The Gross-Pitaevskii Equation and Quantum Vortices

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TOPICS

- **The meaning of the Gross-Pitaevskii equation in the context of many-body quantum mechanics**

  The concept of Bose-Einstein condensation
  The GP limit of the many-body problem

- **The GP description of rapidly rotating Bose gases in anharmonic traps**

  The 2D GP energy functional
  The emergence of vortices
  The three critical velocities and the giant vortex transition

The GP Equation

The **Gross-Pitaevskii (GP) equation** was introduced independently by Eugene P. Gross and Lev P. Pitaevskii in 1961.

It is a nonlinear Schrödinger equation for a complex valued function $\psi(\mathbf{x}, t)$ of space, $\mathbf{x} \in \mathbb{R}^3$, and time, $t \in \mathbb{R}$. As far as this review is concerned, it has the form

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi - \mathbf{\Omega}\cdot\mathbf{L}\,\psi + 2g|\psi|^2\psi$$

with

$\hbar$ = Planck's constant, $m =$ mass, both usually set to 1

V = real valued function, modelling an external potential

$\mathbf{\Omega}$ = angular velocity, $\mathbf{L} = -\mathrm{i}\,\mathbf{x}\times\nabla =$ angular momentum

g = coupling constant for the interaction (here always ≥ 0).

The Stationary Case

If the external potential V is independent of time we may consider solutions of the form

$$\exp(\mathrm{i}\mu t)\psi(\mathbf{x})$$

with $\psi(\mathbf{x})$ satisfying the **time-independent GP equation**

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi - \mathbf{\Omega}\cdot\mathbf{L}\,\psi + 2g|\psi|^2\psi = \mu\psi$$

We shall mostly be concerned with this form of the equation.

Moreover, V will be taken to be confining and we normalize ψ so that

$$\int |\psi(\mathbf{x})|^2 d^3\mathbf{x} = 1.$$

The Meaning of ψ

The function ψ is sometimes called the "superfluid order parameter", or the "macroscopic wave function" of a Bose-Einstein condensate.

The idea is roughly that in a Bose-Einstein condensate a macroscopic number of particles share the same single-particle wave function, so that the quantum mechanical N-particle wave function Ψ of the system with N large should approximately have the form

$$\Psi(\mathbf{x}_1, \dots, \mathbf{x}_N; t) \approx \psi(\mathbf{x}_1, t) \cdots \psi(\mathbf{x}_N, t)$$

and furthermore, ψ should satisfy the GP equation with g related to the particle interaction.

A rigorous mathematical implementation of this idea, which cannot be literally true in general, is not quite simple, however!

The Concept of BEC

One-particle density matrix of an N-particle state with wave function Ψ (t fixed):

$$\rho^{(1)}(\mathbf{x},\mathbf{x}') = N\int \Psi(\mathbf{x},\mathbf{x}_2,\dots\mathbf{x}_N)\Psi(\mathbf{x}',\mathbf{x}_2,\dots\mathbf{x}_N)^* d\mathbf{x}_2\cdots d\mathbf{x}_N.$$

Spectral decomposition:

$$\rho^{(1)}(\mathbf{x},\mathbf{x}') = \sum_i \lambda_i\, \psi_i(\mathbf{x})\psi_i^*(\mathbf{x}')$$

with $\lambda_0 \geq \lambda_1 \geq \dots$ and orthonormal ψ_i, $\sum_i \lambda_i = N$.

BEC in the state Ψ means, by definition, that $\lambda_0 = O(N)$.

Note: For this Ψ need not be a product state!

The Basic Many-Body Hamiltonian

Hamiltonian for N spinless bosons with a pair interaction potential v and V, $\mathbf{\Omega}$, $\mathbf{L}$ as before, operating on *symmetric* functions in $L^2(\mathbb{R}^{3N})$:

$$H = \sum_{j=1}^{N} \left(-\tfrac{1}{2}\nabla_j^2 + V(\mathbf{x}_j) - \mathbf{L}_j \cdot \mathbf{\Omega}\right) + \sum_{1 \leq i < j \leq N} v(|\mathbf{x}_i - \mathbf{x}_j|).$$

The pair interaction potential v is assumed to be radially symmetric, of short range, and repulsive. As written, the Hamiltonian refers to the rotating system of reference.

The Hamiltonian determines the time evolution of the many body states,

$$\Psi(0) \mapsto \Psi(t) = \exp(-\mathrm{i}tH)\Psi(0)$$

but we shall mainly be concerned with its ground state(s).

Many-Body Problems vs. GP

For large N and in a certain dilute limit, called the GP limit, **linear many-body problems** associated with H reduce to **nonlinear one-body problems** associated with the Gross-Pitaevskii equation.

Recall the non-linear term in the GP equation:

$$2g|\psi|^2\psi$$

In the **GP limit** for the many-body system we take $N \to \infty$ and scale the interaction potential v in such a way that

$$g = 2\pi\, Na$$

is kept fixed where a is the *scattering lenght* of v.

Digression: Scattering Lenght

Zero energy scattering equation for the two particle scattering with a potential v:

$$-\frac{\hbar^2}{m}\nabla^2\varphi + v\varphi = 0.$$

Writing $\varphi(\mathbf{x}) = u(r)/r$ with $r = |\mathbf{x}|$ this is equivalent to

$$-\frac{\hbar^2}{m}u''(r) + v(r)u(r) = 0.$$

For r larger than the range of v the solution with $u(0) = 0$ has the form

$$u(r) = (\text{const.})(r - a)$$

with a constant a that is called the *scattering length* of v. Equivalently,

$$a = \lim_{r\to\infty}\left[r - \frac{u(r)}{u'(r)}\right]$$

and this is finite if v decreases at least as $r^{-(3+\varepsilon)}$ at infinity.

The Physical Meaning of the GP limit

The scattering length is a joint measure of the range and the strength of the interaction potential v:

For a hard core potential have $a = \text{radius of the hard core}$, while for a soft potential $a \approx \text{(const.)} \int v(\mathbf{x}) d^3\mathbf{x}$ (1st Born approximation).

The GP limit amounts to scaling the interaction potential,

$$v(r) \to N^2 v(Nr),$$

and taking $N \to \infty$.

The Physical Meaning of the GP limit (cont.)

For dilute gases at $T = 0$ a natural measure of diluteness is the *gas parameter* ρa^3 with ρ the *particle density* ; in a box of fixed side length this is $O(N^{-2})$ in the GP scaling. **The GP limit is a special case of a dilute limit.**

Basic result: To leading order in the gas parameter the energy per particle is $\sim \rho a$.

In a box of fixed side length this is $O(1)$ in GP scaling. Thus:

The GP limit is characterized by the property that the energy per particle due to the interaction is of the same order as the gap in the energy without interaction.

Remark: Other Scalings

Besides the (natural!) GP scaling $N^2 v(N\mathbf{x})$ of the interaction potential, one may consider more generally

$$v(\mathbf{x}) \to N^{3\beta-1} v(N^\beta \mathbf{x})$$

with $0 \leq \beta \leq 1$. If the interaction potential is formally a delta function, $v(\mathbf{x}) = c\delta(\mathbf{x})$, all these scalings lead to the same result, $(c/N)\delta(\mathbf{x})$, but for "genuine" potentials v given by measurable functions this is not so. The case $\beta = 1$ is the GP scaling and $\beta = 0$ is the mean-field Hartree scaling leading to the nonlinear term

$$\int |\psi(\mathbf{y})|^2 v(\mathbf{x} - \mathbf{y}) d^3\mathbf{y}\ \psi(\mathbf{x})$$

in the GP equation.

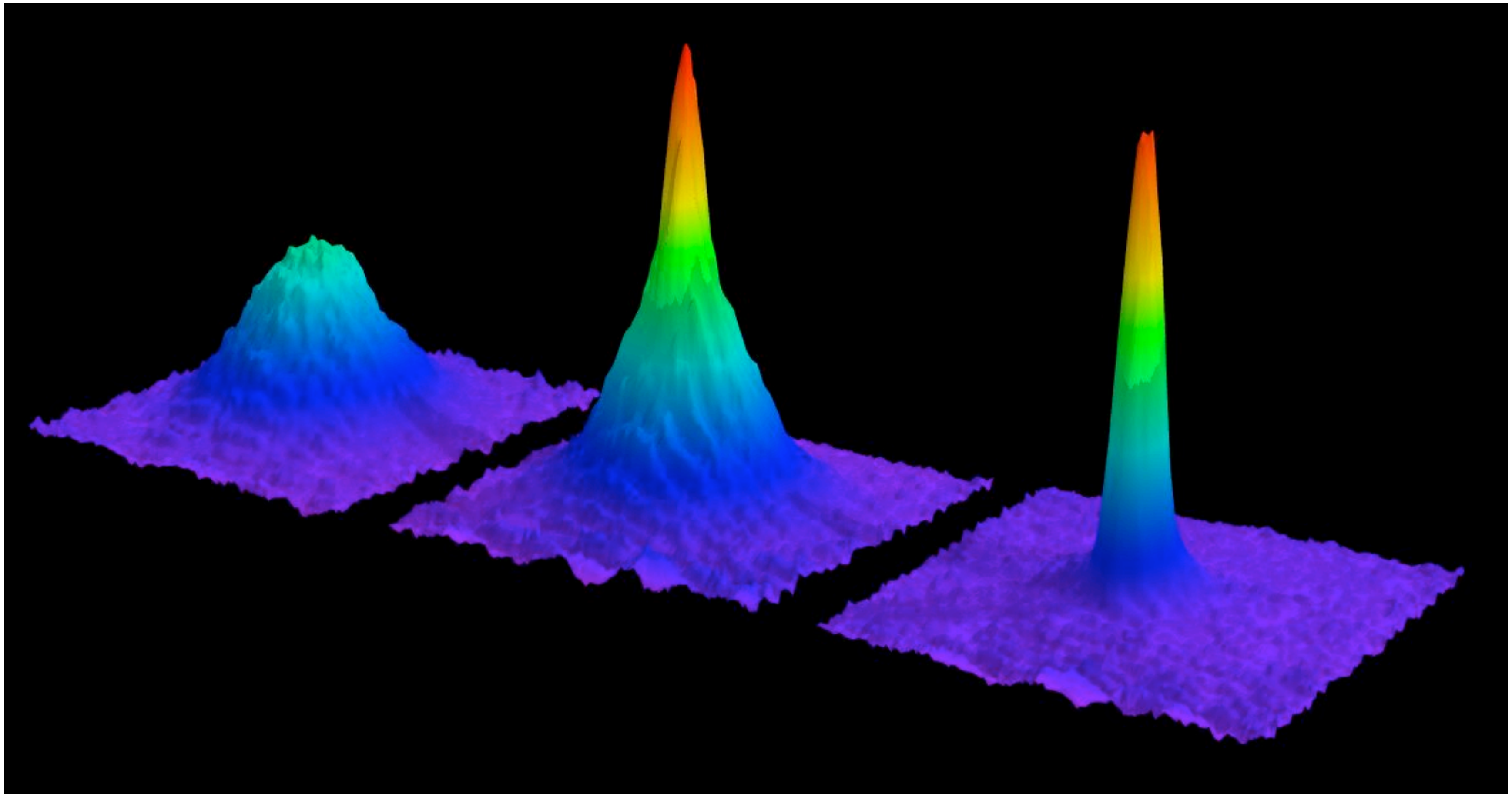
For $0 < \beta < 1$ one obtains the term in the standard form $c|\psi(\mathbf{x})|^2\psi(\mathbf{x})$ but c is proportional to $\int v$ rather than the scattering lenght of v.

The GP Limit of the Many-Body Problem, Results

The basic rigorous results on the relation of the GP equation to the many-body Hamiltonian can be summarized as follows:

- The many-body ground state shows Bose-Einstein condensation in the GP limit. The wave function of the condensate is a solution of the time-independent GP equation.
- The same holds at nonzero temperatures below the BEC critical temperature for an ideal Bose gas.
- If the external potential confining a Bose Einstein condensate is turned off, the many body state remains condensed and the wave function of the condensate follows the time-dependent GP equation.

The mathematical proofs of these results are due to the cumulative effort of many people during the past two decades. See references at the end!



The Gross-Pitaevskii Energy Functional

The time-independent GP equation is obtained by minimizing the energy functional

$$\mathcal{E}^{\rm GP}[\psi] = \int_{\mathbb{R}^3} \left\{ \tfrac{1}{2}|\nabla\psi|^2 + V|\psi|^2 - \psi^* \,\mathbf{\Omega}\cdot\mathbf{L}\,\psi + g|\psi|^4 \right\} d\mathbf{x}$$

with the normalization condition $\int_{\mathbb{R}^3}|\psi|^2 = 1$. A minimizer, which is a solution of the time independent GP equation, will be denoted by $\psi^{\rm GP}$ and the corresponding energy by $E^{\rm GP}$.

In the non-rotating case, $\mathbf{\Omega} = 0$, the minimizer is unique, up to a constant phase factor which may be chosen so that $\psi^{\rm GP}$ is positive.

For $\mathbf{\Omega} \neq 0$ and V rotationally symmetric the minimizer need not be unique (symmetry breaking due to vortices).

GP Functional, Magnetic Version

The GP functional can alternatively be written in the form

$$\mathcal{E}^{\mathrm{GP}}[\psi] = \int_{\mathbb{R}^3} \left\{ \tfrac{1}{2}|(\mathrm{i}\nabla + \mathbf{A})\psi|^2 + (V - \tfrac{1}{2}\Omega^2 r^2)|\psi|^2 + g|\psi|^4 \right\} d\mathbf{x}$$

with

$$\mathbf{A}(\mathbf{x}) = \boldsymbol{\Omega} \times \mathbf{x} = \Omega r\, \mathbf{e}_\theta$$

and r=distance from the rotation axis.

This corresponds to the splitting of the rotational effects into Coriolis and centrifugal forces. The Coriolis force has formally the same effect as a constant magnetic field $\mathbf{B} = 2\boldsymbol{\Omega}$ with vector potential $\mathbf{A}(\mathbf{x})$.

Reduction to 2D

If the external potential is strongly confining in the direction of the rotational axis (z-direction), a two-dimensional description is appropriate.

The same applies to the opposite case, i.e., when the trap potential is almost constant in the z-direction. In this case 2D GP functional describes the ground state energy per unit length in the z-direction.

The coupling constant in the 2D GP functional is in both cases the dimensionless parameter

$$\gamma = g/L = 2\pi Na/L$$

with L a length in the z-direction.

Reduction to 2D (cont.)

It is custormary and convenient to write

$$\gamma = \frac{1}{\varepsilon^2}.$$

The 2D GP functional we consider is thus

$$\mathcal{E}^{\mathrm{GP}}[\psi] = \int_{\mathbb{R}^2} \left\{ \tfrac{1}{2}|(\mathrm{i}\nabla + \mathbf{A})\psi|^2 + (V - \tfrac{1}{2}\Omega^2 r^2)|\psi|^2 + \frac{1}{\varepsilon^2}|\psi|^4 \right\} \mathrm{d}^2\mathbf{r}$$

We are in particular be interested in large γ which means small ε. Moreover, $\mathbf{\Omega}$ is $\perp$ to the 2D plane. We consider the asymptotic parameter regime $\varepsilon \to 0$, $\Omega \to \infty$. To ensure stability the external potential V must thus increase faster than r^2 at $r \to \infty$.

The Meaning of ε

The healing length ℓ_h is defined by the condition that the kinetic energy associated with ℓ_h equals the interaction energy per particle, i.e.,

$$\frac{1}{\ell_\mathrm{h}^2} \sim \frac{1}{\varepsilon^2} \int |\psi|^4.$$

In a trap of effective radius R we have $|\psi|^2 \sim R^{-2}$ by the normalization condition, and thus

$$\int |\psi|^4 \sim R^{-2}.$$

Hence

$$\varepsilon \sim \ell_\mathrm{h}/R.$$

The TF Density and Energy

Dropping the kinetic energy term $\frac{1}{2}|(\mathrm{i}\nabla + \mathbf{A})\psi|^2$ from the GP energy functional lead to the so-called TF functional of the density $\rho = |\psi|^2$:

$$\mathcal{E}^{\mathrm{TF}}[\rho] = \int \left\{ (V - \tfrac{1}{2}\Omega^2 r^2)\rho + \frac{1}{\varepsilon^2}\rho^2 \right\} \mathrm{d}^2\mathbf{r}$$

The minimizer under the normalization condition $\int \rho = 1$, denoted by ρ^{TF}, is explicitly given as

$$\rho^{\mathrm{TF}}(\mathbf{r}) = \frac{\varepsilon^2}{2}\left[\mu^{\mathrm{TF}} - V(\mathbf{r}) + \tfrac{1}{2}\Omega^2 r^2\right]_+$$

where μ^{TF} is a chemical potential and $[\cdot]_+$ denotes the positive part. The corresponding energy is denoted by E^{TF}.

The Concept of a Quantum Vortex

For small ε the density ρ^{TF} essentially determines the **global profile** of the condensate. The term $\frac{1}{2}|(\mathrm{i}\nabla + \mathbf{A})\psi|^2$ on the other hand may lead to the generation of **quantum vortices**.

Quantum vortices are associated with zeros of the GP wave function $\psi(\mathbf{r}) = |\psi(\mathbf{r})| \exp(\mathrm{i}\varphi(\mathbf{r}))$ and singularities of the phase $\varphi(\mathbf{r})$.

The GP equation implies that the velocity field associated with ψ is

$$\mathbf{v}(\mathbf{r})_{\mathrm{vort}} = \nabla\varphi(\mathbf{r}).$$

If $\psi(\mathbf{r})$ has a zero at at $\mathbf{r} = \mathbf{r}_0$ the winding number (degree) of the vortex at $\mathbf{r} = \mathbf{r}_0$) is

$$d = (2\pi)^{-1} \oint_{\mathcal{C}} \nabla\varphi \cdot d\ell$$

where $\mathcal{C}$ is a curve enclosing $\mathbf{r}_0$ (but no other zeros of ψ).

The Emergence of Vortices

For small Ω the condensate is at rest in the inertial system and thus rotates (with angular velocity $-\boldsymbol{\Omega}$) in the rotating system. (This is due to the superfluidity of the conensate; a normal fluid would rotate with the trap and thus be at rest in the rotating system.)

More precisely: The velocity operator in the rotating system is $-\mathrm{i}\nabla - \mathbf{A}(\mathbf{r})$. The constant function is for small Ω the ground state and has the velocity distribution

$$\mathbf{v}_{\mathrm{rot}}(\mathbf{r}) = -\mathbf{A}(\mathbf{r}) = -\boldsymbol{\Omega} \times \mathbf{r} = -\Omega r\, \mathbf{e}_{\theta}.$$

The corresponding kinetic energy is exacly compensated by the centrifugal energy term $-\frac{1}{2}\Omega^2 r^2$.

The Emergence of Vortices (cont.)

At higher rotational velocities vortices may partly compensate the term $-\mathbf{A}$ of the velocity and hence reduce the kinetic energy. This reduction is necessarily accompanied by a redistribution of the density and hence some increase in interaction energy which determines the size of the vortex cores.

Consider the case of small ε and a trap with effective radius $R \sim \rho^{-1/2}$. A *vortex of degree* d located at the origin, has approximately the form

$$\psi(r,\theta) = f(r)\exp(\mathrm{i}\theta d)$$

with

$$f(r) \sim \begin{cases} r^d & \text{if} \quad 0 \le r \lesssim r_\mathrm{v} \\ R^{-1} & \text{if} \quad r_\mathrm{v} \lesssim r \le R \end{cases}$$

with r_v is the radius of the vortex core where the density is small.

The Emergence of Vortices (cont.)

The total velocity $\mathbf{v}(\mathbf{r}) = \mathbf{v}_{\rm vort} + \mathbf{v}_{\rm rot}$ is

$$\mathbf{v}(\mathbf{r}) = \left(\frac{d}{r} - \Omega\, r\right) \mathbf{e}_\theta.$$

Change in kinetic energy compared to the vortex free case, $d = 0$:

$$\Delta E_{\rm kin} \sim R^{-2} \int_{r_{\rm v}}^{R} [(d/r)^2 - 2d\,\Omega]\, r\, dr + O(1)$$

$$= R^{-2} d^2 |\log(r_{\rm v}/R)| - d\,\Omega + O(1).$$

Increase in interaction energy through the creation of the vortex:

$$\Delta E_{\rm int} \sim \frac{1}{\varepsilon^2} (r_{\rm v}/R)^2 R^{-2}.$$

Optimizing the total energy change w.r.t. $r_{\rm v}$ gives

$$r_{\rm v} \sim \varepsilon R = \ell_{\rm h} \quad \text{and} \quad \Delta E_{\rm int} \sim R^{-2} \sim \rho.$$

The Emergence of Vortices (cont.)

The energy change due to the vortex is thus

$$\Delta E \sim \rho d^2 |\log \varepsilon| - d\,\Omega + O(1).$$

A vortex of degree $d = 1$ becomes energetically favorable when

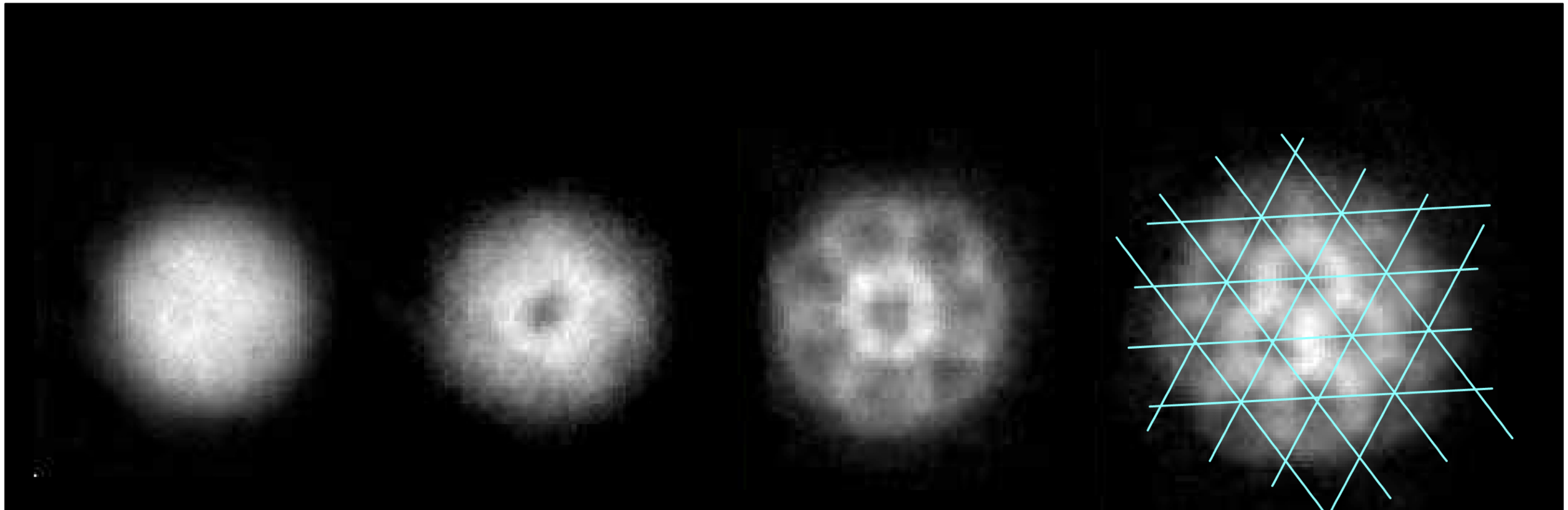
$$\rho |\log \varepsilon| - \Omega + O(1) < 0$$

which for $\varepsilon \ll 1$ means

$$\Omega \gtrsim \rho |\log \varepsilon|.$$

We also see that d vortices of degree 1, ignoring their interaction, have energy $\sim d(\rho|\log \varepsilon| - \Omega)$ while a vortex of degree d has energy $\rho d^2 |\log \varepsilon| - d\,\Omega$. Hence it is energetically favorable to 'split' a d-vortex into d pieces of 1-vortices, breaking the rotational symmetry.

Creation of quantized vortices in a rotating container

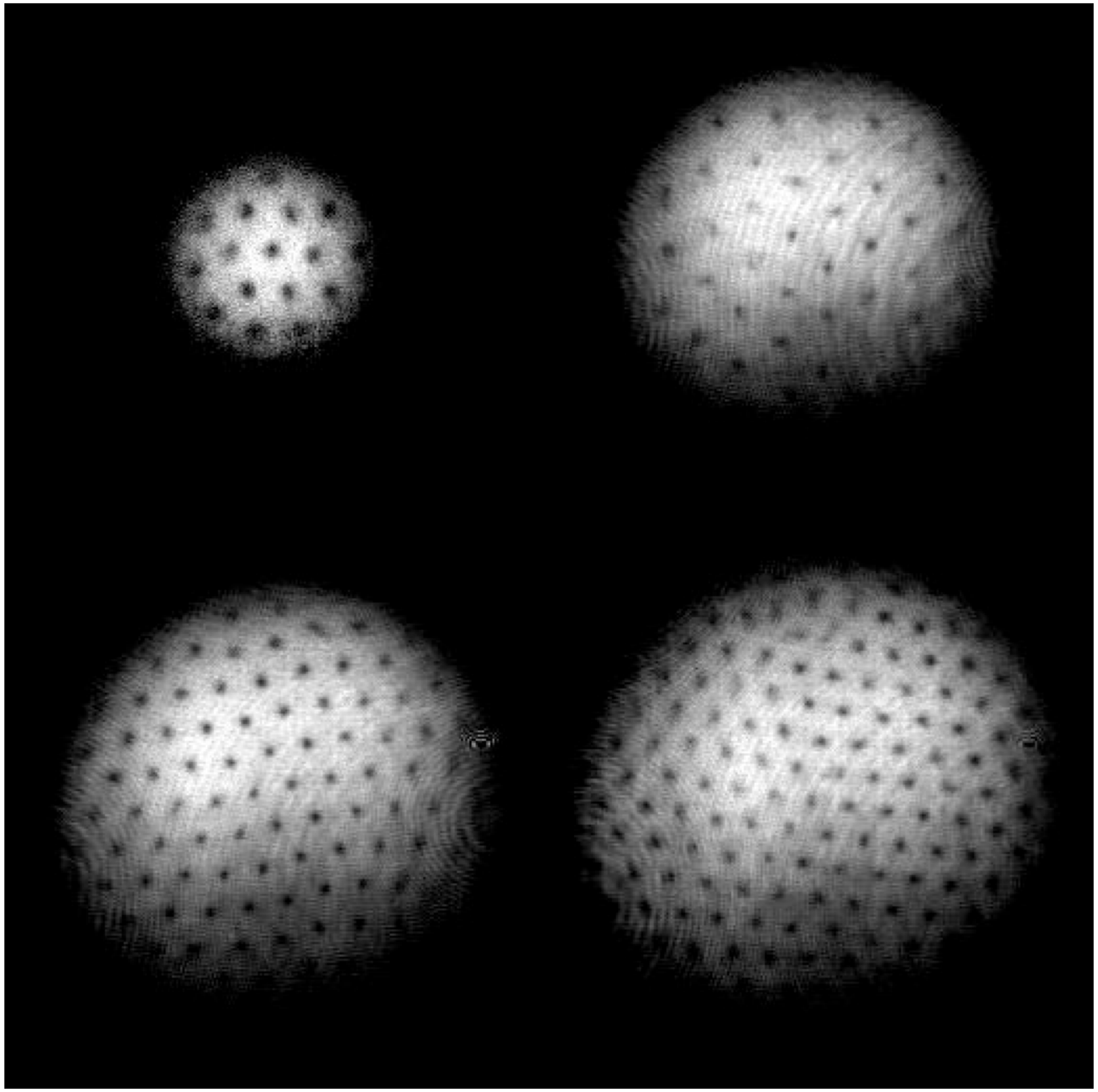


Just below the critical frequency

Just above the critical frequency

Notably above the critical frequency

Even larger freqs.: Abrikosov lattice



Remark:

While the preceeding heuristic discussion is adequate as a first orientation, it ignores some finer points that are important to take into account in a precise analysis:

- Inhomogeneities of the background density are in general significant.
- When there are several vortices their long-range interaction due to the nonlinearity of the GP functional may also be relevant.

In an inhomogeneous background, precisely defined cost functions that go beyond the rough approximation $\rho\,|\log\varepsilon| - \Omega + O(1)$ have to be considered.

2D Gross-Pitaevskii Theory in Anharmonic Traps

We now consider the 2D energy functional

$$\mathcal{E}^{\mathrm{GP}}_{2\mathrm{D}}[\psi] = \int_{\mathbb{R}^2} \left\{ \tfrac{1}{2} \left| (\mathrm{i}\nabla + \mathbf{A})\,\psi \right|^2 + (V - \tfrac{1}{2}\Omega_{\mathrm{rot}}^2 r^2)|\psi|^2 + \frac{|\psi|^4}{\varepsilon^2} \right\}$$

with an anharmonic trap potential of the form (for simplicity)

$$V(r) = kr^s$$

with $s > 2$, $k > 0$. Then Ω_{rot} can be arbitrary large.

The limiting case $s \to \infty$ corresponds to a 'flat' trap. The effective potential is then simply $-\frac{1}{2}\Omega_{\mathrm{rot}}^2 r^2$ and the integration is limited to the unit disc in $\mathbb{R}^2$.

Heuristics

The analysis of the GP minimizer is guided by the following heuristics:

- Vortices reduce the kinetic energy by compensating partly the velocity field generated by $\mathbf{A}(\mathbf{x}) = \mathbf{\Omega}_{\rm rot} \times \mathbf{x}$.
- A vortex causes also a change in the density (mass is moved from the vortex core to the bulk) that increases the interaction energy. This increase depends on the density at the potential location of the vortex. The energy balance decides whether or not a vortex is favorable, and if that is the case, the size of the vortex core.
- A vortex is the more costly the higher the density. At sufficiently high rotational velocities the compression due to centrifugal forces creates a 'hole' and the density in the bulk increases until, at some point, vortices in the bulk become too costly. Then a phase transition to a giant vortex in the 'hole' takes place.
- The task is to turn this heuristics into mathematics!

Scaling of the Energy Functional

The potential $(kr^s - \frac{1}{2}\Omega_{\rm rot}^2 r^2)$ has a unique minimum at $r = (\Omega_{\rm rot}^2/(sk))^{1/(s-2)}$. Taking this as a length unit we obtain the scaled energy functional

$$\mathcal{E}^{\rm GP}[\psi] = \int_{\mathbb{R}^2} \left\{ |(\mathrm{i}\nabla + \Omega x \mathbf{e}_\theta)\psi|^2 + \Omega^2 W(x)|\psi|^2 + \varepsilon^{-2}|\psi|^4 \right\}$$

where $x = |\mathbf{x}|$ and

$$W(x) = \left(\tfrac{1}{s}x^s - \tfrac{1}{2}x^2\right).$$

The scaled potential has a minimum at $x = 1$, independent of the (scaled) rotational frequency Ω.

Main results

Assume $2 < s < \infty$.

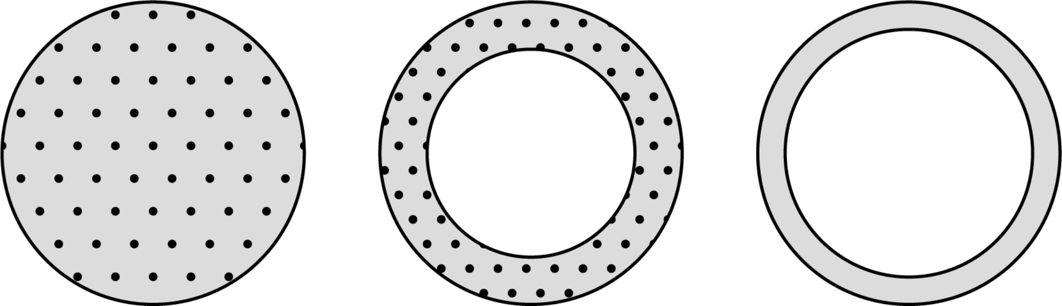
As Ω increases there are three critical velocities:

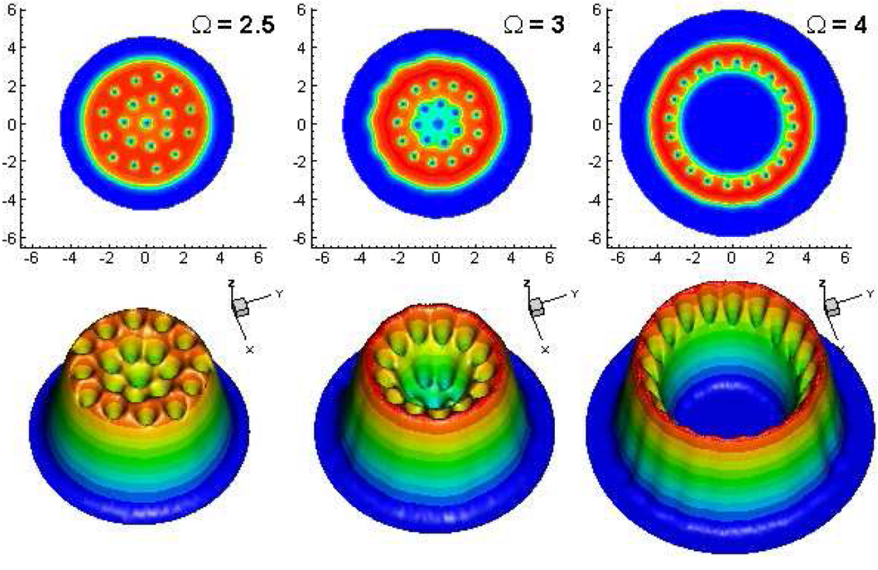
- $\Omega'_{c1} \sim |\log \varepsilon|$ marking the appearance of the first vortex.
- $\Omega_{c2} \sim \varepsilon^{-1}$ marking the creation of a 'hole' by the centrifugal forces.
- $\Omega_{c3} \sim \varepsilon^{-4}$ marking the transition to a 'giant vortex'

Here $\Omega \sim \Omega_{\rm rot}^{(s+2)/(s-2)}$, $\Omega' \sim \varepsilon^{-4/(s+2)}\Omega_{\rm rot}$.

For $\Omega_{c1} \ll \Omega \ll \Omega_{c3}$ the vorticity is uniformly distributed (in the form of a triangular vortex lattice) in the bulk.

For $\Omega > \Omega_{c3}$ the bulk is free of vortices but a macroscopic circulation around the origin remains.





Ω = 2.5
Ω = 3
Ω = 4
6
4
2
0
-2
-4
-6
-6 -4 -2 0 2 4 6
Z
Y
X

The Vortex Lattice Regime

The ground state energy for $\Omega_{c1} \ll \Omega \ll \Omega_{c3}$ can be computed exactly to subleading order:

THEOREM (Energy between Ω_{2c} and Ω_{3c})

If $\varepsilon^{-1} \lesssim \Omega \ll \varepsilon^{-4}$ *as* $\varepsilon \to 0$, *then*

$$E^{\rm GP} = E^{\rm TF} + \tfrac{1}{6}\Omega|\log(\varepsilon^4\Omega)|(1+o(1)).$$

Here $E^{\rm TF}$ is the energy without the kinetic term.

THEOREM (Energy between Ω_{1c} and Ω_{2c})

If $|\log\varepsilon| \ll \Omega' \lesssim \varepsilon^{-1}$ *as* $\varepsilon \to 0$, *then*

$$E^{\rm GP'} = E^{\rm TF'} + \tfrac{1}{2}\Omega'|\log(\varepsilon^2\Omega')|(1+o(1)).$$

The Giant Vortex Regime

Consider a variational ansatz for the wave function of the form

$$\psi(\mathbf{x}) = f(\mathbf{x}) \exp(\mathrm{i}\Omega\theta)$$

with a real valued function f, normalized such that $\int f^2 = 1$. (Assume that Ω is an integer). This gives

$$\mathcal{E}^{\mathrm{GP}}[\psi] = \int_{\mathbb{R}^2} \left\{ \tfrac{1}{2}|\nabla f|^2 + \tfrac{1}{2}\Omega^2 (x - x^{-1})^2 f^2 \right.$$
$$\left. + \Omega^2 \left(\tfrac{1}{s} x^s - \tfrac{1}{2} x^2 \right) f^2 + \varepsilon^{-2} f^4 \right\} \equiv \mathcal{E}^{\mathrm{gv}}[f].$$

The unique positive minimizer f_{gv} of $\mathcal{E}^{\mathrm{gv}}$ is rotationally symmetric, Corresponding energy: E^{gv}.

Energy Estimates and Absence of Vortices

THEOREM [Energy in the giant vortex regime]

There is a constant $0 < \bar{\Omega}_0 < \infty$ *such that for* $\Omega = \Omega_0\, \varepsilon^{-4}$ *with* $\Omega_0 > \bar{\Omega}_0$ *the ground state energy is*

$$E^{\mathrm{GP}} = E^{\mathrm{gv}} + O(|\log \varepsilon|^{9/2}).$$

THEOREM [Absence of vortices in the bulk]

There is a constant $c > 0$ *such that for* $\Omega = \Omega_0\, \varepsilon^{-4}$ *with* $\Omega_0 > \bar{\Omega}_0$ *and* ε *sufficiently small the minimizer* ψ^{GP} *is free of zeros in the annulus*

$$\mathcal{A} = \{\mathbf{x} : |1 - x| \leq c\Omega^{-1/2} |\log \varepsilon|^{1/2}\}.$$

On the proof of the GV transition

The main issue is a precise lower bound to the energy. Restrict $\mathcal{E}^{\rm gv}$ to the annulus $\mathcal{A}$, obtaining a positive minimizer f. Define $u(\mathbf{x})$ on the annulus by writing

$$\psi^{\rm GP}(\mathbf{x}) = f(x)u(\mathbf{x})\exp(\mathrm{i}\Omega\theta).$$

The function u contains all possible zeros of $\psi^{\rm GP}$ in the annulus. The variational equation for f leads to the lower bound

$$E^{\rm GP} \geq E^{\rm gv}_{\mathcal{A}} + \mathcal{E}_{\mathcal{A}}[u]$$

with a functional of Ginzburg-Landau type with f^2 as weight

$$\mathcal{E}_{\mathcal{A}}[u] = \int_{\mathcal{A}} f^2 \left\{\tfrac{1}{2}|\nabla u|^2 - \mathbf{B}\cdot\mathbf{J}(u) + \varepsilon^{-2} f^2(1-|u|^2)^2\right\}$$

where $\mathbf{B} = \Omega(x - x^{-1})\,\mathbf{e}_\theta$ and $\mathbf{J}(u) = \frac{\mathrm{i}}{2}(u\nabla u^* - u^*\nabla u)$.

One needs to estimate the negative term $-\int f^2\mathbf{B}\cdot\mathbf{J}(u)$.

On the proof (cont.)

Write $f^2\mathbf{B} = \nabla^\perp F$ with $\nabla^\perp = (-\partial_{x_2}, \partial_{x_1})$ and a potential function F. Integration by parts and estimates of F (key point!) give

$$\int_{\mathcal{A}} f^2 \left\{ \tfrac{1}{2}|\nabla u|^2 - \mathbf{B}\cdot\mathbf{J}(u) \right\} \geq -C\Omega_0^2 |\log\varepsilon|^{3/2}$$

leading to the lower energy bound.

A consequence of this bound, combined with the variational upper bound $E_{\mathcal{A}}^{\mathrm{gv}} \leq 0$ is an upper bound on the interaction term for large Ω_0:

$$\int_{\mathcal{A}} \varepsilon^{-2} f^4 (1-|u|^2)^2 \leq C\Omega_0^2 |\log\varepsilon|^{3/2}$$

Together with he Gagliardo-Nirenberg inequality this implies that u must be close to 1, in particular free of zeros.

Comparison with the 'flat' case

The flat case, $s = \infty$, differs from the case $s < \infty$ in several respects:

- The GV transition takes place at $\Omega \sim \varepsilon^{-2}|\log \varepsilon|^{-1}$ rather than $\Omega \sim \varepsilon^{-4}$
- The density profile in the GV regime is of TF type in the 'flat' case, but for $s < \infty$ it is gaussian around $x = 1$.
- The 'last' vortices before the GV transition have size $\sim \varepsilon^{3/2}$ that is much smaller than the thickness of the annulus $\sim \varepsilon|\log \varepsilon|$. For $s < \infty$ the size of vortices, $\sim \varepsilon^2$ and the size of the annulus, $\sim \varepsilon^2|\log \varepsilon|^{1/2}$, are almost comparable.

The techniques of proof in the two cases are also by necessity different: While *vortex ball constructions* and subsequent *jacobian estimates* for the potential function are applicable for the 'small' vortices in a 'flat' trap they are useless for $s < \infty$ and new ideas are required.

Circulation and symmetry breaking

Below the onset of the second vortex the GP minimizer has rotationally symmetric density, but a vortex lattice clearly breaks the symmetry. On the other hand, the giant vortex variational ansatz, that gives an excellent approximation to the energy for $\Omega_0 > \bar{\Omega}_0$, is an eigenfunction of angular momentum. A true minimizer does not have this property, however:

THEOREM (Circulation and rotational symmetry breaking)

In the giant vortex regime $\Omega_0 > \Omega_1$ the circulation of any GP minimizer is $2\pi\,\Omega + O(1)$, but no minimizer is an eigenfunction of angular momentum.

These result holds for all $s < \infty$ and an analogous result also for $s = \infty$.

Summary

The study of the GP equation for dilute Bose gases in rotating, anharmonic traps reveals a surprising rich landscape, both from the mathematical and physical point of view. Detailed analysis can be carried out in an asymptotic regime where both the coupling constant and the rotational speed are large.

Among the results found are:

- Energy asymptotics corresponding to a distribution of vorticity in a lattice of vortices for $\Omega_{c1} \ll \Omega \ll \Omega_{c3}$.
- Emergence of a 'hole' with strongly depleted density above a critical rotation speed $\Omega_{c2} \sim \varepsilon^{-1}$.
- Transition to a 'giant vortex' state above $\Omega_{c3} \sim \varepsilon^{-4}$ where the vortex lattice disappears from the bulk and all vorticity resides in the 'hole', creating a macroscopic circulation in the bulk.
- Breaking of rotational symmetry, also in the giant vortex regime.

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