization and Interpretation

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- "Time change formula" follows from limit description and a few manipulations or directly from tail measure (see Clement's talk).

$\Rightarrow$ Is there also a probabilistic interpretation of this formula?

- TCF completely characterizes the class of spectral tail processes of stationary time series (Dombry et al. (2017), Janßen (2017), Planinić & Soulier (2017))

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Another interpretation - A few definitions first

"Time change formula" - Property (TCF)

We say that a process $(\Psi_t)_{t\in\mathbb{Z}}$ with $\Psi_t \in \mathbb{R}^d$ satisfies Property (TCF) if it satisfies the time change formula and $\|\Psi_0\| = 1$ a.s.

Summability assumption - Property (SC)

We say that a process $(\Psi_t)_{t\in\mathbb{Z}}$ with $\Psi_t \in \mathbb{R}^d$ satisfies Property (SC) if

$$0 < \sum_{i\in\mathbb{Z}} \|\Psi_i\|^\alpha < \infty \text{ a.s..}$$

If Property (TCF) holds, then Property (SC) is satisfied for many processes. In particular, this follows already from $\|\Psi_t\| \to 0$ a.s. for $|t| \to \infty \Rightarrow$ (Janßen (2017), Planinić & Soulier (2017))

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The "RS process"

Definition: The "RS process" of $(\Psi_t)_{t\in\mathbb{Z}}$ (random shift, rescaled)

Let $(\Psi_t)_{t\in\mathbb{Z}}$ be a process which satisfies Property (SC). The corresponding RS process is the process for which

$$(\Psi_t^{\mathrm{RS}})_{t\in\mathbb{Z}} \stackrel{d}{=} \left(\frac{\Psi_{t+K(\Psi)}}{\|\Psi_{K(\Psi)}\|}\right)_{t\in\mathbb{Z}},$$

where

$$P(K(\Psi)=k|(\Psi_t)_{t\in\mathbb{Z}}) = \frac{\|\Psi_k\|^\alpha}{\sum_{i\in\mathbb{Z}}\|\Psi_i\|^\alpha}, \quad k\in\mathbb{Z}.$$

A closer look at the definition

- The distribution of $(\Psi_t^{\mathrm{RS}})_{t\in\mathbb{Z}}$ does not change if we multiply $(\Psi_t)_{t\in\mathbb{Z}}$ with a (random) scalar or apply a (random) shift in time. Thereby, it only depends on the "patterns" that we see in $(\Psi_t)_{t\in\mathbb{Z}}$ (see Basrak et al. (2016)). Below is a bivariate process, first component in red and second in black.

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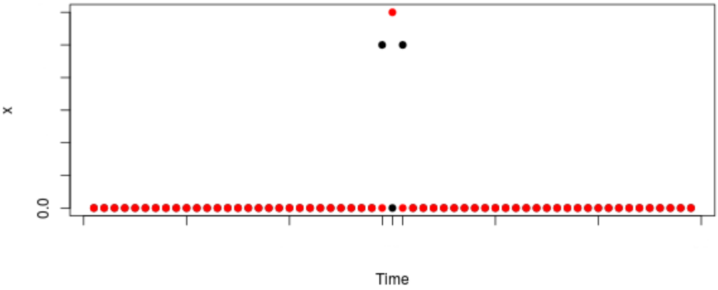

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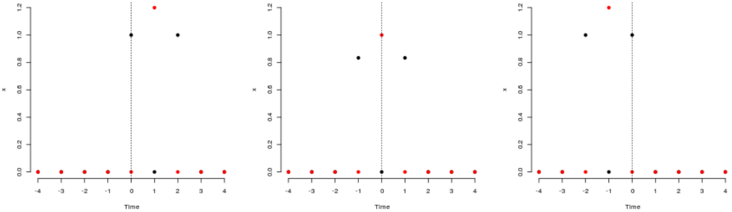

Corresponding probabilities for $\alpha = 1$:

$$\frac{1}{1+1.2+1} = \frac{5}{16}, \qquad \frac{1.2}{1+1.2+1} = \frac{3}{8}, \qquad \frac{1}{1+1.2+1} = \frac{5}{16}$$

An equivalent statement of the time change formula

Theorem (J. (2017))

Let $(\Theta_t)_{t\in\mathbb{Z}}$ satisfy Property (SC). Then the following two statements are equivalent:

1. $(\Theta_t)_{t\in\mathbb{Z}}$ satisfies Property (TCF) (i.e. the "time change formula" $+ \|\Theta_0\| = 1$ a.s.).
2. $(\Theta_t^{\mathrm{RS}})_{t\in\mathbb{Z}} \stackrel{d}{=} (\Theta_t)_{t\in\mathbb{Z}}$.

Corollary - How to generate a spectral tail process?

Let $(\Theta_t)_{t\in\mathbb{Z}}$ satisfy Property (SC). Then $(\Theta_t^{\mathrm{RS}})_{t\in\mathbb{Z}}$ satisfies property (TCF).

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Max-stable processes

Generation of max-stable process from Poisson point processes:

Let $(U_i, T_i)_{i\in\mathbb{N}}$ be an enumeration of points from a Poisson point process with intensity

$$\alpha u^{-\alpha-1}\mathrm{d}u \otimes \mathrm{Count}(dt)$$

(Count is counting measure on $\mathbb{Z}$) and $(S_t^i)_{t\in\mathbb{Z}}$ be i.i.d. copies from a non-negative process that satisfies property (SC), independent of $(U_i, T_i)_{i\in\mathbb{N}}$. Then,

$$(X_t)_{t\in\mathbb{Z}} := \left(\max_{i\in\mathbb{N}} U_i \frac{S^i_{t+T_i}}{(\sum_{z\in\mathbb{Z}} \|S_z\|^\alpha)^{1/\alpha}}\right)_{t\in\mathbb{Z}}$$

is a stationary max-stable process and $(S_t^{\mathrm{RS}})_{t\in\mathbb{Z}}$ is the corresponding spectral tail process.

Cf. Engelke et al. (2014) for a related result in a similar context.

Max-stable processes: Example

Generation of max-stable process from Poisson point processes:

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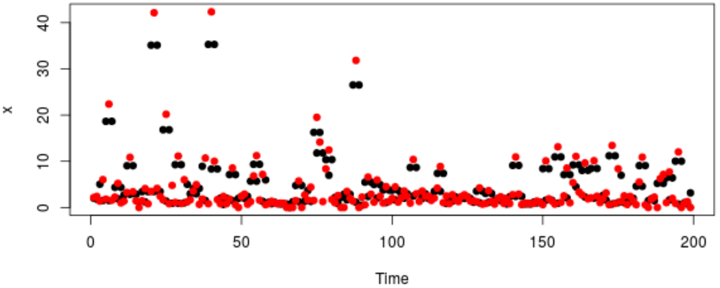
The example for our previous RS-process:



Construction of max-stable process without Property (SC)

Let $(\Theta_t)_{t\in\mathbb{Z}}$ with values in $[0,\infty)^d$ be a stochastic process which satisfies Property (TCF).

For $j \in \mathbb{N}_0$ let $(U_i^{(j)}, (\Theta_t^{(j,i)})_{t\in\mathbb{Z}})_{i\in\mathbb{N}}$ be point from a PPP with intensity

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Then the stochastic process

$$X_0 = \left(\bigvee_{i\in\mathbb{N}} U_i^{(0)} \Theta_0^{(0,i)} \right)$$

is a stationary and max-stable process with corresponding forward spectral tail process $(\Theta_t)_{t\in\mathbb{N}_0}$.

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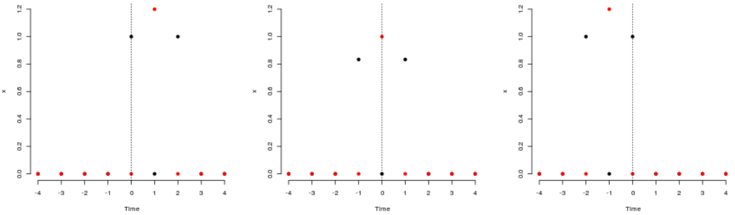
Two complementary views on extremal behavior

Under stationarity, if we know extremal behavior given exceedance at time 0...



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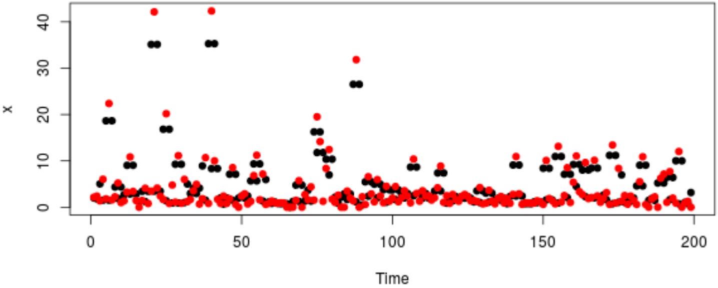
Two complementary views on extremal behavior

... then we also know "global" behavior as approximated by max-stable process



Statistical application

How is extremal dependence best reflected in estimators?

Assume we are interested in inference for Θ_0, for example estimator for $P(\Theta_0 \in A)$, A set on unit sphere.

- For estimator construction, one can use all observations with norm over given threshold, variance is influenced by extremal dependence.
- In univariate extreme value theory it was suggested by Davison and Smith (1990) to use clusterwise maxima as approximately i.i.d. realisations for GPD-fitting.
- However, extending this concept to the multivariate setting will introduce a bias, since we then infer about cluster maximum $\neq \Theta_0$.

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Cluster maximum vs. Θ_0

In a way, the cluster maximum is still "the best guess" given the following theorem...

Conditional law of $(\Theta_t)_{t\in\mathbb{Z}}$ given its "pattern", J. (2017)

Assume $(\Theta_t)_{t\in\mathbb{Z}}$ satisfies Property (SC) and (TCF) and set

$$\|\Theta^*\| = \sup_{t\in\mathbb{Z}} \|\Theta_t\|, \quad T^* = \inf\{t \in \mathbb{Z} : \|\Theta_t\| = \|\Theta^*\|\}.$$

Then,

$$\mathcal{L}\left((\Theta_t)_{t\in\mathbb{Z}} \middle| \left(\frac{\Theta_{T^*+t}}{\|\Theta^*\|}\right)_{t\in\mathbb{Z}}\right) = \sum_{k\in\mathbb{Z}} \frac{\|\Theta_{T^*+k}\|^\alpha}{\sum_{s\in\mathbb{Z}} \|\Theta_s\|^\alpha} \delta_{\left(\frac{\Theta_{T^*+k+t}}{\|\Theta_{T^*+k}\|}\right)_{t\in\mathbb{Z}}},$$

where δ_x denotes the Dirac measure in $x \in (\mathbb{R}^d)^{\mathbb{Z}}$.

Short: Conditional distribution is RS-process of observed pattern.

General idea for estimator

Conditional distribution of Θ_0

... which gives

$$P\left(\Theta_0 \in A \middle| \left(\frac{\Theta_{T^*+t}}{\|\Theta^*\|}\right)_{t\in\mathbb{Z}}\right) = \sum_{k\in\mathbb{Z}} \frac{\|\Theta_{T^*+k}\|^\alpha}{\sum_{s\in\mathbb{Z}}\|\Theta_s\|^\alpha} \mathbb{1}_A\left(\frac{\Theta_{T^*+k}}{\|\Theta_{T^*+k}\|}\right)$$

$$= \sum_{k\in\mathbb{Z}} \frac{\|\Theta_k\|^\alpha}{\sum_{s\in\mathbb{Z}}\|\Theta_s\|^\alpha} \mathbb{1}_A\left(\frac{\Theta_k}{\|\Theta_k\|}\right) \quad (1)$$

and the highest weight is attained at cluster maximum.

Idea: Try to identify clusters (that is an i.i.d. sample from the distribution of $(\Theta_{T^*+t}/\|\Theta^*\|)_{t\in\mathbb{Z}}$) and use (1) for estimator construction.

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General idea for estimator

- But how to identify "clusters"?
- Very simple approach: Split time series into blocks of size c.
- Estimate α and take the $(X_{k_i c+1}, \ldots, X_{(k_i+1)c}), i = 1, \ldots, m$ with largest values of $\sum_{s=k_i c+1}^{(k_i+1)c} \|X_s\|^{\hat{\alpha}}$ as manifestations of patterns.
- Apply

$$\hat{P}_{k_i}(\Theta_0 \in A) = \sum_{j=k_i c+1}^{(k_i+1)c} \frac{\|X_j\|^{\hat{\alpha}}}{\sum_{s=k_i c+1}^{(k_i+1)c} \|X_s\|^{\hat{\alpha}}} \mathbb{1}_A\left(\frac{X_j}{\|X_j\|}\right).$$

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Does this work?

Example 1: Max-stable process from before.

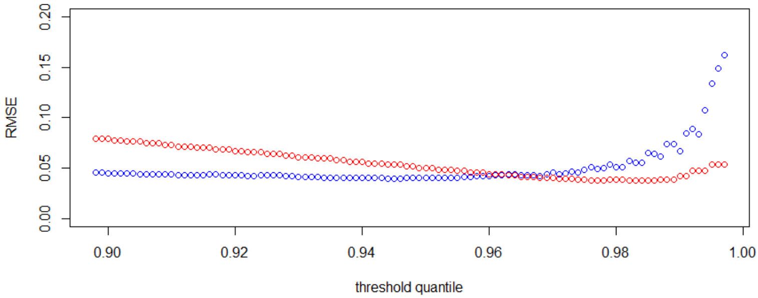
Recipe from before with deterministic spectral process

$$\dots, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{3.2} \end{pmatrix}, \begin{pmatrix} \frac{1.2}{3.2} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{3.2} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \dots$$

Want to estimate $P(\measuredangle\Theta_0 \in [0.4\pi, 0.6\pi])$.

Does this work?

RMSE from 1000 simulations of a time series of length 1000.



Blue: All observations with norm over corresponding quantile
Red: As previously explained, cluster length 10, $\hat{\alpha}$ is Hill estimator for norms at 90% quantile.

Quantiles at x-axes are correct for old estimator, and adjusted by extremal index for cluster based estimator, such that approximately the same number of extremal observations is used for vertically aligned dots.

Does this work?

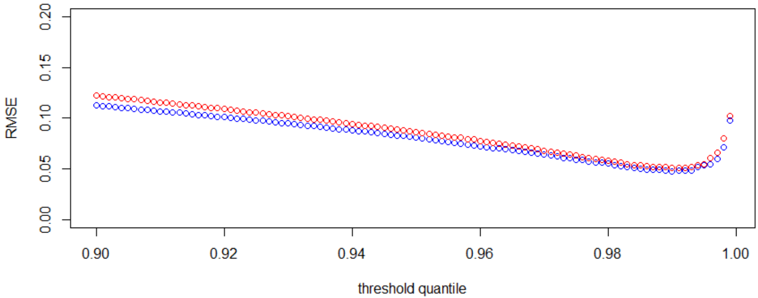
Example 2: Random Difference Equation

$$\mathbf{X}_t = Z \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \mathbf{X}_{t-1} + Y \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

where $\mathbf{X}_{t-1} \in \mathbb{R}^2$, $Z \sim \mathcal{N}(0,1), \theta \sim \mathcal{U}[0, 2\pi]$, $Y \sim \text{Par}(3)$ all independent.
Then, $\measuredangle\Theta_0 \sim \mathcal{U}[0, 2\pi], \alpha = 2$.
Estimate again $P(\measuredangle\Theta_0 \in [0.4\pi, 0.6\pi])$.

Does this work?

RMSE from 1000 simulations of a time series of length 10000.



Blue: All observations with norm over corresponding quantile

Red: As previously explained, cluster length 10, $\hat{\alpha}$ is Hill estimator for norms at 90% quantile.

Quantiles at x-axes are correct for old estimator, and adjusted by extremal index for cluster based estimator, such that approximately the same number of extremal observations is used for vertically aligned dots.

Summary

- The time change formula can be expressed in terms of invariance under the RS-transformation (for summable tail processes).
- Furthermore, RS-process relates spectral functions of max-stable process and the corresponding tail process.
- Perhaps a better understanding of cluster behavior can even lead to improved estimators for multivariate quantities?

Thank you for your attention!

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