Fractional PDEs and steady states for aggregation-diffusion models

Edoardo Mainini

Department of Mechanical Engineering, University of Genoa

BIRS-CMO Workshop
New Trends in Nonlinear Diffusion: a Bridge between PDEs, Analysis and Geometry

7 September 2021

Aggregation-diffusion model

$$\begin{cases} \partial_t \rho = \Delta \rho^m - \nabla \cdot (\rho \nabla K_s * \rho) & x \in \mathbb{R}^N, \quad t > 0, \\ \rho(0) = \rho_0 \geq 0 \end{cases}$$

where $m > 1$ (slow diffusion) and $K_s(x) = c_{N,s}|x|^{2s-N}$, $s \in (0, N/2)$ (Riesz potential)

The classical Patlak-Keller-Segel model in dimension two is obtained with $m = s = 1$.

Free energy:

$$\mathcal{F}[\rho] = \mathcal{H}_m[\rho] - \mathcal{W}_s[\rho]$$

$$\mathcal{H}_m[\rho] = \frac{1}{m-1} \int_{\mathbb{R}^N} \rho^m(x)\, dx\,, \qquad \mathcal{W}_s[\rho] = \frac{1}{2} \iint_{\mathbb{R}^N \times \mathbb{R}^N} K_s(|x-y|)\rho(x)\rho(y)\, dxdy\,.$$

Objective: We wish to characterize radial stationary states

Aggregation vs diffusion

$\mathcal{H}_m$ and $\mathcal{W}_s$ are homogeneous by taking dilations $\rho^\lambda(x) = \lambda^N \rho(\lambda x)$

$$\mathcal{F}[\rho^\lambda] = \lambda^{N(m-1)}\mathcal{H}_m[\rho] - \lambda^{N-2s}\mathcal{W}_s[\rho]\,.$$

Critical exponent $m_c := 2 - 2s/N$. $m_c \in (1,2)$

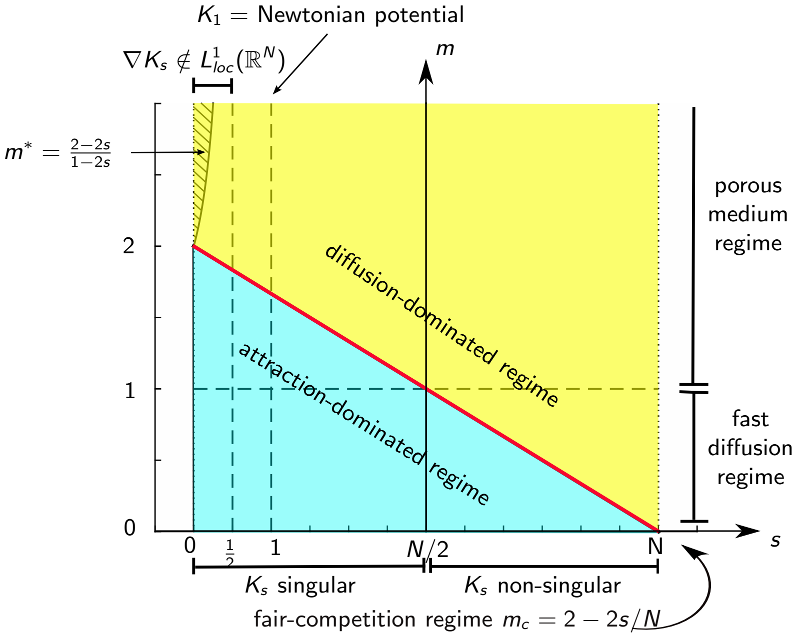
- $m = m_c$: fair competition regime (critical mass appears)
- $m > m_c$: diffusion dominated regime
- $m < m_c$: attraction dominated regime

Dynamics in the Newtonian case: global-in-time solutions exist for $m > m_c$ and also for $m = m_c$ if the initial mass is subcritical [Calvez, Carrillo 2006], [Sugiyama 2007], [Blanchet, Carrillo, Laurencot 2009]

Analysis of stationary states:

- $m = m_c$ [Calvez, Carrillo, Hoffmann 2016, 2017]
- $m > m_c$ with Newtonian potential interaction [Kim, Yao 2012], [Bian, Liu 2013], [Carrillo, Castorina, Volzone 2015], [Carrillo, Hittmeir, Volzone, Yao 2019]
- $m > m_c$ with Riesz potential: [Carrillo, Hoffmann, M., Volzone 2018]

The different regimes for $N \geq 3$



Stationary solutions

Aggregation diffusion model

$$\partial_t \rho = \Delta \rho^m - \nabla \cdot (\rho \nabla K_s * \rho) = \nabla \cdot (\rho \mathbf{v}), \qquad \mathbf{v} := \tfrac{m}{m-1} \nabla \rho^{m-1} - \nabla K_s * \rho$$

A stationary solution $\rho \geq 0$ formally satisfies

$$\tfrac{m}{m-1} \rho^{m-1} - K_s * \rho + C = 0$$

in each connected component of $\{\rho > 0\}$ (C is a constant that may take different values in each connected component).

Radial stationary solutions:

$$\tfrac{m}{m-1} \rho^{m-1} - K_s * \rho + C = 0 \qquad \text{in } B_R(0)$$

A radiality result of EVERY stationary solution is proven for the Newtonian potential by [Carrillo, Hittmeir, Volzone, Yao 2019].

The result is extended to Riesz potential for $m^* > m > m_c := 2 - 2s/d$ in [Carrillo, Hoffmann, M., Volzone 2018]. Here $m^* := \frac{2-2s}{1-2s}$ if $s < 1/2$ and $m^* = +\infty$ o.w.

Outline

1) Minimization of the free energy and regularity properties of stationary states in the diffusion dominated regime

2) The fractional plasma problem and uniqueness of radial stationary states in the different regimes

Global minimizers: diffusion dominated regime

$$\mathcal{F}[\rho] = \frac{1}{m-1}\int_{\mathbb{R}^N} \rho^m(x)\,dx - \frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} K_s(|x-y|)\rho(x)\rho(y)\,dxdy$$

$$\mathcal{Y}_M := \left\{\rho \in L^1_+(\mathbb{R}^N)\cap L^m(\mathbb{R}^N)\,,\ ||\rho||_1 = M\,,\ \int_{\mathbb{R}^N} x\rho(x)\,dx = 0\right\}$$

Theorem (Carrillo, Hoffmann, M., Volzone 2018)

Let $s \in (0, N/2)$, $m > m_c := 2 - 2s/N$, and $M > 0$. There is a minimizer of $\mathcal{F}$ over $\mathcal{Y}_M$.

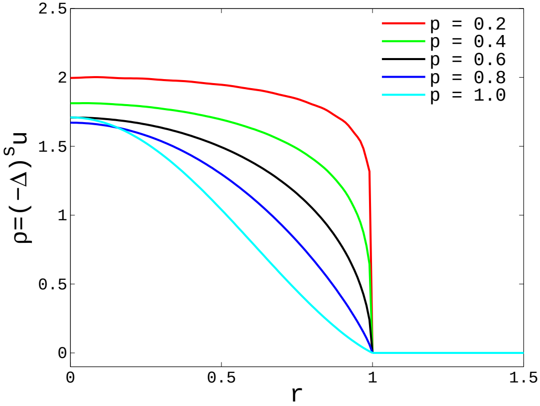
If ρ is a minimizer, then

- *ρ is radially decreasing (by Riesz rearrangement inequality),*
- *ρ is bounded and compactly supported, and it satisfies the equilibrium equation*

$$\rho^{m-1}(x) = \tfrac{m-1}{m}\,\left(K_s * \rho(x) - C\right)_+ \qquad \text{in } \mathbb{R}^N$$

  where $C > 0$ is a constant (it can be explicitly written in terms of $\mathcal{F}[\rho]$ and M)
- *If $m_c < m < m^* := \frac{2-2s}{1-2s}$, then $\rho^{m-1} \in W^{1,\infty}(\mathbb{R}^N)$, thus $\rho \in C^{0,\alpha}(\mathbb{R}^N)$ with $\alpha = \min\{1, \frac{1}{m-1}\}$. Moreover $\rho \in C^1$ if $m < 2$.*
- *If $m \geq m^*$ (only if $s < 1/2$), then $\rho^{m-1} \in C^\alpha(\mathbb{R}^N)$ for any $\alpha < \frac{2s(m-1)}{m-2} \leq 1$.*

Numerical simulations: $s = 1/2$, $N = 2$, $m \geq 2$



The radial solution ρ for $N = 2$, $s = 1/2$ and different values of $m \geq 2$.

$p = \dfrac{1}{m-1}$ is the Hölder exponent.

Radius of the support is 1, masses are different.

Uniqueness of radial steady states

Uniqueness of radial steady states in the diffusion dominated regime is known with Newtonian kernels [Kim-Yao 2012], [Carrillo, Castorina, Volzone 2015].

In the case of Riesz kernels $K_s(x) = c_{s,N}|x|^{2s-N}$, uniqueness is proved for $N = 1$, $s \in (0, 1/2)$ and $m > m_c$ in [Carrillo, Hoffmann, M., Volzone 2018]

For $N > 1$, the task is more complicated. Recent results on this topic are contained in

- [Calvez, Carrillo, Hoffmann 2020]: $m \geq m_c := 2 - 2s/N$, $s \in (0, 1)$.
- [Delgadino, Yan, Yao 2020]: $m \geq 2$, $s \in (0, N/2)$ (and other general potentials)
- [Chan, Gonzalez, Huang, M., Volzone 2020]: case $1 < m \leq 2$, $s \in (0, 1)$

Fractional plasma problem

Let

$$u = (-\Delta)^{-s}\rho, \quad s \in (0,1), \quad p = \frac{1}{m-1}, \quad a = \left(\frac{m-1}{m}\right)^{\frac{1}{m-1}}.$$

Since

$$\rho(x)^{m-1} = \tfrac{m-1}{m}\left(K_s * \rho(x) - C\right)_+, \quad x \in \mathbb{R}^N$$

we may rewrite the equilibrium equation in terms of u as

$$\begin{cases} (-\Delta)^s u = a(u-C)_+^p & \text{in } \mathbb{R}^N, \\ u(x) \to 0 & \text{as } |x| \to \infty \end{cases}$$

a fractional plasma problem (FPP).

Uniqueness is studied in
[Chan, Gonzalez, Huang, M. Volzone 2020] for $p \geq 1$, $C \geq 0$, $a > 0$, $s \in (0, 1 \wedge \frac{N}{2})$.

Local case $s = 1$ is studied by [Flucher, Wei 1998]: for $1 < p < \frac{N+2}{N-2}$, $N \geq 3$ with an ODE argument

Relation between stationary states and the FPP

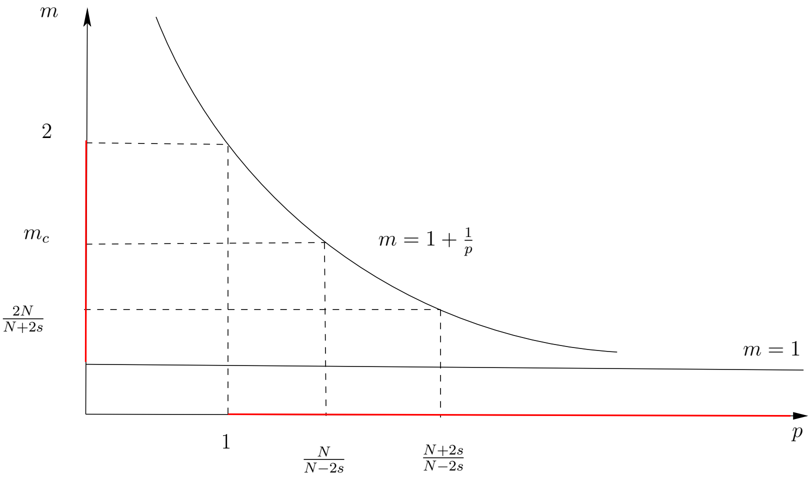


FIGURE 1. Sub and supercritical regimes in terms of m and p

Existence and uniqueness for the FPP

The case $s \in (0,1)$ is more challenging: no ODE technique can be used.

Theorem (Chan, González, Huang, M., Volzone 2020)

Let $1 \le p < (N+2s)/(N-2s)$ (subcritical case). Let $a > 0$, $C > 0$. There exists a unique radially decreasing solution for the problem

$$\begin{cases} (-\Delta)^s u = a\,(u - C)_+^p & \text{in } \mathbb{R}^N, \\ u(x) \to 0 & \text{as } |x| \to \infty. \end{cases}$$

Existence: by considering the energy

$$\mathcal{G}[u] := \frac{1}{2}\|u\|^2_{\dot{H}^s(\mathbb{R}^N)} - \int_{\mathbb{R}^N} F(u), \qquad F(t) := \int_0^t a(\tau - C)_+^p \, d\tau$$

- mountain pass solutions [Ikoma 2020]
- variational arguments à la [Berestycki-Lions 1983]

Remark: the theorem holds true for $0 < p < 1$ as well thanks to the results in [Delgadino, Yan, Yao 2020] and [Calvez, Carrillo, Hoffmann 2020]

Uniqueness for the FPP

Let $a = 1$. Let u_1, u_2 be two radially decreasing solutions with $C > 0$ fixed.
Let $v_i = u_i - C$, $i = 1, 2$. A scaling argument for the equation $(-\Delta)^s v = v_+^p$ allows to reduce to the case $v_1(0) = v_2(0)$. ($v_i^{(\lambda)}(x) := \lambda v_i(\lambda^{\frac{p-1}{2s}} x)$ satisfies same equation as v_i).

The difference $w = v_1 - v_2$ satisfies

$$(-\Delta)^s w = \mathcal{V} w, \qquad w(0) = 0, \tag{$*$}$$

where the nonnegative potential $\mathcal{V}$, in the radial variable $r > 0$, is given by

$$\mathcal{V}(r) = \frac{g(v_1(r)) - g(v_2(r))}{v_1(r) - v_2(r)}, \qquad \text{where} \quad g(t) = t_+^p.$$

For given radially decreasing potential $\mathcal{V}$, the results of [Frank, Lenzmann, Silvestre 2016], [Cabré, Sire 2014], based on the proof of a monotonicity formula for $(-\Delta)^s$, shows that the only radial solution to $(*)$ that vanishes at ∞ is the trivial solution.

With analogous arguments, we obtain $w \equiv 0$.
Key point: thanks to the convexity of g, we have $\mathcal{V}'(r) \leq 0$.

Supercritical result

Theorem (Chan, González, Huang, M., Volzone 2020)

Let $s \in (0,1)$ and $p \geq (N+2s)/(N-2s)$. Let $a > 0$ and $b > 0$. There exists a unique bounded radially decreasing solution to

$$\begin{cases} (-\Delta)^s u = a\, u^p & \text{in } \mathbb{R}^N, \\ u(x) \to 0 & \text{as } |x| \to \infty. \end{cases}$$

such that $u(0) = b$.

Remarks

- Existence theorem in [Ao, Chan, González, Wei 2020]
- We must have $C = 0$ (using the Pohozaev identity for $\mathcal{G}$ [Ros-Oton, Serra 2014])
- if $p > (N+2s)/(N-2s)$ the solution is not in $\dot{H}^s(\mathbb{R}^N)$. Bounded non-radial solutions exist in this regime [Chen, Li, Ou 2005]
- if $p = (N+2s)/(N-2s)$ the solution (for $a = 1$) is

$$u(x) = Q \left(\frac{(Q/b)^{\frac{2}{N-2s}}}{(Q/b)^{\frac{4}{N-2s}} + |x|^2} \right)^{\frac{N-2s}{2}},$$

where $Q = Q(s,N)$ is an explicit constant [Chen, Li, Ou 2006]

Summary about radial steady states

Consequences of the above results in terms of the original equation of steady states

$$\rho^{m-1} = \tfrac{m-1}{m}\,(K_s * \rho - C)_+, \qquad C \geq 0$$

- Let $m > m_c$. For any $M > 0$ there is a unique radial steady state of mass M (minimizing $\mathcal{F}$ over $\mathcal{Y}_M$). There is a one-to-one relation between $M > 0$ and $C > 0$.
- Let $m = m_c$. There is a critical mass $\overline{M} > 0$ such that all radial steady states (one for each value of $C > 0$) are dilations of each other and have mass $\overline{M}$.
- Let $m \in (2N/(N+2s), m_c)$ (aggregation dominated). For any $M > 0$ (M, C are one-to-one) there is a unique radial steady state of mass M (but $\inf_{\mathcal{Y}_M} \mathcal{F} = -\infty$).
- Let $1 < m \leq \frac{2N}{N+2s}$ (critical and supercritical regimes). Then there exists a unique radial steady state ρ such that $\rho(0) = 1$. The family of functions $\{\rho_\lambda\}_{\lambda>0}$, where

$$\rho_\lambda(x) := \lambda^{\frac{1}{m-1}}\,\rho\left(\lambda^{\frac{2-m}{2s(m-1)}}\,x\right),$$

  is the set of all radial steady states. In this case $C = 0$.

Results agree with local case (where ODE techniques are available) from [Bian, Liu 2013].

Thanks for the attention