CMO-Workshop on
Symbolic Dynamical Systems
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Final report

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1 Overview of the field.

Symbolic dynamics is a branch of mathematics which plays a fundamental role in dynamical systems and ergodic theory. Symbolic systems have served not only as important (counter) examples in ergodic theory but have also proved to be fundamental tools for understanding more general dynamical systems. Sinai, Ruelle and Bowen laid out the framework of a phenomenally successful program for using symbolic dynamical systems to understand hyperbolic dynamics via symbolic coding; as an example, countable Markov shifts are used to model more general families of differentiable dynamical systems, including non-uniformly hyperbolic settings (see e.g. [20, 50, 36, 35]).

In recent years, there has been a resurgence of activity in thermodynamic formalism. Ideas introduced by Rufus Bowen [10] have been polished and sharpened to apply to an array of new symbolic dynamical systems with weakened forms of specification. Work of Climenhaga, Cyr, Kwietniak, Pavlov, Thompson and others has expanded the limits of what can be done in these directions [17, 34, 40, 18, 1, 16]. Related work of Petersen, Quas, Yoo and collaborators has initiated a field of relative thermodynamic formalism [47, 53, 2, 54]. A satisfying structure theory is emerging, and this work has connections to questions arising in geometric measure theory.

Multi-dimensional ($\mathbb{Z}^d$) symbolic dynamics has brought a whole new array of research questions involving undecidability issues and higher-dimensional geometric considerations, see e.g. [38]. For multi-dimensional systems, a rich theory had already been developed in the study of algebraic dynamics in the 1980's and 1990's [51], but it was widely believed that outside of the algebraic setting, extensions of results from one-dimensional symbolic dynamics to $\mathbb{Z}^d$ subshifts for $d > 1$ were typically trivial or hopelessly difficult. This paradigm was shattered around 2010 by the work of Hochman and Meyerovitch [28, 29]: Lind had characterized the entropies of one-dimensional shifts of finite type (SFTs) as the collection of logarithms of the Perron numbers [37] while Hochman and Meyerovitch [29] characterized the entropies of multidimensional SFTs as the class of non-negative right recursively enumerable numbers. This completely unexpected connection between dynamics and computability theory opened the door to attack many of the previously obviated questions and has led to an avalanche of developments in the field [12, 7, 44, 33, 32, 45, 46, 41, 42].

Another classic topic which has seen an intense revival during the last years is the study of symmetries of symbolic systems in terms of their automorphism groups. While it has been known that mixing shifts of finite type have very large and complicated automorphism groups both in the one and multidimensional setting [11, 27], the recent focus is on subshifts of low complexity [19, 22, 21, 24, 25, 49] where those groups can be constructed or at least described explicitly giving examples of groups with new combinations of properties leading to solutions of long-standing conjectures in geometric group theory and providing an important and profitable current direction of research.

Probably the youngest area of symbolic dynamics is the study of subshifts on finitely generated groups where the acting group $\mathbb{Z}^d$ is replaced by other (non-abelian) countable groups. A first look at some of the issues of this general theory can be found in [15]. A growing group of researchers including Aubrun, Barbieri, Coven, Jeandel, Sahin and Schraudner are studying the existence of aperiodic subshifts (of finite type) and the implications on the decidability of fundamental questions in this setting [14, 30, 31, 48, 9, 4, 3]. Research is revealing a rich interplay between the combinatorial and geometric structure of the acting group and the dynamical properties of the symbolic system.

2 Motivation and objective of the workshop.

New developments in the field are carried out and flourish in collaborative interdisciplinary environments. The exchange of ideas from different subject areas and points of view is fundamental for providing new insights as well as identifying new research problems. A key motivation for proposing a workshop on symbolic dynamical systems at CMO in México was to create a suitable atmosphere to allow this sort of cross-fertilization.

Hence our main objective was to gather active collaborative research groups from the entire region in a central location like Oaxaca to promote further pan-American development of the theory of symbolic dynamics, while fostering existing and creating new connections to areas that included complexity and computability theory, actions of finitely generated groups and geometric group theory, Gibbs measures and phase transitions, finite and long range interactions, thermodynamic formalism in general and subjects alike.

3 Overview of the workshop.

There were 22 talks on several topics. The overall panorama of their content is discussed next. This discussion is based on notes taken by the organizers, so should not be taken as definitive(!).

3.1 Positive entropy.

Tim Austin talked about thermodynamic formalism for $2d$-regular trees, actions of the free group $\mathbb{F}_d$ (on d generators) on increasing sets of vertices and sofic entropy. He pointed out lack of convergence of sofic entropy along locally tree-like graphs (a sequence of $2d$-regular graphs $G_n = (V_n, E_n)$ is *locally tree-like* if $|\{v \in V_n : N_r(v) \simeq B_r^{\mathbb{F}_d}(e)\}| = (1 - o(1))|V_n|$, where $N_r(v)$ is the r-neighborhood of v and $B_r^{\mathbb{F}_d}(e)$ the r-ball in the regular $2d$ tree with root e, which is just the Cayley graph of $\mathbb{F}_d$). More precisely, the definition of sofic entropy explained by Austin is an expression like

$$h_{\langle \sigma_{n_i} \rangle_i}(\mu) = \inf_{\varepsilon, n} \limsup_{i \to \infty} \underbrace{\frac{1}{|V_{n_i}|} \log \left| \left\{ \underline{x} \in \mathcal{A}^{V_n} : \; ||P_{\underline{x}}^{\sigma_n, r} - \mu_{B_r^{\mathbb{F}_d}(e)}|| < \varepsilon \right\} \right|}_{(*)}$$

(above $\mathcal{A}$ is a given alphabet, so that $\underline{x}$ is configuration, and $P_{\underline{x}}^{\sigma_n, r} = \frac{1}{|V_n|} \sum_{v \in V_n} \delta_{\underline{x}|_{N_r(v)}}$ is the *empirical distribution*). There exist examples where there are subsequences (n_i) for which the $\limsup(*)$ in the above expression is positive whereas the $\liminf(*)$ is $-\infty$. One problem stated by Austin was if there is an instance for which the $\limsup(*)$ and $\liminf(*)$ differ but with $\liminf(*) > -\infty$. The main goal of the talk was to state the problem of convergence of $\frac{1}{|V_n|} \mathbb{E} \log Z(\sigma_n, f)$, where Z denotes the partition function and f an observable (some cases are known, e.g. for the Ising model see [23]).

Sebastián Barbieri Lemp addressed the characterization of the entropies of $\mathbb{G}$-SFTs depending on the structure of the group $\mathbb{G}$. His work extends both the aforementioned characterizations of Lind [37] for $\mathbb{Z}$-SFTs and Hochman and Meyerovitch [28, 29] for $\mathbb{Z}^d$-SFTs. For any polycyclic-by-finite group $\mathbb{G}$, Barbieri shows the tricotomy of the sets of possible entropies of $\mathbb{G}$-SFTs according to whether the Hirsch index $h(\mathbb{G})$ equals zero or one or else is ≥ 2. In particular for $h(\mathbb{G}) = 1$ the resulting set of entropies coincides with Lind's characterization for $\mathbb{Z}$-SFTs, and for $h(\mathbb{G}) \geq 2$ the resulting set of entropies coincides with Hochman and Meyerovitch's characterization for $\mathbb{Z}^2$-SFTs.

Uijin Jung presented work on the existence of invariant Gibbs measures on balanced subshifts. The starting point was the work of Baker and Ghenciu [8], who defined balancedness as the property of having a bounded away from zero proportion of n-follower sets with respect to the amount of n-blocks. This is a one (or right) sided definition and it admits its counterpart version. Baker and Ghenciu showed that being (right)-balanced is equivalent to the existence of a (non-invariant) Gibbs measure for the null potential. The work of Jung together with Kim shows that there exists an *invariant* Gibbs measure for the null potential if and only if the subshift is *both* left and right-balanced.

Dominik Kwietniak addressed problem 32 on Bowen's problem book: classify shift spaces with specification.

On the general category of Polish spaces, Borel order reduction $\leq_{\mathcal{B}}$ and Borel equivalence $\sim_{\mathcal{B}}$ establish the following regimes: (1) the *smooth* regime (Borel equivalences reduced to $\mathsf{id}_{\mathbb{R}}$, the identity equicalence relation in $\mathbb{R}$), followed by (2) the *hyperfinite* regime $E_0 \subset \{0,1\}^\infty \times \{0,1\}^\infty$ defined by the rule xE_0y if and only if $x, y \in \{0,1\}^\infty$ are such that $\{n \geq 0 : x(n) = y(n)\}$ is cofinite, and on top of all the *countable* Borel equivalence relations (an equivalence relation is *countable* if each equivalence class is countable) is (3) the E_∞ regime, the universal Borel equivalence relation $E_\infty \subset \{0,1\}^{\mathbb{F}_2} \times \{0,1\}^{\mathbb{F}_2}$ defined by the rule $xE_\infty y$ if and only if $x, y \in \{0,1\}^{\mathbb{F}_2}$ are such that there exists $g \in \mathbb{F}_2$ such that $y = gx$ (here gx denotes the natural shift action $\mathbb{F}_2 \curvearrowright \{0,1\}^{\mathbb{F}_2}$ of the free group on two elements $\mathbb{F}_2$ on the full two $\mathbb{F}_2$-shift $\{0,1\}^{\mathbb{F}_2}$).

Kwietniak shows that the isomorphism relation on subshifts with specification is Borel equivalent to E_∞: the isomorphism problem within shifts with specification is the hardest in the above hierarchy of countable Borel equivalence relations on Polish spaces.

Yinon Spinka discussed finitary coding of k-dependent measures in the setting of transitive amenable graphs. k-dependent measures on S^G for a graph G are measures such that if V and W are subsets of the vertices of G such that each vertex in V is at least k-distant from all vertices in W, then the configuration seen on V is independent of that seen on W. One natural way to build k-dependent measures is to take finite range factors of Bernoulli measures on G. For quite a long time, it was not known whether all k-dependent measures were of this type, even on the graph $\mathbb{Z}$. This question was resolved in the negative by Burton, Goulet and Meester [13]. In this talk, Yinon showed in the more general context of transitive amenable graphs that although k-dependent measures are not finite range factors of a Bernoulli process, they are, in fact, finitary factors of a Bernoulli process. That is, factors where for any point in the domain and any vertex, a finite (but not bounded) portion of the Bernoulli process is sufficient to determine the image coding at any vertex.

Siamak Taati presented a relative version of the Dobrushin-Landford-Ruelle (DLR) theorem, which relates Gibbs (local) vs equilibrium (global) measures, in a general setting of actions of countable amenable groups. Among the applications/corollaries are an extention of results of Allahbakhshi and Quas [2], and also an extension to the relative setting of Meyerovitch's theorem [43], which is a generalization of the Landford-Ruelle (LR) direction in the DLR theorem (conditions under which equilibrium states are Gibbs measures). In Meyerovitch's theorem, a key concept is that of *exchangeable* patterns (which are related to recent work of García-Ramos and Pavlov [26]), and the proof via the relative LR theorem is achieved by appropriate coding arguments, some of which make use of auxiliary symbolic systems like hard-core subshifts with 'arbitrary shapes' (the ones that support the exchangeable patterns).

Felipe García Ramos talked about his joint work with Ronnie Pavlov [26] on exchangeable patterns and measures of maximal entropy (MME). They generalize exchangeability on $\mathbb{Z}$-subshifts by allowing the shapes (lengths) of the supporting patterns to be different. In this one-dimensional setting, they extend results of Meyerovitch from [43]: for an MME μ and two exchangeable patterns $u, v \in \mathcal{B}(X)$, $\mu(u) = \mu(v)e^{(|v|-|u|)\cdot h_{\text{top}(X)}}$. As an application they obtain a new proof that a synchronized subshift that is entropy minimal possesses a unique MME. For countable amenable groups they also study the exchangeability property for patters with the same shape.

Raimundo Briceño talked about his work on 'pressure representation'. The pressure of a potential for a group action is a complicated object strongly related to convex analysis. In particular, algorithms to compute the pressure at all are subtle, and questions of convergence are a priori unapproachable. The pressure representation gives a remarkable way to express the pressure in terms (as usual) of an integral of potential plus conditional entropy. What is remarkable, however, is that while one would expect the integral to be with respect to an equilibrium state, one can replace the measure by any invariant measure without changing the result. This yields algorithmic approaches to computing the pressure that have reasonable convergence properties. In this talk, pressure representation was discussed in the context of amenable group actions.

Karl Petersen presented joint work with Ibrahim Salama on entropy of tree shifts, which were first introduced and studied in works of Aubrun and Béal (see e.g. [5, 6]). By looking at the complexity function, they give a natural definition of tree shift entropy and first show that the topological entropy of the $\mathbb{Z}$-SFT associated to a square $\{0,1\}$-matrix M is less than or equal to the topological tree shift entropy of the 2-tree shift also associated to M (the inequality is actually strict). Pavlov's strip approximation is used as a main tool in the arguments of the main proofs. Petersen considers MME's, which exist by expansiveness. He points out that in the tree setting MME's are elusive: numerical computations show evidence that for the golden mean shift the Parry like measure defined in the work of Mairesse and Marcovici [39] is not an MME.

Nishant Chandgotia discussed *hom-shifts*. These is a nice sub-class of the class of shifts of finite type consisting of the space of all graph homomorphisms from some infinite graph (often $\mathbb{Z}^d$) to some finite graph. For example if the smaller graph is the complete graph K_n (with no self-loops), this is exactly the space of n-colourings of $\mathbb{Z}^d$. A number of subtle issues arise. For example, if the finite graph is the triangle, one obtains the 'three color checkerboard'. Here it is known that there are subtle 'height functions' controlling mixing and typical configurations, related to work of Kenyon, Okounkov and Propp in the context of dimer models.

Rodrigo Bissacot discussed a local compactification of infinite degree shifts of finite type due to an old paper of Exel and Laca. This local compactification has the remarkable property that one can consider equilibrium states on it. For low values of the temperature (high inverse temperature), the equilibrium states are supported on the original (non-compactified) part of the space, while for high values of the temperature, the equilibrium states are supported on the added parts of the space. This is a very satisfying result because if one does not compactify, one sees that for certain values of the inverse temperature, there are equilibrium states, whereas for other values, there are none. This work shows that in the high temperature regime, the equilibrium states do not disappear; rather they just live on a different part of the compactification.

3.2 Zero entropy

Tom Meyerovitch discussed a relationship between zero topological entropy (for group actions) and 'predictability'. Such a relationship is well-known in the measurable category. Here in the topological category (that is, X is a compact space, Γ is a group action on X, and S is a sub-semigroup of Γ, not containing the identity), *predictability* means that for all $f \in C(X)$, $f \in \mathrm{Cl}(\mathrm{lin}(\{f \circ T_s \colon s \in S\}))$. This extends a number of earlier results. A key tool is that of a random order on a group. This is a slightly subtle notion: some groups admit equivariant total orders, while some do not (in particular any group with torsion does not admit an equivariant total order). A random total order is an ensemble of total orders where the measure on the orders is equivariant (even though the orders themselves may not be). This notion was used to replace the use of an equivariant total order in an earlier result, thereby generalizing the theorem.

Van Cyr presented two theorems: It is known that a minimal symbolic dynamical system with the property that its word complexity has linear growth (or even $\liminf p_X(n)/n < \infty$) supports at most finitely many ergodic invariant measures. In the first, he gave a cutting and stacking construction of a minimal symbolic system supporting uncountably many ergodic measures, with superlinear growth rate as close to linear as one pleases.

For the second result, one uses the slow entropy definition of Katok, counting the smallest number of ε-Hamming balls to cover a set of measure at least $1-\varepsilon$. There are limit-superior and limit-inferior definitions. It is known that if the upper limiting quantity grows exponentially, the system has positive entropy. And similarly, if the lower limiting quantity remains bounded, the system is measurably isomorphic to a rotation. The second result states that one can build a system where the lower limiting quantity diverges arbitrarily slowly and the upper limiting quantity diverges at an arbitrarily large sub-exponential rate.

Sebastián Donoso's talk started from the result (proved by a number of groups of authors, and stated many times during the workshop) that linear complexity of a symbolic minimal dynamical system implies that the automorphism group is virtually $\mathbb{Z}$, and looked for an interpretation in terms of other dynamical concepts.

Specifically, he discussed S-adic dynamical systems. Here one starts with an alphabet $\mathcal{A}$, and a sequence of substitutions $\sigma_1, \sigma_2, \ldots$, each mapping $\mathcal{A}$ to $\mathcal{A}^+$. One can then form $\mathcal{W}_n = \sigma_n \circ \sigma_{n-1} \circ \ldots \circ \sigma_1(\mathcal{A})$ to obtain a collection of $|\mathcal{A}|$ (long) words. An S-adic subshift is then obtained by taking X to consist of those bi-infinite sequences $x \in \mathcal{A}^{\mathbb{Z}}$ such that each finite subword of x is a subword of an element of $\mathcal{W}_n$ for some n. Sebastián discussed the relationship between S-adic dynamical systems and finite rank dynamical systems.

This allowed him to give examples of systems whose automorphism group is virtually $\mathbb{Z}$, but whose complexity is super-linear.

Kitty Yang discussed the mapping class group of a minimal subshift. This is a recent definition where one builds a topological object by the suspension construction and studies the collection of self-homeomorphisms of that object (taking a quotient by the isotopies). This is a rich concept.

In particular, in the case of a primitive substitution, the substitution map gives rise to elements of the mapping class group (and de-substitution likewise).

One striking theorem was that she gave a short exact sequence for the mapping class group of a primitive substitution:

$$0 \to F \to \mathrm{MCG}(X) \to \mathbb{Z} \to 0,$$

where F is a finite group, and the $\mathbb{Z}$ group is essentially counting the number of substitutions or de-substitutions.

Sarah Frick discussed essentially faithful codings. One starts with an adic transformation on a generalized Pascal graph: in the Pascal graph, each vertex is joined to two neighbours in the level below. Here, there may be more edges and the number of vertices per level is not constrained to grow by one at each step, but the pattern of edges descending from any vertex is the same as the pattern of vertices descending from any other vertex. The adic transformation is a map on the infinite downward-pointing paths in the graph, sending each path to its lexicographic neighbour (when this is defined): to define the lexicographic partial order (a total order on any subset of paths that are co-final), one totally orders the edges *coming into* a vertex from above. This order is assumed to be the same at each vertex. One then compares two paths by looking at the last edge where they disagree.

The main result presented states that if one codes a downward path by the sequence of the first 8 edges in the path, then if one codes the entire orbit of a path under the adic transformation, the code is sufficient to reconstruct the original path (provided that the original path lies in a set of total measure 1 – that is, a subset that have measure 1 with respect to any invariant measure).

Ville Salo presented the concept of nilrigidity. A continuous map f of a topological space X is said to be *asymptotically nilpotent* (towards some point $0 \in X$) if $f^n(x) \to 0$ for all $x \in X$. It is *uniformly asymptotically nilpotent* if the convergence is uniform over X. A group action on a graph T is called *nilrigid* if for every finite alphabet $\mathcal{A}$, each asymptotically nilpotent function on $\mathcal{A}^T$ is uniformly asymptotically nilpotent.

The lecture presented a number of examples of nilrigidity.

Scott Schmieding presented an intriguing analogy between automophism groups of full shifts and algebraic K-theory: In the K-theory context, the commutator of $\mathrm{GL}_\infty(R)$ is generated by elementary matrices while the commutator subgroups of the $\mathrm{GL}_n(R)$ are much harder to describe.

The automorphism groups of full shifts are known to be extremely complicated. Denote σ_n for the full shift on n symbols. For example, it is known that for $n > 1$, $\mathrm{Aut}(\sigma_n)$ contains copies of the free group on 2 generators, every finite group and a subgroup isomorphic to $\mathrm{Aut}(\sigma_m)$ for any $m > 1$, but $\mathrm{Aut}(\sigma_2)$ and $\mathrm{Aut}(\sigma_4)$ are not isomorphic.

Scott introduced a new invariant: $\mathrm{Aut}^\infty(\sigma) = \bigcup_k \mathrm{Aut}(\sigma^k)$. The dimension group of σ_n is isomorphic to $\mathbb{Z}[1/n]$; and there is a surjective *dimension representation* $\pi_n\colon \mathrm{Aut}(\sigma_n) \to \mathrm{Aut}(\mathbb{Z}[1/n])$. This may be extended to a representation $\pi_n^\infty\colon \mathrm{Aut}^\infty(\sigma_n) \to \mathrm{Aut}(\mathbb{Z}[1/n])$. He then explained that the kernel of this map is precisely the commutator subgroup of $\mathrm{Aut}(\mathbb{Z}[1/n])$, which is analogous to the K-theory result.

As a consequence, he showed that (for example) $\mathrm{Aut}^\infty(\sigma_2)$ is not isomorphic to $\mathrm{Aut}^\infty(\sigma_6)$.

Álvaro Bustos described a general theory of spatial symmetries of multi-dimensional shifts of finite type with hierarchy. An archetypal example is the famous Robinson tiling example (one of the early examples of an aperiodic shift of finite type, with a relatively small set of tiles: 56). This shift of finite type is easily seen graphically to possess D_4 symmetry.

María Isabel Cortez's talk introduced some algebraic invariants of minimal Cantor systems, as the topological full group and the group of automorphism (the *centralizer*), and presented results about realization. The automorphism group of a Cantor aperiodic minimal system defined by the action of a countable group is realized as a subroup of the automorphism group of a Cantor minimal system defined by a $\mathbb{Z}$-action, hence it is locally embeddable into finite (semi-)group (see [52, 15]).

Alonso Castillo Ramírez discussed general properties of rank and relative rank of monoids. The focus of the talk was on computing the size of generating sets of CA$(\mathbb{G}; \mathcal{A})$, the collection of cellular automata with alphabet $\mathcal{A}$ over a group $\mathbb{G}$ and ICA$(\mathbb{G}; \mathcal{A})$, the collection of invertible cellular automata with alphabet $\mathcal{A}$ over a group $\mathbb{G}$. This is also called the automorphism monoid, and is a subgroup of CA$(\mathbb{G}; \mathcal{A})$. He was able to write down exact ranks of CA and ICA for a number of such monoids for certain groups $\mathbb{G}$. He also used his techniques to show that for some classes of infinite groups (including all infinite Abelian groups) that CA and ICA are infinitely generated.

Bryna Kra discussed joint work with John Franks on polygonal shifts. These are two-dimensio-nal shift spaces X with a 'convex' polygonal subset P of $\mathbb{Z}^2$ such that for any vertex v of P, and any $x \in X$, the symbols of x at any translate of $P \setminus \{v\}$ determine the symbols of x at the translate of v. This is closely related to (a one-sided version of) the notion of directional expansiveness due to Boyle and Lind. The goal was then to characterize those two-dimensional subshifts possessing this polygonal property.

4 New projects, scientific progress and more testimonials.

The workshop allowed participants to collaborate actively. Several groups were able to work and progress on their projects. Also some participants were able to visit institutions in México City and San Luis Potosí, either before or after the workshop, to continue collaborations and initiating new joint projects. Synergy to organize further forums was achieved: new projects for events like international schools on symbolic dynamics in the Pan-american region are being proposed.

The public testimonials of many participants express many aspects of the outcome of workshop. Further feedback we have received include the following:

- *I learned a lot about algebraic aspects in symbolic dynamics which I was only superficially aware of. The discussions with colleagues were very enlightening, and they will very likely give place to some collaborations. It was nice to see how the research in the thermodynamic formalism is still so active.*

  Edgardo Ugalde

- *Van Cyr's talk involved a construction similar to something Ayse Sahin and I had done in the past, so this has spurred a joint project involving how to extend the Cry/Kra result to involve loosely Bernoulli. So that was a very fun offshoot of the conference for me!!!*

  Aimee Johnson

So the workshop was an efficient forum for participants and collaborators to continue and initiate new project. For example, Cyr, Johnson, Kra and Sahin, started a project to study the simplex of invariant measures in low complexity, in particular to determine when there are or are not loosely Bernoulli measures. Other authors were able to resume ongoing projects:

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  - *I was able to connect with a co-author to bring to conclusion an ongoing project which had stalled.*
  - *I was able to connect with Kostya Medynets and María Isabel Cortez to ask them about some projects I had considered related to their recent work on topological orbit equivalence and Maria Isabel's talk.*
  - *I was excited to hear about Nishant's work that has answered a question resulting from early work of Robbie Robinson and myself: they were able to show the domino tilings are a universal model, something that was not possible using Robbie and my machinery. I am excited to learn this new technology. I spoke briefly with Nishant and look forward to continuing the conversation.*

– *All the talks were of very good quality, even those that were far from my area of expertise gave me some new piece of knowledge that I am interested in.*
– *I was able to make other, less well developed, connections for future visits and work ideas with some other folks as well, Domink Kwietniak in particular.*

Ayse Sahin

5 Summary of the Meeting

The workshop gathered a group of researchers specialized in symbolic dynamics, many of them were collaborators and they were able to continue working on their projects. The variety of subjects was broad and included thermodynamic formalism, automorphism groups, nilrigidity, coding, mapping class groups, complexity, entropy, multidimensional shifts, tree shifts, polygonal shifts and countable state Markov shifts. Also, general diversity was achieved:

- Participants came from the U.S.A., Canada, México, Brazil, Chile, Finland, the Netherlands, Poland, Israel and Korea.
- Six participants were women, and four of them gave talks.
- The participants ranged from consolidated and internationally recognized experts to young researchers and some students.
- There were several Mexican participants, including some from Guadalajara and México City and a large number from the consolidated group in San Luis Potosí. This benefited researchers in these regions in their development of this research area.

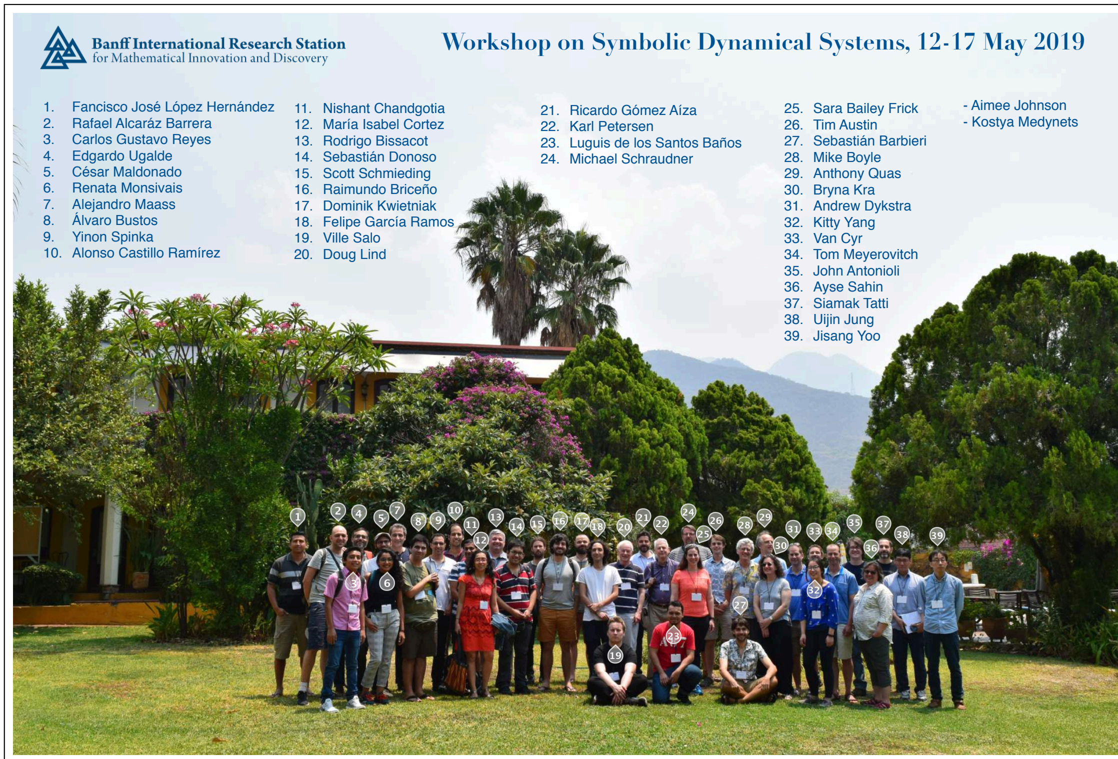

Figure 1: Participants of the CMO Workshop on Symbolic Dynamical Systems, Oaxaca 2019.

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