G_2 Geometry and Related Topics

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1 Introduction

G_2 Geometry is a vibrant and rapidly growing field which interacts with numerous areas in differential and algebraic geometry, as well as with other mathematical research areas such as topology and analysis, and even beyond mathematics to aspects of modern theoretical physics. One of the primary drivers for this expansion of G_2 Geometry is the large number of PhD students, postdoctoral researchers and other early career researchers who have joined the field in recent years. As a consequence, we deliberately made the meeting a forum for researchers at an early career stage: the vast majority of the speakers and participants were PhD students, with the rest of the speakers either postdocs or researchers who had recently been postdocs. This gave the meeting a vigorous energy which led to lively discussions, especially since most of the work presented was in progress and was explained at a level that was understandable to the PhD students present. To facilitate these discussions, as well as to provide plenty of opportunities for informal meetings, we ran a successful Open Problem session early in the workshop, and an invaluable Review session towards the end of the workshop. It was clearly evident that there was a great level of interaction between the PhD students and the more senior attendees at the workshop, which was markedly more than at a typical workshop.

2 Overview of the Field

The holonomy group associated to a Riemannian metric is generated by the parallel transport maps around loops in the manifold. The holonomy group is generically the special orthogonal group $\mathrm{SO}(n)$ if the manifold is n-dimensional, and reductions in the holonomy group are equivalent to various special structures on Riemannian manifolds. For example, on $2m$-dimensional manifolds, a reduction of the holonomy group to the unitary group $\mathrm{U}(m)$ corresponds to the metric being Kähler, and a reduction of the holonomy group to $\mathrm{SU}(m)$ corresponds to the metric being Calabi–Yau, which implies that the metric is Ricci-flat and Kähler.

The modern study of G_2 Geometry began with Berger's celebrated result in 1955 which classified the possible non-trivial holonomy groups which can occur for a Riemannian n-manifold:

$\mathrm{SO}(n)$	(any n);
$\mathrm{U}(m)$	($n = 2m$);
$\mathrm{SU}(m)$	($n = 2m$);
$\mathrm{Sp}(k)$	($n = 4k$);
$\mathrm{Sp}(k)\mathrm{Sp}(1)$	($n = 4k$);
G_2	($n = 7$);
$\mathrm{Spin}(7)$	($n = 8$).

We see immediately that the only non-trivial possibility for a reduction of the holonomy group in odd dimensions occurs in dimension 7 and the holonomy group in this case must be the exceptional Lie group G_2. A Riemannian metric with holonomy G_2 is necessarily Ricci-flat, and so finding holonomy G_2 metrics gives the only currently known method to construct non-trivial examples of Ricci-flat metrics in odd dimensions.

However, a key point is that Berger's list of holonomy groups does not say that these groups actually occur as holonomy groups, but only that they *can* occur. In fact, Berger's original list contained the additional possibility of $\mathrm{Spin}(9)$ as a holonomy group, but it was shown that this in fact only ever arises in a trivial way: more accurately, any Riemannian manifold with $\mathrm{Spin}(9)$ holonomy must be a (locally) symmetric space. Therefore the first challenge in the field, which continues to drive research in the area, is to find examples of holonomy G_2 metrics.

2.1 G_2-manifolds

After Bryant first proved the local existence of metrics with holonomy G_2 in 1985, Bryant and Salamon soon constructed the first examples of complete metrics with holonomy G_2: these metrics are asymptotically conical and play a crucial role in the field. These examples justified the notion of G_2-manifold: a manifold endowed with a Riemannian metric whose holonomy is contained in G_2.

Then in 1996, Joyce constructed the first compact examples of holonomy G_2-manifolds, which was a fundamental breakthrough in the field, and the analytic theory developed by Joyce underpins all known methods to construct compact G_2-manifolds. In 2003 Kovalev gave a new construction for compact holonomy G_2-manifolds, based on a idea of Donaldson; this construction was later extended by Corti–Haskins–Nordström–Pacini. Based on these constructions, there are now known to be many examples of compact G_2-manifolds.

2.2 G_2-structures

The key to understanding and constructing G_2-manifolds goes via G_2-structures: 3-forms on 7-manifolds satisfying a certain positivity condition. A G_2-structure determines a metric and an orientation on a 7-manifold, and the condition for the G_2-structure to define a metric with holonomy contained in G_2 is the so-called *torsion-free* condition: namely that the 3-form is parallel for the Levi-Civita connection of the metric it defines or, equivalently, that it is closed and co-closed (again, using the metric and orientation that it defines). This is a nonlinear differential equation for the 3-form.

Although the main interest is in torsion-free G_2-structures, one can also consider splitting the torsion-free condition into two sub-cases: those which are closed and those which are co-closed. In fact, the co-closed condition is essentially vacuous: on any 7-manifold (compact or otherwise), a G_2-structure can be deformed to a co-closed one by the h-principle. By contrast, the closed condition is vital for all known constructions of compact G_2-manifolds, and is poorly understood.

2.3 Gauge theory and calibrated geometry

Donaldson–Thomas and Donaldson–Segal pioneered the notion of gauge theory in higher dimensions, and in particular in the setting of G_2 geometry. In particular, they defined G_2-instantons, which are connections generalising the more familiar anti-self-dual instantons from 4-dimensional geometry. Specifically, G_2-instantons are connections whose curvature satisfies the condition that its 2-form part lies pointwise in the Lie algebra $\mathfrak{g}_2$ of G_2, viewed as a subspace of the 2-forms. On G_2-manifolds, G_2-instantons are automatically Yang–Mills connections. That is, they are critical points of the Yang–Mills functional. The proposal is to try to build enumerative invariants for compact G_2-manifolds by "counting" G_2-instantons.

There is a close relationship between G_2 gauge theory and a "dual" theory of certain submanifolds. On a G_2-manifold, the G_2-structure and its Hodge dual are *calibrations*; that is, they are closed differential forms with comass one. The submanifolds calibrated by these calibrations (those submanifolds on which the forms restrict to be the volume form) are called *associative* and *coassociative* submanifolds, and they are automatically homologically volume-minimizing. There are also conjectures suggesting that one can build enumerative invariants using calibrated submanifolds.

2.4 Related geometries

There are two close cousins to G_2 geometry: $SU(3)$ geometry in 6 dimensions and $Spin(7)$ geometry in 8 dimensions.

Of particular relevance in 6 dimensions are *Calabi–Yau 3-folds* which have metrics with holonomy $SU(3)$, and *nearly Kähler* 6-manifolds which have the property that the Riemannian cone on them has a torsion-free G_2-structure. In these contexts one has associated problems in gauge theory (namely (pseudo-)Hermitian–Yang–Mills connections) and in calibrated geometry (namely (pseudo-)holomorphic curves and special Lagrangian submanifolds).

In 8 dimensions the most important geometry comes from metrics with holonomy $Spin(7)$, giving $Spin(7)$ manifolds: these include Calabi–Yau 4-folds and hyperkähler 8-folds as special cases. This yields some corresponding geometries in 7 dimensions: for example, nearly G_2-manifolds, which have a co-closed G_2-structure (called a *nearly parallel* G_2-structure) with the property that the Riemannian cone on them has holonomy contained in $Spin(7)$; and Sasaki–Einstein 7-manifolds, where the Riemannian cone on it is Calabi–Yau. More generally, one can try to understand classes of $Spin(7)$-structures, which are defined a certain type of very restrictive nondegenerate 4-form on an 8-manifold. In particular, closed $Spin(7)$-structures are necessarily torsion-free and so define a metric with holonomy contained in $Spin(7)$.

2.5 Physics

Another key direction of interest in G_2 geometry comes from theoretical physics. Compact G_2-manifolds, and compact 7-manifolds with other types of G_2-structures, appear when compactifying String Theory and M-Theory, as well as in the study of anomaly cancellation in heterotic String Theory. In this context, G_2-instantons on compact G_2-manifolds are important because they minimize the Yang–Mills action, and calibrated submanifolds play a crucial role because they minimize volume.

There are several groups of researchers in theoretical physics actively pursuing G_2 geometry, and the physics perspective motivates multiple research directions in G_2 geometry for pure mathematics. In particular, the physics viewpoint leads to various predictions which remain conjectural mathematically.

3 Recent Developments and Open Problems

3.1 G_2-manifolds

Recently, there have been various successful generalisations of the known constructions of compact G_2-manifolds which could lead to further examples, including by Joyce–Karigiannis (who extend the Joyce construction) and Nordström (who extends the Kovalev construction).

In another direction, there has been progress in the rigorous construction of complete non-compact G_2-manifolds which had been predicted by physicists. This work by Foscolo–Haskins–Nordström produces infinitely many cohomogeneity one examples which are asymptotically conical and *asymptotically locally conical*: the latter are asymptotic to a circle bundle over a Calabi–Yau cone. Foscolo–Haskins–Nordström have also produced infinitely many asymptotically locally conical G_2-manifolds which have at most an $\mathcal{S}^1$-symmetry. In general, this is contrary to predictions from physics.

The key problem in the study of holonomy G_2 metrics remains open:

- which compact 7-manifolds admit holonomy G_2 metrics?

Our understanding of this problem is incredibly limited, but there has been some progress on defining topological and analytic invariants of G_2 structures by Crowley–Goette–Nordström.

3.2 G_2-Laplacian flow

One key problem with current technology for producing compact G_2-manifolds is that one must start with a closed G_2-structure which is very close to torsion-free and then perturb using Joyce's analytic technique. An alternative approach to the problem of finding torsion-free G_2-structures was suggested by Bryant in 1992: a geometric flow of closed G_2-structures called the G_2-Laplacian flow. This flow provides the possibility of making large deformations of closed G_2-structures to torsion-free ones. Moreover, it also allows the opportunity to gain further insight into which 7-manifolds that admit closed G_2-structures support torsion-free G_2-structures.

Despite the G_2-Laplacian flow having been introduced almost 30 years ago, the analytic theory has only been developed recently by Bryant–Xu and, most notably, by Lotay–Wei. This has provided the impetus for a major increase in activity in geometric flows of G_2-structures, and particularly the G_2-Laplacian flow.

Of particular note is that one can obtain surprisingly strong analytic results for the G_2-Laplacian flow, which surpass, for example, those of Ricci flow in general dimension. Amongst the most impressive of these results is by Fine–Yau: for a 7-manifold that is a product of a 3-torus and a 4-manifold, so that the 3-tori are associative, the flow exists as long as the torsion (equivalently the scalar curvature) is bounded. Another set of important results (now for a product of a 4-torus and 3-manifold so that the 4-tori are coassociative) by Lambert–Lotay [6] was reported on in this meeting.

The key issue in this area is that one needs a closed G_2-structure to start the flow and so a major open problem is:

- which compact 7-manifolds admit closed G_2-structures?

A related natural problem, which is central to the field, is:

- can a compact 7-manifold admit an *exact* G_2-structure? For example, does the 7-sphere admit a closed (and hence exact) G_2-structure?

3.3 G_2-instantons

An area where there has been a large amount of activity and recent progress is in the study and construction of G_2-instantons.

Building on the earlier gluing results of Walpuski, Sá Earp, and Sá Earp–Walpuski for G_2-instantons on the Joyce and Kovalev examples of compact G_2-manifolds, there has been a great deal of study of the relationship between G_2-instantons and associative 3-folds, and the Seiberg–Witten equations with multiple spinors on 3-manifolds. In particular, there have been significant results by Haydys, Walpuski, Haydys–Walpuski, and Doan–Walpuski.

In another direction, Oliveira, Clarke, and Lotay–Oliveira have constructed new examples and have studied the moduli space of cohomogeneity one G_2-instantons on cohomogeneity one G_2-manifolds, including the Bryant–Salamon G_2-manifolds and asymptotically locally conical G_2-manifolds. Moreover, Ball–Oliveira have constructed homogeneous G_2-instantons on Aloff–Wallach spaces (which are nearly G_2-manifolds), and have used them to distinguish between nearly parallel G_2-structures on the same Aloff–Wallach space.

In general, the key open problem in the field of G_2-instantons, aside from the many analytic issues, is:

- can G_2-instantons be used to distinguish between compact G_2-manifolds? For example, can they be so used for the known compact G_2-manifolds?

4 Presentation Highlights

The research presented at the meeting can be broadly be described using 5 main interrelated themes.

- Instantons
- Symmetries
- Special Structures
- Geometric Flows
- Calibrated Submanifolds

Many of the results discussed touched on more than one of these themes.

4.1 Instantons

The presentations on gauge theory in higher dimensions focused on classification results, deformation theory and construction methods for instantons.

4.1.1 DT-instantons on almost complex 6-manifolds

The first talk on instantons was by Goncalo Oliveira, who described joint work with Gavin Ball which focused on the definition of a notion of DT-instantons on any almost complex 6-manifold. This definition generalised the familiar Hermitian–Yang–Mills connections on Kähler manifolds.

The key result presented related to a study of these DT-instantons on the manifold $\mathbb{F}_2$ of flags in $\mathbb{C}^3$, which can be viewed as the homogeneous space $\mathrm{SU}(3)/T^2$. It is well-known, since $\mathbb{F}_2$ is the twistor space of $\mathbb{CP}^2$, that it has two natural invariant almost complex structures: one which is integrable and one which is not integrable (in fact the former is part of a Kähler structure and the latter is part of a nearly Kähler structure). The main result was a classification theorem for invariant DT-instantons with respect to each of the aforementioned almost complex structures.

4.1.2 G_2-instantons on nearly G_2-manifolds and SU(3)-instantons on Sasaski–Einstein 7-manifolds

There are many examples of nearly G_2-manifolds, and G_2-instantons can exist on them. In fact, they are always endowed with a G_2-instanton known as the "canonical connection". Thus it is natural to study these instantons. Sasaki–Einstein 7-manifolds are particular examples of nearly G_2-manifolds and Ragini Singhal explained how one can naturally define a notion of $\mathrm{SU}(3)$-instantons on Sasaki–Einstein 7-manifolds, which can be compared to G_2-instantons.

Singhal's focus was on the deformation theory of the two types of instantons, and her main result was progress towards understanding deformations of the canonical connection for homogeneous nearly G_2 and Sasaki–Einstein 7-manifolds.

4.1.3 Deformation theory of G_2-instantons on asymptotically conical G_2-manifolds

Given the recent progress made in constructing asymptotically conical G_2-manifolds, and in constructing G_2-instantons on them, it is natural to continue to pursue the study of gauge theory on asymptotically conical G_2-manifolds. Of particular interest is the question of whether the known G_2-instantons, which typically have a large symmetry group (a key aspect of their construction), are unique in some appropriate sense.

Joe Driscoll considered the more general question of deforming asymptotically conical G_2-instantons on asymptotically conical G_2-manifolds: these are the G_2-instantons which converge to a dilation invariant G_2-instanton at infinity, which is equivalent to a pseudo-Hermitian–Yang–Mills connection (or $\mathrm{SU}(3)$-instanton) on the nearly Kähler link of the asymptotic cone. A simple but important example of such a G_2-instanton is the so-called "standard instanton" on $\mathbb{R}^7$, first constructed by Fairlie and Nuyts, which has gauge group G_2 and $\mathrm{SO}(7)$-symmetry.

Driscoll was able to give a local description of the moduli space of asymptotically conical G_2-instantons, and his main result was to use this deformation theory to show that the standard instanton on $\mathbb{R}^7$ is locally unique.

4.1.4 G_2-instantons on Joyce–Karigiannis G_2-manifolds

After providing an overview of the Joyce and Joyce–Karigiannis constructions of compact G_2-manifolds and Walpuski's construction technique for G_2-instantons on Joyce's compact G_2-manifolds, Daniel Platt described work in progress towards generalising Walpuski's construction to provide examples of G_2-instantons on the Joyce–Karigiannis G_2-manifolds.

Of particular interest is that, unlike in the original construction of Walpuski, one would expect to be able to produce many examples of G_2-instantons, thus providing a rich gauge theory. Platt explained how one may be able to achieve these examples by considering the situation where the G_2-manifold is a product of a circle with a Calabi–Yau 3-fold, and relating G_2-instantons to stable bundles on the Calabi–Yau 3-fold.

4.2 Symmetries

The talks concerning symmetries focused on generalisations of toric geometry, closed G_2-structures, and cohomogeneity one methods.

4.2.1 Toric geometry of exceptional holonomy manifolds

Toric geometry has been a very useful tool in the study of Kähler manifolds. Thomas Madsen described his joint work with Andrew Swann [7, 8] where they consider toric geometry of G_2 and $\mathrm{Spin}(7)$-manifolds: that is, where the G_2 or $\mathrm{Spin}(7)$-manifold admits an action by a 3 or 4-dimensional torus preserving the ambient structure, respectively.

In particular, Madsen explained the definition of a multi-moment map (generalising the standard notion of moment map from symplectic geometry) and how one obtains a trivalent graph in the image of the multi-moment map, which describes the singular orbits of the torus action.

4.2.2 Closed G_2-structures with symmetry

Although holonomy G_2 metrics on compact manifolds cannot admit continuous symmetries, it is natural to study closed G_2-structures with symmetry to see whether one can find interesting examples of compact 7-manifolds with closed G_2 structures, in particular to study the question of whether one can have an exact G_2-structure on a compact 7-manifold.

Alberto Raffero first described his joint work with Fabio Podestà [10], which gave strong restrictions on the automorphism group of a closed G_2-structure on a compact manifold. In particular, the main result is that there are no non-trivial homogeneous or cohomogeneity one compact closed G_2-structures.

Raffero then described joint work with Marisa Fernández and Anna Fino [5] which studied left-invariant closed G_2-structures on solvable Lie groups. The main result is that, in contrast to the symplectic setting, they find unimodular examples admitting closed G_2-structures, and give a classification result for such examples. As a consequence they find an example of an expanding G_2-Laplacian soliton. Moreover, they find a unimodular example whose Lie algebra has $b_3 = 0$, so that it has exact left-invariant G_2-structures, but they prove it does *not* admit any compact quotient.

4.2.3 Cohomogeneity one manifolds with exceptional holonomy

Cohomogeneity one techniques have proved to be a powerful tool in geometry, and have been essential in the construction of complete metrics with G_2 and $\mathrm{Spin}(7)$ holonomy going back to the first examples of such metrics by Bryant and Salamon. Fabian Lehmann provided a detailed overview of cohomogeneity one methods and how they can be used to construct complete metrics with exceptional holonomy.

The main results Lehmann described were one of the constructions of asymptotically locally conical and asymptotically conical G_2 manifolds due to Foscolo–Haskins–Nordström, and his own work constructing new examples of asymptotically locally conical and asymptotically conical $\mathrm{Spin}(7)$ manifolds.

4.2.4 Toric nearly Kähler manifolds

In a similar spirit to Thomas Madsen's talk, Kael Dixon studied toric geometry of nearly Kähler 6-manifolds; namely, nearly Kähler 6-manifolds admitting an action of the 3-torus preserving the ambient structure.

One main result was a complete description of the standard homogeneous nearly Kähler $\mathcal{S}^3 \times \mathcal{S}^3$ in toric terms, which built on work of the speaker in [3]. The other key theorem Dixon presented was a local description of all toric nearly Kähler 6-manifolds in terms of a certain second order nonlinear partial differentiation equation.

4.3 Special Structures

In the study of special structures in G_2 geometry and related topics, the presentations focused on balanced $\mathrm{Spin}(7)$-structures and closed G_2-structures.

4.3.1 A spinorial approach to balanced Spin(7)-structures

$\mathrm{Spin}(7)$-structures on 8-manifolds can be equivalently be described using nowhere vanishing spinors instead of certain 4-forms. Lucía Martín-Merchán described her work in [9] which gave a spinorial classification of $\mathrm{Spin}(7)$-structures, and in particular identified so-called *balanced* $\mathrm{Spin}(7)$-structures with harmonic unit spinors.

Martín-Merchán then described her joint work with Giovanni Bazzoni and Vicente Muñoz in [2] which studied 8-manifolds given as the product of a 5 or 6-dimensional nilmanifold with a 3 or 2-dimensional torus, respectively. The main result here was a classification of left-invariant balanced $\mathrm{Spin}(7)$-structures when choosing a 5-dimensional nilmanifold, and examples and a partial classification for the 6-dimensional nilmanifold case.

4.3.2 Quadratic closed G_2-structures

Gavin Ball described his work on what are known as *quadratic* closed G_2-structures: closed G_2-structures whose torsion (which can be identified with a 2-form) has the property that its exterior derivative is a 3-form that is quadratic in the torsion itself. These generalise the torsion-free G_2-structures (for which the torsion is zero) and the *extremally Ricci pinched* closed G_2-structures introduced by Bryant.

Ball's main study was on quadratic closed G_2-structures whose pointwise stabilizer can be identified with one of the two conjugacy classes of $\mathrm{U}(2)$ in G_2. This led to: new examples of extremally Ricci pinched closed G_2-structures; Weierstrass formulae which classify some quadratic closed G_2-structures; another classification result involving links to semi-flat T^4-fibrations and maximal spacelike submanifolds contained in a certain quadric in $\mathbb{R}^{3,3}$; and new examples of G_2-Laplacian solitons including new examples of gradient solitons.

4.4 Geometric Flows

The talks on geometric flows focused on two different areas: the G_2-Laplacian flow with symmetries and a flow of isometric G_2-structures.

4.4.1 G_2-Laplacian flow and spacelike mean curvature flow

Given the recent important on the G_2-Laplacian flow, both in general and in special cases, one is strongly motivated to other situations beyond the general setting where one can get potentially stronger results.

Ben Lambert described joint work with Jason Lotay [6] looking at a special case of the G_2-Laplacian flow, where the 7-manifold is the product of a 4-torus with a 3-manifold, and the closed G_2-structure defines a semi-flat coassociative fibration. In this case, the G_2-Laplacian flow may be identified with the mean curvature flow of spacelike 3-dimensional submanifolds in $\mathbb{R}^{3,3}$.

The main result Lambert described was when the 3-manifold is $\mathbb{R}^3$, where one obtains long-time existence for spacelike mean curvature flow in $\mathbb{R}^{3,3}$ (and thus the G_2-Laplacian flow) for *any* initial data. This is a very surprising result since it is the first of its kind for the G_2-Laplacian flow when it is a nonlinear partial differential system which makes no smallness assumption for the initial data or curvature/torsion assumption along the flow.

4.4.2 $\mathcal{S}^1$-invariant G_2-Laplacian flow

In contrast to Ben Lambert's talk, where one assumes a large amount of symmetry, here Udhav Fowdar described work on the G_2-Laplacian flow with the least amount of continuous symmetry, namely $\mathcal{S}^1$-symmetry. Fowdar first showed that the $\mathcal{S}^1$-invariant G_2-Laplacian flow is equivalent to a coupled system for $\mathrm{SU}(3)$-structures on a 6-manifold and connections on an $\mathcal{S}^1$-bundle over the 6-manifold. This generalised the well-known work of Apostolov–Salamon on $\mathcal{S}^1$-quotients of G_2-manifolds. The main results were then a study of the $\mathcal{S}^1$-invariant G_2-Laplacian flow and the coupled system on the quotient on two particular examples.

First, for a left-invariant closed G_2-structure studied by Fernàndez and Bryant, which is on a T^4-bundle over T^3, for which the G_2-Laplacian flow exists for all time, Fowdar showed that the flow of $SU(3)$-structures on the $\mathcal{S}^1$-quotient has constant symplectic form but the almost complex structure degenerates as time goes to infinity.

Second, Fowdar looked at closed G_2-structures on the product of $\mathbb{R}^+$ with a 6-dimensional nilmanifold (endowed with a 1-parameter family of left-invariant structures) which is a T^2-bundle over T^4. Here, the $\mathcal{S}^1$-quotient turns out to be Kähler and the Kähler condition is preserved along the flow, leading to the question of whether or not this is a general phenomenon.

4.4.3 A flow of isometric G_2-structures

Shubham Dwivedi gave a comprehensive overview of his work with Panagiotis Gianniotis and Spiro Karigiannis in [4] on a flow of G_2-structures which preserves the underlying metric that the G_2-structures define, hence the term *isometric* G_2-structures. This flow is a gradient flow of the L^2-norm of the torsion, restricted to the class of isometric G_2-structures.

The main results were Shi-type estimates, an ϵ-regularity theorem, a control on the size of the singular set at a finite time singularity of the flow, and the fact that Type I singularities are modelled by self-shrinkers.

4.5 Calibrated Submanifolds

The talks on calibrated submanifolds covered a range of ambient geometries: nearly Kähler 6-manifolds, hyperkähler 4-manifolds, and 7-manifolds equipped with G_2-structures with torsion.

4.5.1 Pseudoholomorphic curves in nearly Kähler 6-manifolds

Holomorphic curves are an essential part of complex (and particularly Kähler) geometry and pseudoholomorphic curves play a crucial role in symplectic geometry. Nearly Kähler 6-manifolds are neither complex nor symplectic, but their pseudoholomorphic curves are nonetheless important, particularly since the cone on a pseudoholomorphic curve is associative in the G_2 cone over the nearly Kähler 6-manifold. That said, the general theory of pseudoholomorphic curves in nearly Kähler 6-manifolds has not previously been studied.

Benjamin Aslan described his study of this general theory, with a focus on twistor spaces, namely $\mathbb{CP}^3$ and the flag manifold $\mathbb{F}_2$. The main result was a classification of $\mathcal{S}^1$-invariant pseudoholomorphic curves in the nearly Kähler $\mathbb{CP}^3$.

4.5.2 The minimal sphere in the Atiyah–Hitchin manifold

The (double cover of the) Atiyah–Hitchin manifold is a key example of a hyperkähler 4-manifold which is topologically $\mathbb{CP}^2 \setminus \mathbb{RP}^2$ and is closely related to monopoles on $\mathbb{R}^3$. In the Atiyah–Hitchin manifold there is a central 2-sphere that is well-known to be minimal, but not complex for any of the hyperkähler structures.

Chung-Jun Tsai described his work with Mu-Tao Wang in [11] which proves that the minimal 2-sphere is in fact area-minimizing by showing that it is a calibrated submanifold. Moreover, Tsai showed that the minimal 2-sphere is the unique compact minimal submanifold of the Atiyah–Hitchin manifold of dimension 2 or 3, and that the minimal 2-sphere is stable under mean curvature flow.

4.5.3 Minimality and local non-existence of calibrated submanifolds

In the final talk of the meeting, Jesse Madnick described joint work with Gavin Ball in [1], which investigated associative 3-folds and coassociative 4-folds in general 7-manifolds with G_2-structures. Associative and coassociative submanifolds are minimal (in fact, volume-minimizing) and have good local existence theory in G_2-manifolds, so it is interesting to ask when these properties persist in other G_2-structures.

The main result was to give necessary and sufficient conditions on a G_2-structure for all associative, and respectively all coassociative, submanifolds to be minimal, by providing a formula for the mean curvature determined by the torsion of the G_2-structure. In addition, Madnick described an obstruction to even the local existence of coassociative 4-folds for certain G_2-structures.

5 Scientific Progress Made

We summarize the scientific progress made in each of the main themes highlighted in the previous section.

5.1 Instantons

There has clearly been a significant increase in our understanding of higher-dimensional gauge theory beyond the established settings of compact Calabi–Yau, G_2, and $\mathrm{Spin}(7)$-manifolds. There have been extensions to non-integrable structures, such as almost complex 6-manifolds, nearly G_2-manifolds, and Sasaki–Einstein 7-manifolds, and in the study of the non-compact setting of asymptotically conical G_2-manifolds. In particular, we have seen classification and deformation theory results.

In the compact G_2-manifold setting, which holds the greatest interest in G_2 geometry, there has been exciting progress towards potentially constructing a large number of G_2-instantons on the new examples of G_2-manifolds due to Joyce–Karigiannis.

5.2 Symmetries

The progress made in the use of symmetries to understand exceptional holonomy and related geometries has yielded both positive and negative results. On the one hand, there is a better understanding of toric geometry in this context and there are new examples of complete metrics with exceptional holonomy. On the other hand, it is now clear that there are significant challenges to using symmetry techniques to understand compact manifolds with closed G_2 structures, now that the standard methods have been shown to not provide any non-trivial examples.

5.3 Special Structures

Special structures have received relatively little detailed attention and are generally quite poorly understood. The results presented in the meeting clearly show a marked improvement in our ability to study and understand balanced $\mathrm{Spin}(7)$-structures and closed G_2-structures. In particular, the results provided new examples of and classification results for such structures.

5.4 Geometric Flows

There were some interesting results concerning the G_2-Laplacian flow with symmetries, building on the general theory developed in recent years. Specifically there were some impressive long-time existence results in the setting of trivial semi-flat coassociative torus fibrations, and some intriguing potential relations between $\mathcal{S}^1$-invariant G_2-Laplacian flow and Kähler geometry. Both of these results certainly merit further examination and reveal exciting future research avenues for investigation.

The analytic foundations were developed for a flow of isometric G_2-structures which is a new research topic that has links to several research groups in G_2 geometry, and so will certainly continue to be studied.

5.5 Calibrated Submanifolds

There was notable progress made in the study of calibrated submanifolds outside of the setting of well-known areas of manifolds with special holonomy equipped with their usual calibrations. In particular, the techniques developed to study and classify pseudoholomorphic curves in nearly Kähler 6-manifolds, classify minimal submanifolds in the Atiyah–Hitchin manifold, and analyse the properties of calibrated submanifolds in G_2-structures with torsion, are certainly to yield further results in related areas.

6 Outcome of the Meeting

The key outcome of the meeting was the increase in communication and collaboration between researchers in G_2 geometry, which has and will continue to lead to exciting new research directions and results. It is particularly worth emphasizing the positive outcome of the meeting for early career researchers present, mainly for PhD students but also some postdocs and other participants, who unanimously expressed how enjoyable and productive the meeting was for them. Senior researchers also remarked on how refreshing it was to have so many early career researchers interacting significantly with them and each other, which provided a unique opportunity to learn about and to offer input towards the research avenues pursued by the next generation of researchers in the field.

More specifically, the Open Problem session identified several interesting research problems that the participants considered worth pursuing, which we describe below.

6.1 Gauge theory

Benoit Charbonneau explained a result of Lewis that states that if a bundle E over a compact Calabi–Yau 4-fold admits a Hermitian–Yang–Mills connection, then any Spin(7)-instanton on that bundle must be Hermitian–Yang–Mills. He therefore posed the problem:

- find a Spin(7)-instanton which is not Hermitian–Yang–Mills on some bundle over a compact Calabi–Yau 4-fold, or prove there are no such Spin(7)-instantons.

As a follow-up, he posed the related problem:

- find a G_2-instanton on a bundle over a product of a circle with a compact Calabi–Yau 3-fold which is not the pullback of a Hermitian–Yang–Mills connection, or prove that there are none.

Based on the known relationship between stable bundles and Hermitian–Yang–Mills connections, Spiro Karigiannis asked:

- is there potentially some analogue of the Donaldson–Uhlenbeck–Yau or Hitchin–Kobayashi correspondence for instantons on compact G_2/Spin(7)-manifolds?

Derek Harland remarked that while we now know much more about pseudoholomorphic curves on nearly Kähler 6-manifolds, we know much less about instantons on nearly Kähler 6-manifolds. He therefore suggested to try to:

- find more examples of instantons on nearly Kähler 6-manifolds;
- prove an analogue of Walpuski's gluing result for G_2-instantons in the setting of nearly Kähler 6-manifolds;
- classify homogeneous instantons on nearly Kähler 6-manifolds, with all possible structure groups.

He also considered a 6-dimensional nearly Kähler twistor space, where one can take bundles E whose restriction to each twistor fibre is trivial, and asked:

- are there instantons on E which are not pulled back from the base of the twistor fibration?

Finally he asked:

- are there any smooth instantons on $\mathcal{S}^6$ with structure group SU(2)?

6.2 G_2-manifolds

Considering the known examples of complete non-compact G_2-manifolds, which are asymptotically cylindrical, asymptotically conical, or asymptotically locally conical and thus respectively have $O(r)$, $O(r^6)$, or $O(r^7)$ volume growth for geodesic balls of radius r as $r \to \infty$, Benoit Charbonneau asked:

- what are the possible volume growths for geodesic balls of radius r as $r \to \infty$ for complete non-compact G_2-manifolds?

Spiro Karigiannis recalled the work of Madsen–Swann on toric G_2-manifolds which produced incomplete holonomy G_2 metrics. He therefore asked:

- can we construct compact holonomy G_2-manifolds by gluing building blocks that have incomplete holonomy G_2 metrics?

Jason Lotay responded to this by pointing out the fundamental work by Gross–Wilson, which produced hyperkähler metrics on K3 surfaces by gluing in the incomplete Ooguri–Vaga metric, and the recent work by Hein–Sun–Viaclovsky–Zhang, which also produced metrics on K3 surfaces by gluing in an incomplete hyperkähler metric, now on an interval times a 3-dimensional nilmanifold. He therefore suggested:

- find an incomplete holonomy G_2 metric on an interval times a 6-dimensional nilmanifold which could be used as a building block in a gluing construction for compact G_2-manifolds.

6.3 Special Structures

Henrique Sá Earp posed the following problem:

- formulate the right definition of "extremally Ricci-pinched" for co-closed G_2-structures.

Gavin Ball responded to this problem by suggesting the following analogy of his study of quadratic closed G_2-structures:

- study co-closed G_2-structures such that the exterior derivatives of their torsion forms are quadratic in the torsion.

Following on from this, Ball suggested the following problem:

- find a 1-parameter family of homogeneous co-closed G_2-structures containing a nearly parallel G_2-structure (that is, a G_2-structure whose exterior derivative is a constant multiple of its Hodge dual), but so that all other members of the 1-parameter family are *not* nearly parallel.

A solution to this problem would show that there is unlikely to be a satisfactory answer to Sá Earp's question. Finally, Ball asked:

- can a co-closed G_2-structure which is purely of "type τ_3" (that is, such that its exterior derivative has zero component in the direction of its Hodge dual), be Einstein?

Sá Earp explained that the condition to be a critical point for the flow of isometric G_2-structures is equivalent to a certain natural map associated to isometric G_2-structures be harmonic. He explain how this harmonic condition could be extended to other situations by analogy and in particular he suggested:

- study *harmonic* $\mathrm{Spin}(7)$-structures.

6.4 Calibrated Submanifolds

Jesse Madnick asked the fundamental question:

- are there any topological obstructions for complete, embedded special Lagrangian 3-folds in $\mathbb{C}^3$, or can every topological type occur?

Jason Lotay recalled that Harvey–Lawson proved that, given any real analytic surface in $\mathbb{R}^7$, there is a (locally unique) associative containing that surface. He therefore posed the question:

- when is a real analytic surface in $\mathbb{R}^7$ the boundary of a compact associative 3-fold?

He also posed the related problem:

- given a map u from the boundary of a domain in $\mathbb{R}^3$ to $\mathbb{R}^4$, when does there exist a map on the domain to $\mathbb{R}^4$, with u as its boundary value, so that the graph of the map is associative?

He suggested that there may be some relation to work of Harvey–Lawson on pluripotential theory for calibrated manifolds, and pointed to the fact that Haskins–Pacini have proved some obstructions for the special Lagrangian boundary value problem.

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