Regularly Varying Random Fields

BIRS-CMO Workshop

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Multivariate Regular Variation

Definition.

A random vector $\mathbf{X}$ *is said to be regularly varying with index* $\alpha \in (0, \infty)$ *if there exists a regularly function* $V(x)$ *with index* $-\alpha$ *(i.e.,* $\lim_{x\to\infty} V(ux)/V(x) = u^{-\alpha}$ *for* $u > 0$*), and a nonzero Radon measure* μ *on* $\overline{\mathbb{R}}^d \backslash \{\mathbf{0}\} = [-\infty, \infty]^d \backslash \{\mathbf{0}\}$ *such that*

$$\frac{\mathbb{P}(x^{-1}\mathbf{X} \in \cdot)}{V(x)} \xrightarrow{v} \mu(\cdot)$$

as $x \to \infty$.

Tail Dependence of Stationary Stochastic Processes

- Leadbetter (1983): extremal index θ

  Assuming the regular variation...

- Basrak and Segers (2009): tail process

  $$\mathcal{L}(x^{-1}\mathbf{X}_s, \ldots, x^{-1}\mathbf{X}_t \,|\, \|\mathbf{X}_0\| > x) \xrightarrow{x \to \infty} \mathcal{L}(\mathbf{Y}_s, \ldots, \mathbf{Y}_t \,|\, \|\mathbf{X}_0\| > x)$$

Questions

- Can we go from $\mathbf{X}_t, t \in \mathbb{Z}$ to $\mathbf{X}(\mathbf{t}), \mathbf{t} \in \mathbb{Z}^k$?
- What is an appropriate way to extend?

Outline

Notations

- $M_X(A) = \max_{\mathbf{t} \in A} \|\mathbf{X}(\mathbf{t})\|$, for an index set $A \subset \mathbb{Z}^k$
- $\mathcal{R}_{\mathbf{n}} = \{\mathbf{i} \in \mathbb{Z}^k : -(n_\ell - 1) \leq i_\ell \leq (n_\ell - 1), \ell = 1, \ldots, k\}$
- $\mathcal{R}_{\mathbf{n}}^+ = \{\mathbf{i} \in \mathbb{Z}^k : 0 \leq i_\ell \leq (n_\ell - 1), \ell = 1, \ldots, k\}$

The Spatial Extremal Index

Definition.

Let $(\mathbf{X}(\mathbf{t}) : \mathbf{t} \in \mathbb{Z}^k)$ *be a stationary random field, and* $(\mathcal{R}_\mathbf{n})$ *be a sequence of nondecreasing hypercubes. Suppose for each fixed* $\tau > 0$ *and any sequence* $(u_\mathbf{n}(\tau))$ *satisfying*

$$|\mathcal{R}_\mathbf{n}|\,\mathbb{P}(\|\mathbf{X}(\mathbf{0})\| > u_\mathbf{n}(\tau)) \to \tau$$

as $\mathbf{n} \to \infty$*, it holds that*

$$\mathbb{P}\left(M_X(\mathcal{R}_\mathbf{n}) \le u_\mathbf{n}(\tau)\right) \to e^{-\theta\tau}.$$

Then we say that the extremal index of the random field is θ*.*

Computing θ: Method 1

- Stationary stochastic processes

$$\theta = \lim_{n\to\infty} \mathbb{P}\left(\max_{i=1,\ldots,r_n-1} \|\mathbf{X}_i\| \leq u_n \middle| \|\mathbf{X}_0\| > u_n \right)$$

(O'Brien (1987))

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- Stationary random fields

?

For stationary stochastic processes:

$$\begin{aligned}&\mathbb{P}\left(\max_{i=1,\dots,r_n-1}\|\mathbf{X}_i\|\leq u_n\middle|\|\mathbf{X}_0\|>u_n\right)\\=&\mathbb{P}\left(\max_{i=0,\dots,r_n-2}\|\mathbf{X}_i\|\leq u_n\middle|\|\mathbf{X}_{r_n-1}\|>u_n\right)\end{aligned}$$

For random fields:

Which corner should we condition on?

Example

- $Z(\mathbf{t}), \mathbf{t} \in \mathbb{Z}^2$: iid standard Pareto random variables
- $X(\mathbf{t}) = \max\{Z(\mathbf{t} - \mathbf{1}), Z(\mathbf{t})\}$

Example

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Computing θ: Method 2

- Stationary stochastic processes, assuming vanish condition

$$\theta = \lim_{\mathbf{n}\to\infty} \frac{\mathbb{P}(M_X(\mathcal{R}^+_{r_n}) > u_n)}{r_n\,\mathbb{P}(\|\mathbf{X}_0\| > u_n)} \quad \text{(Basrak and Segers (2009))}$$

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- Stationary random fields, assuming $\Delta(u_{\mathbf{n}})$-condition

$$\theta = \lim_{\mathbf{n}\to\infty} \left(\mathbb{E}\left[\sum_{\mathbf{t}\in\mathcal{R}^+_{\mathbf{r_n}}} \mathbb{1}(\|\mathbf{X}(\mathbf{t})\| > u_{\mathbf{n}}) \mid M_X(\mathcal{R}^+_{\mathbf{r_n}}) > u_{\mathbf{n}}\right]\right)^{-1}$$

$$= \lim_{\mathbf{n}\to\infty} \frac{\mathbb{P}(M_X(\mathcal{R}^+_{\mathbf{r_n}}) > u_{\mathbf{n}})}{\left(\prod_{\ell=1}^{k} r_{n_\ell}\right)\mathbb{P}(\|\mathbf{X}(\mathbf{0})\| > u_{\mathbf{n}})} \quad \text{(Pereira et al (2017))}$$

Condition: Vanish Condition

- $\lim_{\mathbf{m}\to\infty} \limsup_{\mathbf{n}\to\infty} \mathbb{P}\left(M_X(\mathcal{R}_{\mathbf{r_n}} \backslash \mathcal{R}_{\mathbf{m}}) > u_{\mathbf{n}} \mid \|\mathbf{X}(\mathbf{0})\| > u_{\mathbf{n}}\right) = 0$
- $\mathbf{Y}(\mathbf{t}) \stackrel{\text{a.s.}}{\to} \mathbf{0}$ as $\mathbf{t} \to \infty$, $\theta > 0$

Condition: $\Delta(u_{\mathbf{n}})$-condition

- The coordinatewise mixing condition (Choi (2002))

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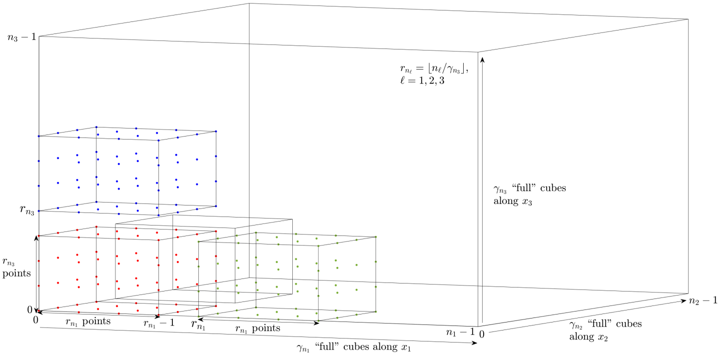

Outline

Existence Of The Tail Field

Theorem 1.

Let $(\mathbf{X}(\mathbf{t}) : \mathbf{t} \in \mathbb{Z}^k)$ *be a stationary random field, and* $\alpha \in (0, \infty)$. *Then it is jointly regularly varying with index* α*, if and only if there exists a random field* $(\mathbf{Y}(\mathbf{t}) : \mathbf{t} \in \mathbb{Z}^k)$ *such that*

$$\mathcal{L}\left(x^{-1}\mathbf{X}(\mathbf{t}) : \mathbf{t} \in \mathbb{Z}^k \mid \|\mathbf{X}(\mathbf{0})\| > x\right) \to \mathcal{L}\left(\mathbf{Y}(\mathbf{t}) : \mathbf{t} \in \mathbb{Z}^k\right)$$

as $x \to \infty$, *and* $\mathbb{P}(\|\mathbf{Y}(\mathbf{0})\| > y) = y^{-\alpha}$ *for* $y \geq 1$.

Properties Of The Tail Field

1. Let $\mathbf{\Theta}(\mathbf{t}) = \mathbf{Y}(\mathbf{t})/\|\mathbf{Y}(\mathbf{0})\|$, then $\|\mathbf{Y}(\mathbf{0})\| \perp (\mathbf{\Theta}(\mathbf{t}) : \mathbf{t} \in \mathbb{Z}^k)$.

2. For any bounded measurable $g : (\overline{\mathbb{R}}^d)^{\mathbb{Z}^k} \to \mathbb{R}$, and $\mathbf{s} \in \mathbb{Z}^k$:
   - $$\mathbb{E}\left[g(\mathbf{Y}(\cdot - \mathbf{s}))\mathbb{1}\left(\mathbf{Y}(-\mathbf{s}) \neq \mathbf{0}\right)\right] = \int_0^\infty \mathbb{E}[g(r\mathbf{\Theta}(\cdot))\mathbb{1}(r\|\mathbf{\Theta}(\mathbf{s})\| > 1)]\, d(-r^{-\alpha})$$
   - $$\mathbb{E}\left[g(\mathbf{\Theta}(\cdot - \mathbf{s}))\mathbb{1}\left(\mathbf{\Theta}(-\mathbf{s}) \neq \mathbf{0}\right)\right] = \mathbb{E}\left[g\left(\frac{\mathbf{\Theta}(\cdot)}{\|\mathbf{\Theta}(\mathbf{s})\|}\right)\|\mathbf{\Theta}(\mathbf{s})\|^\alpha\right]$$

The Sufficient Condition for The Joint Regular Variation

- Stochastic Processes
  the weak convergence on the set of nonnegative times
  $\Rightarrow$ joint regular variation of the original process
  (Basrak and Segers (2009))

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  the weak convergence on the set in first ‘quadrant’
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  No!
  The weak convergence of the **entire** random field is needed.

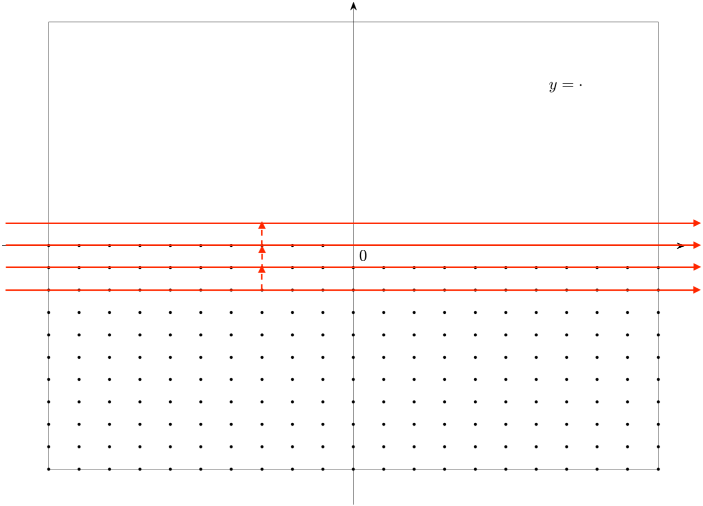
Outline

Computing θ: The Tail Field Expression

$(\mathbf{X}(\mathbf{t}) : \mathbf{t} \in \mathbb{Z}^k)$: stationary, satisfying $\Delta(u_{\mathbf{n}})$-condition, and vanish condition.

- $\theta = \mathbb{P}\left(\max_{\mathbf{t} \prec \mathbf{0}} \|\mathbf{Y}(\mathbf{t})\| \leq 1\right)$

Ordering Examples in 2D



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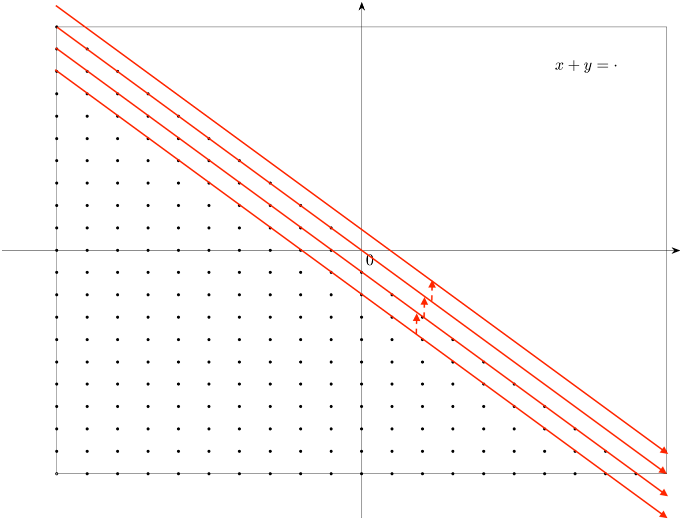

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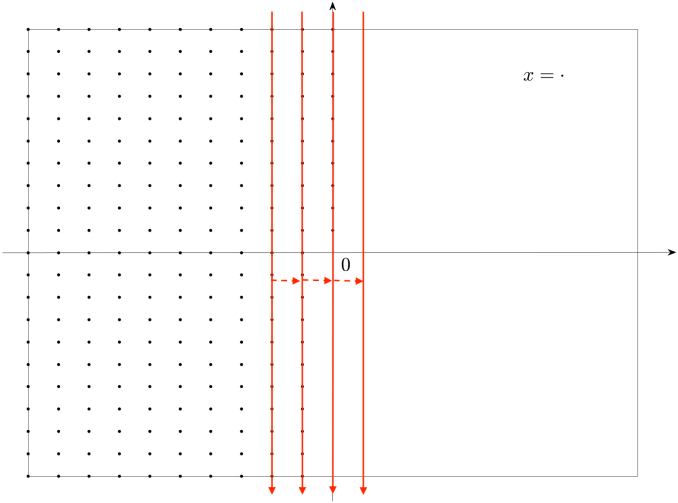

Computing θ: The Tail Field Expression

$(\mathbf{X}(\mathbf{t}) : \mathbf{t} \in \mathbb{Z}^k)$: stationary, satisfying $\Delta(u_{\mathbf{n}})$-condition, and vanish condition.

- $\theta = \mathbb{P}\left(\max_{\mathbf{t} \prec \mathbf{0}} \|\mathbf{Y}(\mathbf{t})\| \leq 1\right)$
- Ordering does not matter!

Outline

Constructing BRRFs

- $(W(\mathbf{t}) : \mathbf{t} \in \mathbb{Z}^k)$: zero-mean Gaussian random field $\mathcal{L}(W(\mathbf{t}+\mathbf{s}) - W(\mathbf{s}))$ does not depend on $\mathbf{s}$
- $(W_i(\mathbf{t}) : \mathbf{t} \in \mathbb{Z}^k)$, $i \in \mathbb{N}$: iid copies of $(W(\mathbf{t}) : \mathbf{t} \in \mathbb{Z}^k)$
- $\sum_{i=1}^{\infty} \delta_{U_i}$: a Poisson process on $\mathbb{R}$ with intensity du/u^2

Constructing BRRFs

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- $\sum_{i=1}^{\infty} \delta_{U_i}$: a Poisson process on $\mathbb{R}$ with intensity du/u^2
- $X(\mathbf{t}) = \max_{i=1,2,\ldots} U_i \exp\{W_i(\mathbf{t}) - \sigma^2(\mathbf{t})/2\}$

Its Tail Field

- The Joint Distribution

$$
\begin{aligned}
&\mathbb{P}(Y(\mathbf{t}_1) < y_1, \ldots, Y(\mathbf{t}_n) < y_n) \\
&= \mathbb{E}\left[\max_{i=1,\ldots,n}\left(\frac{1}{y_i}\exp\left\{W(\mathbf{t}_i) - \frac{\sigma^2(\mathbf{t}_i)}{2}\right\}, \exp\left\{W(\mathbf{0}) - \frac{\sigma^2(\mathbf{0})}{2}\right\}\right)\right] \\
&\quad - \mathbb{E}\left[\max_{i=1,\ldots,n}\frac{1}{y_i}\exp\left\{W(\mathbf{t}_i) - \frac{\sigma^2(\mathbf{t}_i)}{2}\right\}\right]
\end{aligned}
$$

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\end{aligned}
$$

- The Marginal Distribution

$$
\mathbb{P}(Y(\mathbf{t}) < y) = \Phi\left(\frac{\ln y + \gamma(\mathbf{t})}{\sqrt{2\gamma(\mathbf{t})}}\right) - \frac{1}{y}\Phi\left(\frac{\ln y - \gamma(\mathbf{t})}{\sqrt{2\gamma(\mathbf{t})}}\right)
$$

The Spatial Extremal Index

Recall: $\theta = \mathbb{P}\left(\max_{\mathbf{t} \prec \mathbf{0}} Y(\mathbf{t}) \leq 1\right)$

Corollary 2.

Let $(X(\mathbf{t}) : \mathbf{t} \in \mathbb{Z}^k)$ be a BRRF with standard Fréchet margin, satisfying both $\Delta(u_{\mathbf{n}})$-condition and vanish condition. Then

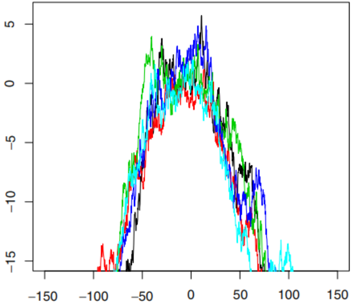
$$\theta = \mathbb{E}\left[\max_{\mathbf{t} \preceq \mathbf{0}} \exp\left\{W(\mathbf{t}) - \sigma^2(\mathbf{t})/2\right\}\right] - \mathbb{E}\left[\max_{\mathbf{t} \prec \mathbf{0}} \exp\left\{W(\mathbf{t}) - \sigma^2(\mathbf{t})/2\right\}\right].$$

In comparison with

$$\theta = \lim_{\mathbf{n} \to \infty} \left(\mathbb{E}\left[\sum_{\mathbf{t} \in \mathcal{R}^+_{\mathbf{r_n}}} \mathbb{1}(X(\mathbf{t}) > u_{\mathbf{n}}) \mid M_X(\mathcal{R}^+_{\mathbf{r_n}}) > u_{\mathbf{n}}\right]\right)^{-1}$$

Simulating BRRFs Is Hard

- Approximation of the Brown-Resnick process based on the definition may result in non-stationarity



Oesting et al (2012). Simulation of Brown-Resnick processes.

Example: Brownian Motions

- $(W_1(t) : t \in \mathbb{Z})$, $(W_2(t) : t \in \mathbb{Z})$: two independent two-sided standard Brownian motions
- $W(t_1, t_2) = W_1(t_1) + W_2(t_2)$

Order	$y = \cdot$	$x + 3y = \cdot$	$x + 2y = \cdot$	$x + y = \cdot$
$\hat{\theta}$	0.0780	0.0786	0.0786	0.0784
$\text{sd}(\hat{\theta})$	0.0008	0.0008	0.0008	0.0008
Order	$2x + y = \cdot$	$3x + y = \cdot$	$x = \cdot$	
$\hat{\theta}$	0.0786	0.0787	0.0781	
$\text{sd}(\hat{\theta})$	0.0008	0.0008	0.0008	

Table: Simulated θ with different ordering

Thank you!