New Trends in Nonlinear Diffusion: a Bridge between PDEs, Analysis and Geometry (Online) (21w5127)

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1 Overview of the Field

Diffusion equations are nowadays an essential tool in the description of several phenomena, covering a large number of fields ranging from Physics and Engineering to Biology, Social Sciences and Finance.
The main goal of this field is to explore and understand the analytic and geometric properties of solutions to local and nonlocal diffusion equations, especially of nonlinear type, for which much less is known.
In the last decades there has been a growing interest towards nonlinear and degenerate/singular diffusions, which give rise to a series of challenging phenomena such as lack of regularity, finite speed of propagation or free-boundary problems. Popular models are represented by the Porous Medium/Fast Diffusion Equation and the parabolic p-Laplacian equation. In fact, the latter naturally pop up in many other analytic, geometric and probabilistic problems: indeed, it is worth mentioning that lots of some recent results treat the connections between these equations and some celebrated functional inequalities, such as Gagliardo-Nirenberg-Sobolev inequalities, through the employment of the successful entropy methods.
The application of Optimal Transport Theory had a tremendous impact on Analysis, Geometry and Probability, since many PDEs can be seen as gradient flows of suitable energy functionals in the space of probability measures, both in Euclidean and non-Euclidean frameworks. However, several issues, like the connections with a negative lower bound on the Ricci curvature of the manifold at hand, are only partially understood for nonlinear diffusion PDEs, where energies of power type are involved.
Another hot topic concerns the geometric flows, which were successfully employed in the proof of the famous Poincaré conjecture. A lot of effort is being made in order to recast the related PDEs as gradient flows in suitable metric spaces, but the research area is still wide open.
Many phenomena in life sciences, from the microscopic level to the macroscopic level exhibit a striking similar structure, composed of a long-range attractive force and a short-range repulsion. The passage among these two levels is dictated by mean-field PDEs of novel type and of nonlocal character. One of the main PDEs coming up from these processes are the commonly called aggregation-diffusion equations. The myriad applications of such PDEs justified their massive study over the last decades, exploiting natural gradient-flow structures, related functional inequalities and suitable entropy methods. Other models arising from the mean-field process lead to kinetic equations of Vlasov type, pertaining to the area of kinetic transport equations. For the latter, powerful methods from hypoelliptic and hypocoercive theory have successfully been employed in the linear setting, and now also some nonlinear cases seem within reach of recent novel tools.

2 Recent Developments and Open Problems

This event gathered leading researchers working on the three main fields of PDEs, Analysis and Geometry. For brevity, we shall only mention a few topics that were covered during the conference, in which recent advances and several open problems were discussed:

- Asymptotics of nonlinear diffusions, entropy methods and connections with functional inequalities;
- Behavior of solutions to certain PDEs affected by the geometry of the space;
- Fine regularity properties for local/nonlocal diffusions and related PDEs;
- Nonlinear nonlocal diffusions in the Euclidean space and on manifolds;
- Recent advances in the theory of geometric flows and related PDEs;
- Gradient flows and global analysis in nonsmooth frameworks;
- Diffusion-aggregation equations or systems;
- Kinetic equations and equations arising from fluid mechanics.

3 Presentation Highlights

We had 39 talks overall, 8 per day (except on Wednesday where we had 7), each one 25 minutes long. Given such a high number, we have decided to select five of them (one per day) which we believe to well represent both the general spirit of the workshop and the main scientific lines of research discussed on that specific day.

Alessio Figalli's talk [1] (Monday 6^{th}) was focused on the so-called one-phase Stefan problem, a very challenging one in the realm of free boundaries and diffusion PDEs. In his thorough discussion, he described a recent result, obtained in collaboration with X. Ros-Oton and J. Serra, achieving a very fine characterization of the singular set in the ice-melting-to-water phenomenon.

Manuel del Pino's talk [2] (Tuesday 7^{th}) concerned the dynamics of solutions to the Euler equation of an inviscid incompressible fluid, tackling a longstanding conjecture originally formulated by Helmholtz. In particular, del Pino and his coauthors established sharp existence and asymptotic results for solutions starting from data whose vorticity is concentrated near isolated points (at least in dimension 2), the latter being of key importance from the physical perspective.

Jean Dolbeault's talk [3] (Wednesday 8^{th}) dealt with the delicate issue of stability in functional inequalities of Gagliardo-Nirenberg or Sobolev type (GNS). His lecture was the third one, following B. Nazaret's and N. Simonov's talks, and discussed an ambitious common project aimed at fully understanding the subtle interaction between quantitative and constructive regularity estimates (of both Harnack and Hölder type) for the fast diffusion equation, entropy-entropy production inequalities and improved forms of GNS inequalities.

José A. Carrillo's talk [4] (Thursday 9^{th}) was devoted to a broad scope overview of recent progress regarding the bifurcation analysis of nonlinear Fokker-Planck equations, which arise in an impressive number of applications such as consensus formation, optimization, granular media, swarming behavior, opinion dynamics and financial mathematics. The main results presented established existence of continuous or discontinuous phase transitions for equations of the type of Cucker-Smale, McKean-Vlasov and Keller-Segel, by means of variational analysis of the right entropy functional associated to the problem at hand. Numerical simulations, shedding light on the stability of the different branches, were also shown.

Juan L. Vázquez's talk [5] (Friday 10^{th}) considered a large class of equations, both elliptic and parabolic, involving nonlinear fractional Laplacian operators, for which several well-posedness and fine asymptotic results have been achieved in the last few years. A very significant instance, known as the fractional p-Laplacian operator, was thoroughly discussed, laying the foundations for the study of a number of challenging open problems.

4 Scientific Progress Made

The workshop covered many "hot topics" in several branches of both analysis and geometry. The talks, only to mention a few subjects, dealt with free boundary problems, fine properties of solutions to diffusion, aggregation-diffusion and diffusion-reaction equations, PDE analysis on Riemannian and non-Riemannian settings (such as Finsler manifolds, Lorentzian manifolds, singular spaces, ...), optimal transport theory and gradient-flow PDEs, non-local nonlinear equations in Euclidean and non-Euclidean frameworks, symmetrization techniques for fractional operators, equations arising from two or three dimensional fluid mechanics, Boltzmann equations, Vlasov-Poisson equations, functional inequalities in connection with nonlinear elliptic and parabolic PDEs, linear and nonlinear Fokker-Planck equations, Keller-Segel type models. Most of the lectures were delivered by leading experts in the field, from young emerging figures to consolidated ones, and allowed to witness the impressive progress on the above mentioned topics and related ones: full characterization theorems, foundations of new branches of theory, significant improvement of previous results and in some cases even solution of longstanding problems or conjectures. Last but not least, the outstanding communication skills of the speakers succeeded to convey in an excellent way such important findings. The new ideas shared in this workshop not only strengthened already well-established collaborations, but also gave the opportunity to create new ones, even between researchers from different areas.

5 Outcome of the Meeting

The workshop was undoubtedly a great success, as most of the speakers acknowledged during the final greetings on the last day of the conference, and also personally to the organizers. We had an average of 40-50 people attending the daily lectures, many of which were young researchers. The general atmosphere was always friendly and interactive, as proved by the high number of questions and remarks that were raised during most of the talks, some of which probably leaded to new collaborations. The interdisciplinary structure of the event (involving analysis, PDEs, geometry, fluid dynamics, ...), which at first glance might have appeared too ambitious, in the end turned out to be one of the strong points of the meeting. Indeed, people seemed to appreciate the wide range of topics covered by the talks, that in particular gave them opportunities to discover connections on new challenging problems lying at the "interface" between different research fields.

We are very grateful to the BIRS and CMO personnel that helped us in the organization in a friendly and efficient environment. Their professional job surely contributed to the success of the event.

To conclude, we strongly believe that a follow-up of the present event, to be carried out in presence, would be of crucial importance to consolidate the achievements and to keep up to date the networking started during the workshop.

References

[1] A. Figalli, X. Ros-Oton, and J. Serra, *The singular set in the Stefan problem*, preprint arXiv:2103.13379.

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[4] J. Carrillo de la Plata, R. Gvalani, G. Pavliotis, and A. Schlichting, *Long-time behaviour and phase transitions for the McKean-Vlasov equation on the torus*, Arch. Rat. Mech. Anal. 235, 635–690 (2019).

[5] J.L. Vázquez, *The fractional p-Laplacian evolution equation in $\mathbb{R}^N$ in the sublinear case*, preprint arXiv:2004.05799.