Geometric and categorical aspects of CFTs

G. Borot (Max Planck Institute for Mathematics, Bonn),
D. Ridout (School of Mathematics and Statistics, University of Melbourne)
A. Ros Camacho (Mathematisch Instituut, Universiteit Utrecht)

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This report summarises the organisational details, highlights of talks, and some of the scientific progress that made during this workshop at the Casa Matemática Oaxaca. We had 38 participants from 14 different countries including at least one from every continent except Africa. These included researchers of all ranks from Ph.D. students right through to full professors, all of them working on aspects of conformal field theory (CFT) with a vast variation in their approaches. During this workshop, we had 4 introductory lectures (mini-courses) that were specifically chosen to be on topics that would make connections between the backgrounds of the participants, complemented by 13 research talks of 40min each on recent advances in the field. We have limited the number of research talks to leave ample time for discussions between the attendants, which have been very lively. The talks were spread over 4 days: Friday was left free for private discussions or departure as some participants were planning to leave through the day. All in all, this event was considered a success by all attendees. The fourth organiser (J.E. Andersen) had to cancel his coming due to last minute issues.

Aims of the workshop

The two main aims that motivated our workshop proposal were:

1. To discuss the interrelations between the algebraic approaches, specifically *vertex operator algebras* (VOAs) *and their representation theory*, and the geometric approaches to two-dimensional CFTs, with a view towards extending the successes of both to also encompass non-semisimple examples.

2. To relate certain constructions relevant to low-dimensional topological quantum field theories, specifically *modular functors*, *cohomological field theories*, *topological recursion* and *factorisation homology*, in which axioms involving the degeneration of surfaces play a prominent role, in order to disseminate mathematical techniques between communities.

The first aim was motivated by the ongoing interest in logarithmic CFT which describes the situation in which conformal invariance interplays with non-semisimple representations of the corresponding VOA. One of the big questions in this area is whether the notion of a modular tensor category (MTC), familiar from categories of modules over rational VOAs, has a useful generalisation to the logarithmic setting. This workshop brought together experts in the categorical aspects of VOAs in order to pool our knowledge concerning this important question. Crucially, new constructions and examples of a geometric flavour are starting to appear and are expected to shed further light on non-semisimple MTCs as well as open the door to utilising the powerful methods of geometry.

To introduce the second aim, let us say that many mathematical notions and constructions have been introduced and developed in close contact to their uses in quantum field theories, and sometimes more specifically to conformal field theories. Some of them evolved in theories that are be studied for themselves, and also find applications in other areas of mathematics. Therefore, the landscape of knowledge is very wide, and at the same time, scattered among different communities. It is important to regularly keep ourselves updated with these developments, in order to benefit from cross-fertilisation but also with the deeper purpose of unifying as much as possible these approaches growing in different directions. This workshop was structured around four such directions: topological recursion, matrix factorisation, factorisation homology and the geometry and representation theory of VOAs.

Overview of the field and progress made

In the following, we will discuss these four directions with a particular focus on the state-of-the-art and the scientific progress made at the workshop.

Topological recursion

Topological recursion is a computational scheme, related to the combinatorics of pair of pants on surfaces, for solutions of certain systems of compatible linear, often infinite-dimensional, PDEs. Initially discovered by Chekhov, Eynard and Orantin in the context of matrix models, it has found many applications in the enumerative geometry of curves and in low-dimensional quantum field theories. Less abstractly, the topological recursion defines quantities $(\omega_{g,n})_{g,n}$ indexed by two non-negative integers g, n, by induction on $2g-2+n > 0$, mimicking the decomposition of surfaces of genus g with n boundaries. The base cases and the recursion operator are specified by the data of a spectral curve, or by certain collections of compatible linear PDEs (the quantum Airy structure recently formalised by Kontsevich and Soibelman). What the $\omega_{g,n}$ in general compute is unknown, but these quantities have many structural properties (holomorphic anomaly equation, variational equation and special geometry, relations to intersection theory on the Deligne–Mumford moduli space of curves, etc.) that often appear in field-theoretic contexts, making this construction interesting *per se*.

For well-adjusted spectral curves or quantum Airy structures, the $\omega_{g,n}$ encode information that solve interesting geometric problems. For instance, with an appropriate choice of initial data, the associated $(\omega_{g,n})_{g,n}$ will be correlation functions of two-dimensional topological quantum field theories, or correlation functions of semi-simple cohomological field theories (CohFTs), or generating series of descendent Gromov–Witten invariants, or tau functions of certain integrable systems, or contain information on conformal blocks of certain CFTs, etc. The definition of topological recursion is axiomatic, but it relies on an algebraic and geometric structures that can (or should more generally) be identified in quantum field theories.

The mini-course of Orantin presented the basic theory of topological recursion and its application to 2d TQFT, while he prepared the background so that the application to CohFTs could be discussed after the lecture. His exposition was praised by many people in the audience who could grasp the underlying ideas. Orantin's mini-course generated discussions with Murfet to import ideas from the theory of the topological recursion to address questions of glueing and descendent insertions in A_∞-category constructions attached to Landau–Ginzburg models.

Belliard presented his work on Toda conformal field theory, which is a CFT with symmetry given by the $(q, t = q^\beta)$-deformed $\mathcal{W}$ algebra at level 1 in the limit $q \to 1$. Together with Eynard and Ribault [3], he showed that the Ward identities for the current correlators in this field theory can be solved — when expanded in the parameter $\epsilon \to 0$ that governs the heavy charge regime — by a generalisation of the topological recursion associated to non-commutative spectral curves (i.e. D-modules on curves). Their work is a strong sign that topological recursion and its possible generalisations should have more to say on conformal field theory in the future.

Borot, Bouchard and Chidambaram continued their collaboration (together with T. Creutzig who was invited but could not attend) during the workshop which resulted in the article [5]. They gave a general construction of initial data for topological recursion using the representation theory of $\mathcal{W}$-algebras of simple simply laced complex Lie algebras at the self-dual level. A particular case of this construction is due to

Milanov, who concluded that the $\omega_{g,n}$ associated to the spectral curve $x = y^r$ by topological recursion are encoding the descendent potential of the A_{r-1}-singularity, or equivalently the intersection theory of Witten's r-spin class on $\overline{\mathcal{M}}_{g,n}$. In particular, [5] shows that the topological recursion for the spectral curves which locally behave like $x^{r-s}y^r = 1$ is well-defined if and only if $r = \pm 1 \mod s$, and in this case, it is equivalent to $\mathcal{W}(\mathfrak{gl}_r)$-constraints.

This also suggests that there should exist an enumerative geometry meaning for the corresponding $\omega_{g,n}$ in these cases. This work can be considered one step less quantum than the work of Belliard and Eynard (as it does not deal with non-commutative spectral curves), but this suggests that many more $\mathcal{W}$-like algebras and modules can be incorporated in the formalism of [5]. The workshop was an occasion for Borot, Bouchard and Chidambaram to initiate discussions with the $\mathcal{W}$-algebraists Arakawa and Moreau about possible relations between this construction and the known quantization procedures of nilpotent orbits.

Cohomological field theories and matrix factorizations

An extra target in our workshop was to better understand geometric realizations of cohomological field theories and possible relations between them. Vaintrob gave a mini-course describing his construction with Polishchuk [13] of a cohomological field theory associated with an equivariant, quasi-homogeneous, non-degenerate Landau–Ginzburg (LG) model. He started with a very nice introduction to differential-graded categories and homological invariants (Chern characters) that can be obtained from them. He then described the differential-graded category of (equivariant) matrix factorizations attached to a Landau–Ginzburg model. Its Hochschild homology yields the (equivariant) Jacobi ring of the LG model — in the non-equivariant case this is a result of Dyckerhoff — where the pairing is given by the Euler characteristic.

Vaintrob then sketched the key idea of his construction: there is a fundamental matrix factorisation over the product of the moduli space of W-spin curves with a certain affine space, which one can carry to a cohomological field theory using the Hochschild–Kostant–Rosenberg map from Hochschild homology to the cohomology of the moduli space. He explained that equivariance is used for technical reasons, to avoid infinite-dimensional issues in the construction of the fundamental matrix factorisation. It is expected that the restriction to the non-equivariant sector coincides with the Fan–Jarvis–Ruan–Witten cohomological field theories (of which the Witten r-spin class is an example) and this is proved for the ADE singularities. In this latter case, it is known that the correlation functions satisfy $\mathcal{W}$-constraints which are obtained as a particular case in [5]. It is hoped that more generally the correlation functions of the CohFTs of Polishchuk and Vaintrob can be computed by topological recursion: discussions between Borot, Orantin and Vaintrob raised the question of whether one could propose $\mathcal{W}$-like constraints for them via an approach inspired by [5].

As of relations between different cohomological field theories, Ros Camacho and Vaintrob started discussing about how to give a geometric description to the notion of "orbifold equivalence", a term first introduced in [6]. This is an equivalence relation between potentials describing different LG models, of which we know very few examples [7, 9–11]. This notion was first introduced in a higher categorical context and possible generalisations of this notion to the framework of cohomological field theories, related to the work of Polischuk–Vaintrob [13], were discussed. These discussions continue over email to date.

Factorisation homology

A factorisation algebra on a topological space X is a functor F from the category of open subsets to a given symmetric monoidal (higher) category satisfying a co-sheaf property for a certain class of covers. It is a good model for the collections of quantum fields in QFT taking into account local to global properties. A factorisation algebra is locally constant if for any retract $U \to V$, then $F(U) \to F(V)$ is an equivalence. On n-manifolds, locally constant factorisation algebras are characterised by the value on $\mathbb{R}^n$ and are equivalent to E_n-algebras.

Factorisation homology — developed by Costello and Gwilliam — is a geometric invariant of an n-dimensional space X using E_n-algebras as "coefficients". It can equally be seen as an invariant attached to the E_n-algebra. For instance, an E_1-algebra is an associative algebra and the value of factorisation homology on the circle retrieves the Hochschild homology. With other input it can also retrieve algebras arising by quantization of certain moduli spaces. An important application of factorisation homology (worked out by Calaque and Scheimbauer using earlier work of Gwilliam and of Lurie) is to construct fully extended topological field theories.

The mini-course of Jordan presented us with the basics and first applications of factorisation homology and a few of its applications. A complement lecture of Williams reviewed the relation to other mathematical structures which appear in various quantum field-theoretic contexts, especially beyond locally constant factorisation algebras. For instance:

- chiral/vertex algebras of Beilinson–Drinfeld (which axiomatise CFTs) determine factorisation algebras.
- one-dimensional σ-models can be used to retrieve, after taking factorisation homology on $\mathbb{S}^1$, the $\hat{A}$-genus.
- holomorphic sigma models can be used to retrieve, after taking factorisation homology on elliptic curves, the Witten genus (Gorbounov–Gwilliam–Williams).
- one can associate vertex algebras to holomorphic factorisation algebras on $\mathbb{C}$. Some participants discussed the impossibility to make a backward construction (from vertex algebras to factorisation algebras).

Geometry and representation theory of VOAs

Logarithmic CFTs and categorical aspects

On the fourth day of the workshop, we focused on non-rational (logarithmic) CFTs, starting with a mini-course courtesy of Ingo Runkel on logarithmic VOAs. This reviewed the challenging task of understanding MTCs and, in particular, non-semisimple generalisations of the Verlinde formula, with the aid of a detailed example. Liang Kong then focused on the question whether one can find a category whose centre is a modular tensor category, proposing a solution inspired by the physical bulk-boundary relation for 2d topological orders with a chiral gapless edge. Jürgen Fuchs' lecture focused on recent progress in rigorously constructing arbitrary correlators for certain classes of logarithmic CFTs. While such a construction is required for the consistency of these theories, it is remarkably difficult to prove that an appropriate system of correlators exists.

The challenge of understanding more general classes of CFTs will undoubtedly benefit from the unification of different approaches to CFT. In this spirit, James Tener contributed a talk on a positivity conjecture for unitary vertex operator algebras that he aims to prove using ideas from conformal nets. Christian Blanchet then reviewed the notion of modified traces and their application to quantum group invariants as well as the growing body of evidence that these apply equally well to VOAs, there playing a central role in understanding the modular properties of certain logarithmic theories. Terry Gannon was in charge of the last lecture of the workshop, advocating for the construction of many new examples of MTCs that arise in the theory of subfactors. He laid down a new challenge, essentially to find a means of realising these new MTCs using VOAs. This aligns with the well-known, but by no means undisputed, conjecture that every MTC may be constructed as the category of modules of some rational VOA.

On the collaborative front, Kanade and Ridout made progress on their efforts to establish a natural dictionary between the seemingly different approaches to the fusion product of VOA modules taken by physicists, à la Nahm–Gaberdiel–Kausch, and mathematicians, á la Huang–Lepowsky–Zhang. This work subsequently appeared in [8]. While discussing at the CMO, Ridout and Wood managed to complete their study of the modularity of a new example of a logarithmic CFT: that associated with the simple affine VOA $L_{-3/2}(\mathfrak{sl}_3)$. This is the first such study of a "higher-rank" logarithmic model and will appear shortly on the arXiv. Krauel and Ros Camacho also discussed with Wood the modular forms that describe the characters of the representations present in the bosonic orbifold of the unitary $N = 2$ minimal models in CFT. These have been known for a long time, but have been recently refined in [4]. Krauel and Ros Camacho are using the results in this article in a joint project.

4d SCFTs and CFTs

The two talks of Arakawa and Pei on applications of 4-dimensional superconformal field theories (4d SCFTs) gave a remarkable example of fruitful interactions between geometry, representation theory and theoretical physics which have occurred in the past three years. The classical phase space of these theories on $\mathbb{R}^4$ contain two factors, the Coulomb and the Higgs branch, which are in general hyperkähler varieties with singularities. The physicists Rastelli and Beem (among others) associated to any such 4d SCFT a VOA [2]. In general, this geometrically defined VOA is non-unitary and is almost always non-rational and non-C_2-cofinite. This new construction therefore gives interesting new examples of these types of VOAs. Progressively, a fine dictionary between the (quantum) geometry of these moduli spaces and the underlying CFT is emerging.

Arakawa focused on certain theories (said to be of class S) which are labelled by a semi-simple group and a Riemann surface. For these theories, there exists a mathematical description of the Higgs branch which was described physically by Moore–Tachikawa and rigorously formulated by Braverman–Finkelberg–Nakajima. This involves a 2d TQFT valued in the symmetric monoidal category V (resp. V_{s}) where objects are algebraic groups, morphisms are vertex algebras (resp. symplectic varieties) with the actions of the algebraic groups compatible with the vertex structure (resp. hamiltonian actions), and composition of morphisms is defined by reduction. 2d TQFTs valued in V or V_{s} should be thought of as a refined geometric setting as compared with the more common 2d TQFTs valued in the category Vect of vector spaces. They are particularly interesting for non-lagrangian field theories as it is not possible to apply the ideas of geometric quantization on the phase space to produce directly a 2d TQFT valued in Vect. In his talk, Arakawa explained the intricate definition of V and he showed [1] that for all class S theories of genus 0, the Higgs branch coincides with the associated variety of the VOA obtained by the construction of Beem–Rastelli *et al.*

Pei advertised the possibility of constructing 3d TQFTs from a broader class of 4d SCFTs called Argyres–Douglas theories. The Coulomb branch of each of these theories on $\mathbb{R}^3 \times \mathbb{S}^1$ carries a complex integrable system whose base is the Coulomb branch of the theory on $\mathbb{R}^4$. In certain situations, the Coulomb branch is a wild character variety and these cases are particularly interesting from the geometric Langlands point of view. The superconformal invariance manifests itself in terms of an $\mathbb{S}^1$-action respecting the integrable system. Let n be the degree of this action on the holomorphic symplectic form. The case $n = 1$ is realised for instance by theories of class S, while $n \geq 2$ is realised by the Argyres–Douglas theories (which do not have a lagrangian description). n is also related to the number of Stokes rays in the description of the moduli space via irregular flat connections. The hyperkähler structure is then preserved by the subgroup $\mathbb{Z}_n \subset \mathbb{S}^1$.

In joint works [12], Pei has studied 3d TQFTs that come from a 4d SCFT on a manifold of the form $M \times_\gamma \mathbb{S}^1$, where $\gamma \in \mathbb{Z}_n^*$ and one applies a topological twist to the 3-manifold M. He put forward the conjecture that the $\mathbb{Z}_n$-action corresponds to the Galois action on the S and T matrices. He illustrated this phenomenon on a small subclass of Argyres–Douglas theories that are labelled by a pair of ADE root systems, in particular showing that the Galois action can transform one into another. As a contribution to the dictionary between the geometric and the VOA side, Pei proposed that the $\mathbb{S}^1$ fixed points should correspond to simple objects in the MTC attached to the corresponding VOA, that the eigenvalues of the T-matrix are related to the value of the moment map (coming from the $\mathbb{S}^1$-action), and that the first row of the S-matrix could be read from the action of $\mathbb{Z}_n$ on the normal bundle at the fixed points. As it was pointed out that almost all of the 4d SCFTs he considered were non-rational and therefore did not give MTCs, this gave further impetus to the question of how to generalise MTCs to the non-semisimple setting. It was likewise proposed that we need to use these new methods to explore the fusion rules and braiding on the CFT side, for example in terms of the Morse flow on the moduli space side.

Elliptic genera and CFTs

Wendland reviewed a hierarchy of quantities that one can attach to a complex d-dimensional manifold M. At the bottom of the hierarchy, the Euler characteristic and the Hirzebruch genus are topological invariants that have an index interpretation. The complex elliptic genus refines both: it arises from the equivariant index of a Dirac operator on the loop space of M and it is generally a weak Jacobi form (perhaps with an anomaly in case M is not Calabi–Yau) in two variables, one taking values in some subset of $\mathbb{C}$ and the other taking values in the upper half-plane. Physicists expect that for any Calabi–Yau d-fold M, there exists a family (over the moduli space of complex structures on M) of $N = 2$ SCFTs of central charge $c = 3d$ whose (suitably interpreted) partition functions give the complex elliptic genus of M. This is known at least for the elliptic curve and for K3 surfaces, though the subtleties of non-rationality have not yet been explored in detail. Moreover, the properties of the SCFT attached to the moduli space of K3 surfaces offers a physical understanding of the phenomenon known as Mathieu moonshine.

Inspired by the introduction of a further refinement, due to Kachru and Tripathy, called the Hodge elliptic genus, Wendland proposed [14] a chiral elliptic genus which is constructed from d copies each of the $\beta\gamma$- and bc-ghost CFTs and asked for its interpretation in terms of the SCFT attached to M. In fact, on the moduli space $\mathcal{X}$ of complex structures on M, the space of states need not have constant rank. Wendland advertised that there should exist a generic space of states H_0, forming a vector bundle over $\mathcal{X}$ (in particular having a given rank), on which one can trace a suitable operator coming from the superconformal algebra, the result then coinciding with the chiral elliptic genus. She then explained how to realise this program for the K3 SCFTs using the Mathieu moonshine module and also showed that the chiral elliptic genus is independent of the K3 moduli (in analogy with the result of Kachru and Tripathy for the Hodge elliptic genus of K3). Several questions remain open even for this K3 example, for instance, it is unknown whether the natural VOA structures on H_0 and on the Mathieu moonshine module agree.

Feedback

Some participants provided feedback comments which certify the excellent environment created at the workshop, the very active discussions between researchers, and the depth of knowledge exchanged during the event.

> "The workshop was very stimulating. I was especially interested by interaction between LCFT which was central topics in the workshop and TFT which is my main interest, and indeed the meeting was excellent for this purpose. I had opportunity to communicate with new people working on problems and structures which are parallel to those I meet in my field. The conditions of the workshop made communications possible at excellent level." — Christian Blanchet

> "I had several intensive discussions that eventually grew into two scientific projects. In particular, I have started a project jointly with Nathan Geer, on TQFT constructions associated to any non-semisimple modular tensor category. We already obtained a few important results in Oaxaca and are currently writing a paper about them", "For me, the CMO–BIRS workshop was a excellent platform for very stimulating discussions with my colleagues, and I also got new contacts and ideas. It was a great meeting!" — Azat Gainutdinov

Conclusions

All in all, we have experienced an exciting workshop which provided a timely and highly desired overview of various aspects of both the geometric and categorical aspects of CFT. The workshop led to intensive scientific discussions among the participants between and after the talks. We therefore regard it as an unqualified success. Feedback from attendees, during the event and after, was unanimously glowing, both for the quality and breadth of the speakers and the format chosen (introductory plus research level seminars). We thank the Casa Matemática Oaxaca for the generous support that made this possible and hope that we will have the opportunity to organise similar events in the future.

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