Dynamics of Polish groups, submeasures, and a new concentration of measure

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Submeasures and their classifications

Submeasures

$\mathcal{C}$ = an algebra of subsets of X

A function $\phi\colon \mathcal{C} \to \mathbb{R}$ is a **submeasure** if

- $\phi(\emptyset) = 0$,
- ϕ is *monotone*, that is, $\phi(A) \leq \phi(B)$ for all $A, B \in \mathcal{C}$ with $A \subseteq B$, and
- ϕ is *subadditive*, that is, $\phi(A \cup B) \leq \phi(A) + \phi(B)$ for all $A, B \in \mathcal{C}$.

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All submeasures ϕ are assumed to be diffused, that is, for every $\epsilon > 0$, there exists a finite subset $\mathcal{B} \subseteq \mathcal{C}$ such that

$$X = \bigcup \mathcal{B} \quad \text{and} \quad \phi(B) \leq \epsilon \text{ for } B \in \mathcal{B}.$$

ϕ a submeasure on $\mathcal{C}$

ϕ is a **measure** if $\phi(A \cup B) = \phi(A) + \phi(B)$ for disjoint $A, B \in \mathcal{C}$.

ϕ is **pathological** if there does not exist a non-zero measure $\mu\colon \mathcal{C} \to \mathbb{R}$ with $\mu \leq \phi$.

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Herer–Christensen (1975), **Popov** (1976), **Erdős–Hajnal** (1967): There exists a pathological submeasure.

Talagrand: There exists an exhaustive pathological submeasure.

A submeasure ϕ on $\mathcal{C}$ induces a (pseudo-)metric on $\mathcal{C}$

$$\mathrm{dist}_\phi(A, B) = \phi(A \triangle B), \text{ for } A, B \in \mathcal{C}.$$

Classification of submeasures

Let $C_1, \ldots, C_m \subseteq X$. Define

$$t(C_1, \ldots, C_m)$$

to be the maximum of $k \in \mathbb{N}$ such that for each $x \in X$

$$|\{i \mid x \in C_i\}| \geq k.$$

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$\frac{t(C_1,\ldots,C_m)}{m}$ is the **covering number** of Kelley of the sequence $(C_1, \ldots, C_m)$.

$\phi\colon \mathcal{C} \to \mathbb{R}$ a submeasure

For $\xi > 0$, let

$$\mathcal{C}_{\phi,\xi} = \{A \in \mathcal{C} \mid \phi(A) \leq \xi\}.$$

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Define $h_\phi\colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ by

$$h_\phi(\xi) = \frac{1}{\xi}\sup\left\{ \tfrac{t(C_1,\ldots,C_m)}{m} \;\middle|\; m \in \mathbb{N},\ m > 0,\ C_1,\ldots,C_m \in \mathcal{C}_{\phi,\xi} \right\}.$$

The asymptotic behavior of h_ϕ at 0 is restricted.

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Theorem (Sch.–S.)

The limit $\lim_{\xi \to 0} h_\phi(\xi)$ *exists (possibly infinite).*

A submeasure ϕ is called

— **elliptic** if $h_\phi(\xi) = O(\xi)$ as $\xi \to 0$,

— **hyperbolic** if $\frac{1}{h_\phi(\xi)} = O(\xi)$ as $\xi \to 0$,

— **parabolic** if ϕ is neither elliptic, nor hyperbolic.

Proposition (Sch.–S.)

Let ϕ be a submeasure.

(i) *The following conditions are equivalent.*

- — ϕ *is hyperbolic;*
- — ϕ *is pathological;*
- — h_ϕ *is unbounded;*
- — $\lim_{\xi \to 0} \xi h_\phi(\xi) = 1$.

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(ii) *If ϕ is parabolic, then $\lim_{\xi \to 0} h_\phi(\xi)$ exists and is finite.*

(iii) *If ϕ is a measure, then $\lim_{\xi \to 0} h_\phi(\xi) = \frac{1}{\phi(X)}$.*

Topological dynamics and groups of the form $L_0(\phi, G)$

Groups of the form $L_0(\phi, G)$

ϕ a submeasure on $\mathcal{C}$ and G a topological group

Let

$$L_0(\phi, G)$$

be the collection of all $f\colon X \to G$, for which there exists a finite partition $\mathcal{B}$ of X into elements of $\mathcal{C}$ with

$$f \text{ is constant on } B \text{ for } B \in \mathcal{B}.$$

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$\delta, r > 0$ determine a neighborhood of $f \in L_0(\phi, G)$ as the set of all $g \in L_0(\phi, G)$ with

$$\phi(\{x \mid d(f(x), g(x)) > \delta\}) < r.$$

This is the **topology of convergence in** ϕ.

Topological dynamics

G a topological group

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A topological group G is **extremely amenable** if each G-flow has a G-fixed point.

G is **amenable** if each G-flow has a G-invariant, regular, Borel probability measure.

The first example of an extremely amenable group

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Herer–Christensen: If ϕ is a **pathological** submeasure, then $L_0(\phi, \mathbb{R})$ is extremely amenable.

Used methods of functional analysis. The proof does not generalize much beyond $G = \mathbb{R}$.

Two general methods for proving extreme amenability

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(A) Ramsey theory

(B) Concentration of measure

Gromov–Milman: The unitary group of a separable, infinite dimensional Hilbert space is extremely amenable.

Glasner, Pestov: If ϕ is a **measure** and G is an **amenable** locally compact Polish group, then $L_0(\phi, G)$ is extremely amenable.

Dynamics of groups of the form $L_0(\phi, G)$

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If ϕ is **parabolic** *or* **hyperbolic** *and G is* **amenable***, then $L_0(\phi, G)$ is* **extremely amenable***.*

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Theorem (Sch.–S.)

If ϕ is **parabolic** *or* **hyperbolic** *and G is* **amenable**, *then $L_0(\phi, G)$ is* **extremely amenable**.

The theorem above generalizes results of Herer–Christensen, Glasner, Pestov, and, to a large degree, Farah–S. and Sabok.

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Proposition (Sch.–S.)

If ϕ *is* **elliptic** *or* **parabolic** *and* G *is* **not amenable**, *then* $L_0(\phi, G)$ *is* **not extremely amenable**.

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Proposition (Sch.–S.)

If ϕ *is* **elliptic** *or* **parabolic** *and* G *is* **not amenable**, *then* $L_0(\phi, G)$ *is* **not extremely amenable**.
In fact, $L_0(\phi, G)$ *is not even amenable.*

Nets of *mm*-spaces

mm-spaces and their nets

$\mathcal{X} = (X, d, \mu)$ is a **metric measure space**, *mm*-**space** for short, if

— X is a standard Borel space,

— d is a Borel pseudo-metric on X, and

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— X is a standard Borel space,

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For a Borel set $A \subseteq X$ and $r > 0$, we write

$$B_r(A) = \{x \in X \mid d(A, x) < r\}.$$

Let $(\mathcal{X})_{i\in I}$ be a net of *mm*-spaces along a directed order I.

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$(\mathcal{X})_{i\in I}$ has **concentration of measure** if, given Borel sets $A_i \subseteq X_i$ and $r > 0$,

$$\inf_{i\in I} \mu_i(A_i) > 0$$

implies

$$\lim_{i\in I} \mu(B_r(A_i)) = 1.$$

Nets of *mm*-spaces associated with a submeasure

ϕ a submeasure on $\mathcal{C}$

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For a partition $\mathcal{P}$ into elements of $\mathcal{C}$ and a set Ω, define a pseudo-metric $\delta_{\mathcal{P},\phi}$ by

$$\delta_{\mathcal{P},\phi}(x, y) = \phi\Big(\bigcup\{P \in \mathcal{P} \mid x_P \neq y_P\}\Big) .$$

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Given a standard Borel probability space (Ω, μ), let

$$\mathcal{X}(\mathcal{P}) = \left(\Omega^{\mathcal{P}}, \delta_{\mathcal{P},\phi}, \mu^{\otimes \mathcal{P}}\right).$$

$\mathcal{X}(\mathcal{P})$ is an *mm*-space.

Given two partitions $\mathcal{P}$ and $\mathcal{Q}$ into elements of $\mathcal{C}$, we write

$$\mathcal{P} \preceq \mathcal{Q} \iff \forall Q \in \mathcal{Q} \exists P \in \mathcal{P} \; Q \subseteq P.$$

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$$\mathcal{P} \preceq \mathcal{Q} \Longleftrightarrow \forall Q \in \mathcal{Q} \, \exists P \in \mathcal{P} \; Q \subseteq P.$$

$\preceq$ is a directed order. So

$$(\mathcal{X}(\mathcal{P}))_{\mathcal{P}}$$

is a **net of *mm*-spaces**.

Covering concentration of submeasures

We say that a submeasure ϕ has **covering concentration** if the associated with it net $(\mathcal{X}(\mathcal{P}))_{\mathcal{P}}$ of *mm*-spaces has concentration of measure.

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The connection with extreme amenability is given by the following proposition.

Proposition (Sch.–S.)

If ϕ has covering concentration and G is amenable, then $L_0(\phi, G)$ is extremely amenable.

The following theorem is our main result on covering concentration. It implies extreme amenability of $L_0(\phi, G)$ for ϕ hyperbolic or parabolic and G amenable.

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Theorem (Sch.–S.)

Every **hyperbolic** *or* **parabolic** *submeasure has covering concentration.*

The previous theorem does not extend to elliptic submeasures.

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Theorem (Sch.–S.)

There is a submeasure (necessarily elliptic) that does not have covering concentration.

Concentration of measure in products

N a finite non-empty set and $m > 0$

$\mathcal{C} = (C_i)_{1 \leq i \leq m}$ a cover of N, and $w = (w_i)_{1 \leq i \leq m}$ where $w_i \geq 0$

$(\Omega_j)_{j \in N}$ a family of non-empty sets

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Define the pseudo-metric $d_{\mathcal{C},w}$ on $\prod_{j \in N} \Omega_j$ by

$$d_{\mathcal{C},w}(x, y) = \inf \left\{ \sum_{i \in I} w_i \mid \{j \in N \mid x_j \neq y_j\} \subseteq \bigcup_{i \in I} C_i \right\}.$$

The metric $d_{\mathcal{C},w}$ generalizes the Hamming metric on product spaces in a direction that seems "orthogonal" to an important generalization due to Talagrand.

Theorem (Sch.–S.)

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Then, for every $r > 0$,

$$\Big(\bigotimes_{j \in N} \mu_j\Big)\left(\{x \mid f(x) - \mathbb{E}(f) \geq r\}\right) \leq \exp\left(-\frac{kr^2}{4(w_1^2 + \cdots + w_m^2)}\right).$$

The proof uses entropy (extending methods due to Ledoux and Marton, involving “Herbst argument”) and is inspired by a Loomis–Whitney-type theorem due to Bollobás–Thomason and Finner.