Monotone theories

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QE for coloured orders

Theorem (Simon)

The theory of a linearly ordered structure $(M, \leqslant, P_i, R_j)$, *where all* $\varnothing$*-definable unary sets and all* $\varnothing$*-definable monotone relations are named, eliminates quantifiers.*

Definition

- A relation $R \subseteq A \times B$ between linear orders $(A, <_A)$ and $(B, <_B)$ is *monotone* if: $a' <_A a \; R \; b <_B b' \implies a' \; R \; b'$.

  Equivalently, $(R(A, b) \mid b \in B)$ is an increasing sequence of initial parts of A.
- A formula $\phi(x, y)$ is $<$*-monotone* if it defines a monotone relation between $(\mathfrak{C}, <)$ and $(\mathfrak{C}, <)$.
- By a $um_<$*-formula* we mean a Boolean combination of unary and $<$-monotone formulae.

Monotone theories

We introduce monotone theories as theories of linear orders in which every binary definable set has simple geometric description.

Definition

- An ω-saturated structure $M = (M, \ldots)$ is *monotone* if there is an L-definable linear order $<$ on M such that for all $A \subseteq M$ every L_A-formula in two free variables is equivalent to an L_A-$um_<$-formula. In this case we say M is monotone with respect to $<$.
- A complete theory is *monotone* if it has an ω-saturated monotone model.

Weakly quasi-o-minimal theories

Weakly quasi-o-minimal theories are generalization of both weakly o-minimal and quasi-o-minimal theories.

Definition (Kudaĭbergenov)

A theory T is *weakly quasi-o-minimal* with respect to an L-definable linear order $<$ if every definable subset of any model of T is a finite Boolean combination of convex sets and L-definable sets.

A theory is *weakly quasi-o-minimal* if it is weakly quasi-o-minimal with respect to some L-definable linear order.

Characterisation of weak quasi-o-minimality

Fact

The following are equivalent:

(1) *T is weakly quasi-o-minimal with respect to $<$;*

(2) *for every $p \in S_1(T)$ and definable (with parameters) $D \subseteq \mathfrak{C}$, D has finitely many $<$-convex components on $p(\mathfrak{C})$.*

Each of the convex components of D is relatively definable by an instance of $<$-convex formula, or by a Boolean combination of instances of two $<$-initial formulae. By compactness, D is definable by a Boolean combination of unary L-formulae and instances of $<$-initial formulae (using same parameters).

Definition

A formula $\phi(x, \bar{y})$ is: *$<$-convex* (*$<$-initial*) if $\phi(\mathfrak{C}, \bar{a})$ is $<$-convex ($<$-initial part of $\mathfrak{C}$) for every $\bar{a} \in \mathfrak{C}$.

Monotone $\Longrightarrow$ weakly quasi-o-minimal

Proposition

If T is monotone with respect to $<$, then it is weakly quasi-o-minimal with respect to $<$.

Outline of the proof.

Check (2) by induction on the number of parameters used in the definition of D. □

The converse

Theorem

The converse is also true, i.e. T is monotone with respect to $<$ iff it is weakly quasi-o-minimal with respect to $<$.

Theorem

A theory is weakly quasi-o-minimal with respect to some L-definable linear order iff it is weakly quasi-o-minimal with respect to every L-definable linear order.

Corollary

Monotone = weakly quasi-o-minimal.

Proof strategy

- Weak quasi-o-minimality is preserved under naming parameters, so it suffices to show that every L-formula $\phi(x, y)$ is equivalent to an L-$um_<$-formula.
- Every formula $\phi(x, y)$ is equivalent to a Boolean combination of unary and $<$-initial L-formulae, hence it suffices to prove that every $<$-initial formula $\phi(x, y)$ is equivalent to an L-$um_<$-formula.
- Every $<$-initial formula $\phi(x, y)$ defines a total preorder by $y_1 \preccurlyeq y_2$ iff $\phi(\mathfrak{C}, y_1) \subseteq \phi(\mathfrak{C}, y_2)$.

  Observation: $\phi(x, y)$ defines a monotone relation between $(\mathfrak{C}, <)$ and $(\mathfrak{C}, \preccurlyeq)$.

Definable linear orders

Definition

Let E be a $<$-convex equivalence relation. Define $x <_E y$ by:

$$(E(x,y) \wedge y < x) \vee (\neg E(x,y) \wedge x < y).$$

The relation $<_E$ is a linear order, and if $<$ and E are definable, then $<_E$ is definable too.

Remark

If E' is $<$-convex equivalence relation either finer or coarser than E, then E' is $<_E$-convex equivalence relation. We can iterate the construction: if $\vec{E} = (E_1, \ldots, E_n)$ is a decreasing sequence of $<$-convex equivalence relations, then:

$$<_{\vec{E}} = (<_{(E_1,\ldots,E_{n-1})})_{E_n}.$$

$<_{\vec{E}}$ and weak quasi-o-minimality/monotonicity

Lemma

If T is weakly quasi-o-minimal with respect to $<$ and $\vec{E}$ is a decreasing sequence of definable $<$-convex equivalence relations, then T is weakly quasi-o-minimal with respect to $<_{\vec{E}}$.

Outline of the proof.

Every $<$-convex subset of $p(\mathfrak{C})$ has at most three $<_E$-convex components, for a definable $<$-convex equivalence relation E, so the construction does not change the property of having finitely many convex components on $p(\mathfrak{C})$. □

Lemma

If $\phi(x,y)$ defines a monotone relation between $(\mathfrak{C},<)$ and $(D,<_{\vec{E}})$, where D is L-definable, then $\phi(x,y)$ is equivalent to an $um_<$-formula.

The main technical result

Proposition

Suppose that T is weakly quasi-o-minimal with respect to $<$, $\triangleleft$ is an L-definable linear order and $p \in S_1(T)$. There exists a decreasing sequence $\vec{E}$ of $<$-convex equivalence relations such that $\triangleleft$ and $<_{\vec{E}}$ agree on $p(\mathfrak{C})$.

Outline of the proof

- For $a \models p$, $a \lhd x$, $x \lhd a$ and $x = a$ give a finite $<$-convex partition $\mathcal{P}_<$ of $p(\mathfrak{C})$.
- For consecutive $<$-convex parts different from $\{a\}$ one is determined by $a \lhd x$ and the other by $x \lhd a$.
- Let $L_<(a)$ be the leftmost $<$-convex part, $l_<(a)$ the second leftmost, $R_<(a)$ the rightmost and $r_<(a)$ the second rightmost.
- $L_<(a)$ and $R_<(a)$ are not determined by the same formula.
- There exists a definable $<$-convex equivalence relation $E(x, y)$ which agrees with $L_<(x) < y < R_<(x)$ on $p(\mathfrak{C})$.
- $L_{<_E}(a) = L(a) \cup r(a)$, $R_{<_E}(a) = l(a) \cup R(a)$ and other components don't change, so $|\mathcal{P}_{<_E}| = |\mathcal{P}_<| - 2$ and we can proceed by induction.

Total preorders

If $\preccurlyeq$ is a total preorder, denote by $E_{\preccurlyeq}$ the equivalence relation given by $a \preccurlyeq b \wedge b \preccurlyeq a$.

Corollary

Suppose that T is weakly quasi-o-minimal with respect to $<$, $\preccurlyeq$ is an L-definable total preorder and $p \in S_1(T)$. There exists a decreasing sequence $\vec{E}$ of $<$-convex equivalence relations such that $a \preccurlyeq b$ is equivalent with $E_{\preccurlyeq}(a,b) \vee (\neg E_{\preccurlyeq}(a,b) \wedge a <_{\vec{E}} b)$ on $p(\mathfrak{C})$.

Independence on order

Theorem

Suppose that T is weakly quasi-o-minimal with respect to $<$ and $\preccurlyeq$ is an L-definable total preorder. There exist L-definable partition $\mathfrak{C} = D_1 \cup \ldots \cup D_n$ and decreasing sequences $\vec{E}_1, \ldots, \vec{E}_n$ of $<$-convex equivalence relations such that $a \preccurlyeq b$ is equivalent with $E_{\preccurlyeq}(a,b) \vee (\neg E_{\preccurlyeq}(a,b) \wedge a <_{\vec{E}_i} b)$ on D_i for $i = 1, \ldots, n$.

If $\preccurlyeq$ is a linear order, then $\prec$ agrees with $<_{\vec{E}_i}$ on every D_i.

Corollary

A theory is weakly quasi-o-minimal with respect to some L-definable linear order iff it is weakly quasi-o-minimal with respect to every L-definable linear order.

Outline of the proof of monotonicity

- If $\phi(x, y)$ is an $<$-initial L-formula, then by $a \preccurlyeq b$ iff $\phi(\mathfrak{C}, a) \subseteq \phi(\mathfrak{C}, b)$ is defined a total preorder.
- We have an L-decomposition $\mathfrak{C} = D_1 \cup \ldots \cup D_n$ and decreasing sequences of L-definable $<$-convex equivalence relation $\vec{E}_1, \ldots, \vec{E}_n$ such that $a \preccurlyeq b$ iff $E_{\preccurlyeq}(a, b) \vee (\neg E_{\preccurlyeq}(a, b) \wedge a <_{\vec{E}_i} b)$ on D_i.
- This means that $\phi(x, y) \wedge y \in D_i$ defines a monotone relation between $(\mathfrak{C}, <)$ and $(D_i, <_{\vec{E}_i})$, for every $i = 1, \ldots, n$, so it is equivalent to an L-$um_<$-formula.
- The formula $\phi(x, y)$ is equivalent to an L-$um_<$-formula.