Rates of convergence to non-degenerate asymptotic profiles for fast diffusion equations via an energy method

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Cauchy-Dirichlet problem for the FDE

We shall consider the Cauchy-Dirichlet problem (FDE) $= \{(1)\text{–}(3)\}$ for the Fast Diffusion Equation,

$$
\begin{aligned}
&(1) \qquad \partial_t \left(|u|^{q-2}u\right) = \Delta u \quad \text{in } \Omega \times (0,\infty),\\
&(2) \qquad u = 0 \quad \text{on } \partial\Omega \times (0,\infty),\\
&(3) \qquad u(\cdot,0) = u_0 \quad \text{in } \Omega,
\end{aligned}
$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, under the hypotheses

$$
\textbf{(H)} \qquad u_0 \in H^1_0(\Omega) \setminus \{0\}, \quad 2 < q < 2^* := \frac{2N}{(N-2)_+}.
$$

Physical Background: stability of asymptotic profiles of plasma diffusion (for $q = 3$ in [Okuda-Dawson '73], [Berryman-Holland '80])

Linear diffusion ($q = 2$)

In case $q = 2$, the solution is represented as a Fourier series,

$$u(x,t) = \sum_{n=1}^{\infty} a_n \mathrm{e}^{-\lambda_n t} e_n(x), \quad a_n = (u_0, e_n)_{L^2(\Omega)},$$

where $\{(\lambda_n, e_n)\}_{j=1}^{\infty}$ denote eigenpairs of

$$-\Delta e = \lambda e \ \text{ in } \Omega, \quad e = 0 \ \text{ on } \partial\Omega$$

satisfying $(e_j, e_k)_{L^2(\Omega)} = \delta_{jk}$. Moreover,

$$0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots \le \lambda_n \to +\infty,$$

and hence, as long as $a_1 \neq 0$,

$$u(x,t) \sim a_1 \mathrm{e}^{-\lambda_1 t} e_1(x) \quad \text{for } \ t \gg 1.$$

Finite-time extinction

Proposition 1 (Finite-time extinction with rates)

Let u be the energy solution to (FDE). Then $\forall u_0 \in H_0^1(\Omega) \setminus \{0\}$, $\exists t_* = t_*(u_0) > 0$ and $\exists c_1, c_2 > 0$ such that

(4) $$c_1(t_* - t)_+^{1/(q-2)} \le \|u(\cdot,t)\|_{H_0^1} \le c_2(t_* - t)_+^{1/(q-2)}$$

for all $t \ge 0$. Moreover,

(5) $$\lambda_q \frac{\|u_0\|_{L^q}^q}{\|\nabla u_0\|_{L^2}^2} \le t_*(u_0) \le \lambda_q C_q^2 \|u_0\|_{L^q}^{q-2},$$

where $\lambda_q := \frac{q-1}{q-2} > 0$ and C_q is the best constant of the Sobolev-Poincaré inequality, $\|w\|_{L^q} \le C_q \|\nabla w\|_{L^2}$ for $w \in H_0^1(\Omega)$.

[Berryman-Holland '80] [Kwong '88] [Savaré-Vespri '94]...[A-Kajikiya '13]

Asymptotic profiles of vanishing solutions

Consider the asymptotic profile of $u = u(x,t)$ as follows:

$$\phi(x) := \lim_{t \nearrow t_*} (t_* - t)_+^{-1/(q-2)} u(x,t).$$

To this end, set

$$(6) \qquad v(x,s) := (t_* - t)_+^{-1/(q-2)} u(x,t), \quad s := \log\left(\frac{t_*}{t_* - t}\right).$$

Then v turns out to be an energy solution to (R) $= \{$(7)–(9)$\}$:

$$(7) \qquad \partial_s \left(|v|^{q-2}v\right) = \Delta v + \lambda_q |v|^{q-2} v \quad \text{in } \Omega \times (0,\infty),$$

$$(8) \qquad v = 0 \quad \text{on } \partial\Omega \times (0,\infty),$$

$$(9) \qquad v(\cdot,0) = v_0 \quad \text{in } \Omega,$$

where $v_0 := t_*(u_0)^{-1/(q-2)} u_0 \in H_0^1(\Omega) \setminus \{0\}$ and $\lambda_q := \frac{q-1}{q-2} > 0$.

Rescaled equation (R) as a gradient flow

Then (R) is reduced into the Cauchy problem for

(10) $$\frac{\mathrm{d}}{\mathrm{d}s}\left(|v|^{q-2}v\right)(s) = -J'(v(s)) \ \text{ in } H^{-1}(\Omega), \quad s > 0,$$

where $J' : H_0^1(\Omega) \to H^{-1}(\Omega)$ is the Fréchet derivative of the following energy functional $J : H_0^1(\Omega) \to \mathbb{R}$:

(11) $$J(w) := \frac{1}{2}\|\nabla w\|_{L^2}^2 - \frac{\lambda_q}{q}\|w\|_{L^q}^q \quad \text{for } \ w \in H_0^1(\Omega).$$

Then $J(v(s))$ decreases in time and $v(s)$ converges to a critical point $\phi \in H_0^1(\Omega)$ of $J(\cdot)$, that is,

(12) $$J'(\phi) = 0 \ \text{ in } H^{-1}(\Omega).$$

Asymptotic profiles for vanishing solutions

Theorem 2 (Asymptotic profiles for vanishing solutions)

For every $s_n \to \infty$, there exist a subsequence (n') of (n) and a function $\phi \in H_0^1(\Omega) \setminus \{0\}$ such that

$$v(s_{n'}) \to \phi \quad \text{strongly in } H_0^1(\Omega).$$

Moreover, ϕ solves the following Dirichlet problem (D):

$$-\Delta\phi = \lambda_q |\phi|^{q-2}\phi \ \text{ in } \Omega, \quad \phi = 0 \ \text{ on } \partial\Omega.$$

[Berryman-Holland '80] [Kwong '88] [Savaré-Vespri '94]...[A-Kajikiya '13]

Convergence (along the whole sequence) follows for **isolated** asymptotic profiles (e.g., 1D case, ball domains, "convex domains" for $q \sim 2, 2^*$) and for **positive** asymptotic profiles (by Łojasiewicz-Simon's inequality).

Convergence of non-negative solutions for (R)

As for non-negative solutions $v \geq 0$ to (R), we can further use

- [DiBenedetto-Kwong-Vespri '91] $\forall \varepsilon > 0,\ \exists c, C > 0$;

(13) $$c\, d(x) \leq \frac{v(x,s)}{\phi(x)} \leq C d(x) \quad \text{for } x \in \Omega,\ s \geq \varepsilon,$$

where $d(x) := \mathrm{dist}(x, \partial\Omega)$. $\forall \varepsilon > 0,\ \forall k \in \mathbb{N},\ \exists C_k > 0$;

$$|D^\alpha v(x,s)^{q-1}| \leq C_k d(x)^{q-1-k} \quad \text{for } x \in \Omega,\ s \geq \varepsilon,\ |\alpha| = k.$$

- [Feireisl-Simondon '00] Uniform convergence

(14) $$v(\cdot, s) \to \phi \quad \text{uniformly in } \overline{\Omega}.$$

- [Bonforte-Grillo-Vázquez '12] Relative error convergence

(15) $$\lim_{s\to\infty} \left\| \frac{v(s)}{\phi} - 1 \right\|_{L^\infty} = 0.$$

Rates of convergence to non-degenerate profiles

The aim of this talk is to discuss rate of convergence of $v(s) \to \phi$ as $s \to \infty$ in view of linearized analysis.

To this end, we always assume that

- $\phi = \phi$ is a non-degenerate solution to (D), that is,

$$\mathcal{L}_\phi e := -\Delta e - \lambda_q (q-1)|\phi|^{q-2} e = 0 \text{ in } \Omega, \quad e = 0 \text{ on } \partial\Omega$$

  admits no non-trivial solution. That is,

  - $\mathcal{L}_\phi$ does not have zero eigenvalue ($0 \notin \sigma_{pt}(\mathcal{L}_\phi)$),
  - $\mathcal{L}_\phi$ is invertible.

- If $v = v(x,s)$ is non-negative (hence $\phi > 0$), then

$$\lim_{s\to\infty} \left\| \frac{v(s)}{\phi} - 1 \right\|_{L^\infty} = 0.$$

Analysis of linearized problems (1/3)

Suppose that $v \geq 0$ (and hence, $\phi > 0$). Based on [Bonforte-Figalli '21], set $v = \phi + h$ and formally expand $v^{q-1} \risingdotseq \phi^{q-1} + (q-1)\phi^{q-2}h$. Then

$$\begin{aligned} (q-1)\phi^{q-2}\partial_s h &\risingdotseq \Delta h + \lambda_q (q-1)\phi^{q-2} h && \text{in } \Omega \times (0,\infty), \\ h &= 0 && \text{on } \partial\Omega \times (0,\infty), \\ h(\cdot, 0) &= h_0 := v_0 - \phi && \text{in } \Omega. \end{aligned}$$

Multiply both sides by h and integrate it over Ω to get

$$\frac{q-1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\underbrace{\left(\int_\Omega h^2 \phi^{q-2}\,\mathrm{d}x\right)}_{=:\,\mathsf{E}[h]} + \underbrace{\int_\Omega |\nabla h|^2\,\mathrm{d}x - \lambda_q (q-1)\int_\Omega h^2\phi^{q-2}\,\mathrm{d}x}_{=:\,\mathsf{I}[h]} \risingdotseq 0.$$

Analysis of linearized problems (2/3)

Improved Poincaré Inequality (IPI)

$$(16)\qquad \underbrace{\mu_k \int_\Omega h^2 \phi^{q-2}\,\mathrm{d}x}_{=\mathsf{E}[h]} \le \int_\Omega |\nabla h|^2\,\mathrm{d}x \quad \text{if } h \perp \mathrm{span}\{\psi_j\}_{j=1}^{k-1},$$

where (μ_j, ψ_j) denote eigenpairs of the eigenvalue problem,

$$(17)\qquad -\Delta\psi = \mu\phi^{q-2}\psi \text{ in } \Omega, \quad \psi = 0 \text{ on } \partial\Omega$$

and $0 < \mu_1 < \mu_2 \le \cdots \le \mu_j \to +\infty$ and (ψ_j) forms a CONS of $L^2(\Omega\,;\phi^{q-2}\mathrm{d}x)$ (normalized as $(\psi_i, \psi_j)_{L^2(\Omega\,;\phi^{q-2}\mathrm{d}x)} = \delta_{ij}$).

Let μ_k be the smallest eigenvalue such that $\mu_k > \lambda_q(q-1)$.
Then if $h(s) \perp \{\psi_j\}_{j=1}^{k-1}$ for $s \gg 1$, Improved Poincaré Inequality holds,

$$(\text{IPI})\qquad [\mu_k - \lambda_q(q-1)]\mathsf{E}[h(s)] \le \mathsf{I}[h(s)] \quad \text{for } s \gg 1.$$

Analysis of linearized problems (3/3)

Thus

$$\frac{q-1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\mathsf{E}[h(s)] + [\mu_k - \lambda_q(q-1)]\mathsf{E}[h(s)] \text{ "}\leq\text{" } 0,$$

which implies

Optimal decay rate for the linearized problem

$$\mathsf{E}[h(s)] \text{ "}\leq\text{" } \mathsf{E}[h_0]\mathrm{e}^{-\lambda_0 s}, \quad \lambda_0 := \frac{2}{q-1}[\mu_k - \lambda_q(q-1)] > 0.$$

Here we recall that

$$\mathsf{E}[h(s)] = \int_\Omega h(\cdot, s)^2 \phi^{q-2}\,\mathrm{d}x = \int_\Omega |v(\cdot, s) - \phi|^2 \phi^{q-2}\,\mathrm{d}x.$$

[Bonforte-Figalli '21] introduced "Nonlinear Entropy Method" to justify the analysis of linearization for (R).

Nonlinear entropy method [Bonforte-Figalli '21]

Step 1. Derivation of entropy inequality: Test (R) by $h = v - \phi$.

$$\frac{1}{q'}\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}[v(s)|\phi] + \mathsf{I}[h(s)] = \mathsf{R}[h(s)],$$

$$\mathcal{E}[v|\phi] := \int_\Omega \left[v^q - \phi^q - q'\left(v^{q-1} - \phi^{q-1}\right)\phi\right]\mathrm{d}x \asymp \mathsf{E}[h(s)],$$

$$|\mathsf{R}[h]| \lesssim \left\|\frac{v}{\phi} - 1\right\|_\infty \underbrace{\int_\Omega |h|^2\phi^{q-2}\,\mathrm{d}x}_{=\mathsf{E}[h]}.$$

Step 2. Improved Poincaré Inequality for "almost orthogonality":

$$\mathsf{Q}_j[h(s)] := \frac{|\int_\Omega h(s)\psi_j\phi^{q-2}\,\mathrm{d}x|}{\mathsf{E}[h]^{1/2}} < \varepsilon \quad (\forall j \le k-1) \;\Rightarrow\; \textbf{(IPI)}_\varepsilon$$

Nonlinear entropy method [Bonforte-Figalli '21]

Step 3. Nonlinear flows improve "almost orthogonality": claims that

$$\forall \varepsilon > 0,\ \exists s_\varepsilon > 0\,;\ \sup_{s \ge s_\varepsilon} \mathbf{Q}_j[h(s)] < \varepsilon \quad \text{for} \quad j = 1, 2, \ldots, k-1.$$

Hence $(\mathbf{IPI})_\varepsilon$ yields

$$\frac{1}{q'}\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}[v|\phi] + \underbrace{\Big(\mu_k - \lambda_q(q-1) - C\varepsilon^2 - C\delta\Big)C_1}_{\fallingdotseq [\mu_k - \lambda_q(q-1)](2/q) > 0 \ \text{for}\ \delta,\varepsilon \ll 1,\ s \gg 1} \mathcal{E}[v|\phi] \le 0.$$

Step 4. Sharp rate of convergence: Remove ε and δ to get

Theorem 3 (Sharp rate for the relative entropy [BF '21])

Assume $v \geq 0$. There exists $\kappa_0 > 0$ such that

$$\int_\Omega |v(\cdot, s) - \phi|^2 \phi^{q-2}\,\mathrm{d}x \le \kappa_0 \mathrm{e}^{-\lambda_0 s} \quad \text{for} \quad s \ge 0.$$

Rates of convergence via energy methods

In this talk, we shall reveal rates of convergence based on energy methods.

Theorem 4 (Rates of convergence for the energy [A])

For any constant $\lambda > 0$ satisfying

$$0 < \lambda < \frac{2}{q-1} C_q^{-2} \|\phi\|_{L^q(\Omega)}^{-(q-2)} \min_j \left| \frac{\mu_j - \lambda_q(q-1)}{\mu_j} \right|,$$

where C_q is the best constant of the Sobolev-Poincaré inequality, there exists a constant $C > 0$ depending on the choice of λ such that

$$0 \le J(v(s)) - J(\phi) \le C\mathrm{e}^{-\lambda s} \quad \text{for} \quad s \ge 0.$$

Furthermore, $v(s)$ strongly converges to ϕ in $H_0^1(\Omega)$ at an exponential rate as $s \to +\infty$.

Ingredients of proof

- **Energy identity:** Test (R) by $\partial_s v(s)$ to get

$$c_q \left\| \partial_s \left(|v|^{(q-2)/2} v \right)(s) \right\|_{L^2}^2 + \frac{\mathrm{d}}{\mathrm{d}s} J(v(s)) \le 0$$

with $c_q = 4/(qq')$.

- **Gradient inequality:** For any constant

$$\omega > \|\mathcal{L}_\phi^{-1}\|_{\mathscr{L}(H^{-1}(\Omega), H_0^1(\Omega))}^{1/2} / \sqrt{2},$$

there exists a constant $\delta > 0$ such that

$$|J(w) - J(\phi)|^{1/2} \le \omega \|J'(w)\|_{H^{-1}(\Omega)} \quad \text{for} \quad w \in H_0^1(\Omega),$$

provided that $\|w - \phi\|_{H_0^1(\Omega)} < \delta$.

- **Quantitative estimate for $\|\mathcal{L}_\phi^{-1}\|$ in terms of eigenvalues (μ_j)**

Sharp rate of convergence via energy methods

As for non-negative solutions $v = v(x, s) \geq 0$, we obtain

Theorem 5 (Sharp rate of convergence for the energy [A])

Assume $v \geq 0$. Then there exists $\kappa_1 > 0$ such that

$$0 \leq J(v(s)) - J(\phi) \leq \kappa_1 e^{-\lambda_0 s} \quad \text{for} \quad s \geq 0. \tag{18}$$

Here λ_0 is the decay rate of solutions for the linearized problem.

Theorem 3 follows as a corollary, and moreover, we have

Corollary 6 (Sharp rate of convergence in $H_0^1(\Omega)$ [A])

Assume $v \geq 0$. There exists $\kappa_2 > 0$ such that

$$\int_\Omega |\nabla v(x, s) - \nabla \phi(x)|^2 \, dx \leq \kappa_2 e^{-\lambda_0 s} \quad \text{for} \quad s \geq 0. \tag{19}$$

Outline of proof (1/3)

Step 1. "Refined" gradient inequality:

Lemma 7 ("Refined" gradient inequality)

$$0 \le J(v(s)) - J(\phi) \le \frac{1}{2\nu_k} \|J'(v(s))\|^2_{L^2(\Omega;\phi^{2-q}\mathrm{d}x)} + o\left(\|v(s) - \phi\|^2_{H^1_0}\right).$$

Step 2. Energy inequality: Note that

$$\int_\Omega \left|\partial_s(v^{q-1})(s)\right|^2 \phi^{2-q}\,\mathrm{d}x = \frac{4(q-1)^2}{q^2} \int_\Omega \left|\partial_s(v^{\frac{q}{2}})(s)\right|^2 \left(\frac{v(s)}{\phi}\right)^{q-2} \mathrm{d}x.$$

Then for any $\lambda < \lambda_0$, one can take $s_\lambda > 0$ such that

$$0 \le J(v(s)) - J(\phi) \le -\frac{1}{\lambda}\frac{\mathrm{d}}{\mathrm{d}s} J(v(s)) \quad \text{for } \ s > s_\lambda.$$

Thus we shall obtain the "almost sharp" rate of convergence for $J(v(s))$.

Outline of proof (2/3)

Step 3. Exponential convergence of Sobolev norm:

Lemma 8 (Exponential convergence in $H_0^1(\Omega)$)

Assume that $J(v(s)) - J(\phi) \lesssim \mathrm{e}^{-\lambda s}$ for some $\lambda > 0$. Then

$$\mathbb{E}[h(s)] = \|v(s) - \phi\|^2_{L^2(\Omega;\phi^{q-2}\mathrm{d}x)} \lesssim \mathrm{e}^{-\lambda s},$$
$$\|v(s) - \phi\|^2_{H_0^1} \lesssim \mathrm{e}^{-\lambda s}.$$

Step 4. "Sharp" rate of convergence: We have obtained

$$H(s) := J(v(s)) - J(\phi)$$
$$\leq -\left(\frac{q-1}{2\nu_k} + \varepsilon(s)\right)(1 + \delta(s))^{q-2} \frac{\mathrm{d}}{\mathrm{d}s} J(v(s)).$$

Outline of proof (3/3)

By Lemma 8, (assuming $q \geq 3$ for simplicity) we observe

$$\varepsilon(s) := \frac{o\left(\|v(s) - \phi\|_{H_0^1}^2\right)}{\|v(s) - \phi\|_{H_0^1}^2} \lesssim \|v(s) - \phi\|_{H_0^1} \lesssim \mathrm{e}^{-\frac{\lambda}{2}s}.$$

Moreover, thanks to Lemma 8 with [Theorem 4.1, BF '21], we can prove

$$\delta(s) := \left\|\frac{v(s)}{\phi} - 1\right\|_{L^\infty} \lesssim \mathrm{e}^{-bs} \quad \text{for} \quad s \gg 1$$

for some $b > 0$. Thus

$$\frac{\mathrm{d}H}{\mathrm{d}s}(s) + \lambda_0 H(s) \leq C\mathrm{e}^{-cs} H(s) \quad \text{for} \quad s > s_*$$

for some $c, C, s_* > 0$. Then it follows that

$$H(s) \leq H(s_*)\mathrm{e}^{C/c}\mathrm{e}^{-\lambda_0(s - s_*)} \quad \text{for} \quad s \geq s_*.$$

Remarks for nonnegative solutions

As for the results obtained for $v \geq 0$, we remark that:

- These results seem slightly stronger than Theorem 3 for relative entropy; on the other hand, with aid of the recent regularity result by [Jin-Xiong, to appear], they may also be derived from Theorem 3.
- However, the proof of [A] seems simpler than that of [BF '21]; in particular, we can avoid "Step 3", which may be the most involved part of the proof.

Thank you for your attention !

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Sketch of proof

Let us recall that

Ansatz

- $v = v(x,s) \geq 0$: non-negative solution to (R)
- $\phi = \phi(x) > 0$: non-degenerate positive solution to (D)
- $\delta(s) := \left\| \frac{v(s)}{\phi} - 1 \right\|_{L^\infty} \to 0$ as $s \to +\infty$

Sketch of proof

Weighted eigenvalue problem

$$-\Delta e_j = \mu_j \phi^{q-2} e_j \text{ in } \Omega, \quad \phi = 0 \text{ on } \partial\Omega.$$

- $0 < \mu_1 < \mu_2 \leq \cdots \leq \mu_n \to +\infty$,
- (e_j) forms a CONS of $H^1_0(\Omega)$ such that $(e_i, e_j)_{H^1_0} = \delta_{ij}$,
- $\mu_1 = \lambda_q$, $e_1 = \phi / \|\phi\|_{H^1_0}$,
- $(-\Delta e_j)$ forms a CONS of $H^{-1}(\Omega)$.

Then the linearized operator $\mathcal{L}_\phi = -\Delta - \lambda_q(q-1)\phi^{q-2}$ fulfills

- $\mathcal{L}_\phi e_j = \nu_j \phi^{q-2} e_j$ in Ω, $\quad e_j = 0$ on $\partial\Omega$,
- $\nu_j = \mu_j - \lambda_q(q-1)$ (in particular, $\nu_1 = 1 - q < 0$),
- Let $k \in \mathbb{N}$; $\mu_{k-1} < \lambda_q(q-1) < \mu_k$ (i.e., $\nu_{k-1} < 0 < \nu_k$).

Step 1. "Refined" gradient inequality

Lemma 9 ("Refined" gradient inequality)

$$0 \leq J(v(s)) - J(\phi) \leq \frac{1}{2\nu_k} \|J'(v(s))\|^2_{L^2(\Omega;\phi^{2-q}\mathrm{d}x)} + o\left(\|v(s) - \phi\|^2_{H^1_0}\right).$$

By Taylor's theorem, we have

$$J(v(s)) - J(\phi) = \frac{1}{2} \langle \mathcal{L}_\phi(v(s) - \phi), v(s) - \phi \rangle_{H^1_0} + o\left(\|v(s) - \phi\|^2_{H^1_0}\right),$$

$$J'(v(s)) = \mathcal{L}_\phi(v(s) - \phi) + o\left(\|v(s) - \phi\|_{H^1_0}\right).$$

Step 1. “Refined” gradient inequality

Hence

$$
\begin{aligned}
&J(v(s)) - J(\phi) \\
&= \frac{1}{2} \left\langle J'(v(s)), \mathcal{L}_{\phi}^{-1} \left(J'(v(s))\right) \right\rangle_{H_0^1} + o\left(\|v(s) - \phi\|_{H_0^1}^2 \right).
\end{aligned}
$$

We substitute

$$
J'(v(s)) = \sum_{j=1}^{\infty} \sigma_j(s)(-\Delta e_j).
$$

Then we find that

$$
\begin{aligned}
&J(v(s)) - J(\phi) \\
&= \frac{1}{2} \sum_{j=1}^{\infty} \frac{\mu_j}{\nu_j} \sigma_j(s)^2 + o\left(\|v(s) - \phi\|_{H_0^1}^2 \right).
\end{aligned}
$$

Step 1. "Refined" gradient inequality

Moreover,

$$
\begin{aligned}
&J(v(s)) - J(\phi) - \frac{1}{2}\sum_{j=1}^{k-1}\frac{\mu_j}{\nu_j}\sigma_j(s)^2 \\
&= \frac{1}{2}\sum_{j=k}^{\infty}\frac{\mu_j}{\nu_j}\sigma_j(s)^2 + o\left(\|v(s)-\phi\|_{H_0^1}^2\right) \\
&\leq \frac{1}{2\nu_k}\sum_{j=k}^{\infty}\mu_j\sigma_j(s)^2 + o\left(\|v(s)-\phi\|_{H_0^1}^2\right) \\
&\leq \frac{1}{2\nu_k}\|J'(v(s))\|_{L^2(\Omega;\phi^{2-q}\mathrm{d}x)}^2 + o\left(\|v(s)-\phi\|_{H_0^1}^2\right) \\
&\leq \left(\frac{1}{2\nu_k} + \varepsilon(s)\right)\|J'(v(s))\|_{L^2(\Omega;\phi^{2-q}\mathrm{d}x)}^2 .
\end{aligned}
$$

Thus we have proved Lemma 9. □

Step 2. “Almost sharp” rate of convergence

Lemma 10 (“Almost sharp” rate of convergence)

For any $\lambda < \lambda_0 = \frac{2\nu_k}{q-1}$, there exist $s_\lambda, \kappa_\lambda > 0$ such that

$$0 \leq J(v(s)) - J(\phi) \leq \kappa_\lambda \mathrm{e}^{-\lambda(s-s_\lambda)} \quad \text{for} \quad s \geq s_\lambda.$$

Noting that

$$\partial_s(v^{q-1})(s) = \frac{2(q-1)}{q} |v(s)|^{\frac{q-2}{2}} \partial_s(v^{\frac{q}{2}})(s),$$

we find that

$$\|J'(v(s))\|^2_{L^2(\Omega;\phi^{2-q}\mathrm{d}x)} \overset{(\mathrm{R})}{=} \int_\Omega \left|\partial_s(v^{q-1})(s)\right|^2 \phi^{2-q}\,\mathrm{d}x$$
$$= \frac{4(q-1)^2}{q^2} \int_\Omega \left|\partial_s(v^{\frac{q}{2}})(s)\right|^2 \left(\frac{v(s)}{\phi}\right)^{q-2} \mathrm{d}x.$$

Step 2. "Almost sharp" rate of convergence

Combine this with the last lemma to see that

$$
\begin{aligned}
&J(v(s)) - J(\phi) \\
&\le \left(\frac{1}{2\nu_k} + \varepsilon(s)\right)\frac{4(q-1)^2}{q^2}(1+\delta(s))^{q-2}\left\|\partial_s(v^{\frac{q}{2}})(s)\right\|_{L^2}^2 \\
&\le -\left(\frac{1}{2\nu_k} + \varepsilon(s)\right)\frac{4(q-1)^2}{q^2}(1+\delta(s))^{q-2} c_q^{-1}\frac{\mathrm{d}}{\mathrm{d}s}J(v(s)).
\end{aligned}
$$

Thus for any $\lambda < \lambda_0$, one can take $s_\lambda > 0$ such that

$$
J(v(s)) - J(\phi) \le -\frac{1}{\lambda}\frac{\mathrm{d}}{\mathrm{d}s}J(v(s)) \quad \text{for} \quad s > s_\lambda,
$$

which implies

$$
J(v(s)) - J(\phi) \le [J(v(s_\lambda)) - J(\phi)]\,\mathrm{e}^{-\lambda(s-s_\lambda)} \quad \text{for} \quad s > s_\lambda. \quad \square
$$

Step 3. Convergence of Sobolev norm with rate

Lemma 11 (Convergence in $H_0^1(\Omega)$ with rates)

Assume that $J(v(s)) - J(\phi) \lesssim \mathrm{e}^{-\lambda s}$ for some $\lambda > 0$. Then

$$\mathbf{E}[h(s)] = \|v(s) - \phi\|^2_{L^2(\Omega;\phi^{q-2}\mathrm{d}x)} \lesssim \mathrm{e}^{-\lambda s},$$
$$\|v(s) - \phi\|^2_{H_0^1} \lesssim \mathrm{e}^{-\lambda s}.$$

As a by-product of the argument so far, we obtain

$$\|\partial_s(v^{q-1})(s)\|_{L^2(\Omega;\phi^{2-q}\mathrm{d}x)} \le -C\frac{\mathrm{d}}{\mathrm{d}s}\left[J(v(s)) - J(\phi)\right]^{1/2},$$

whence follows from Lemma 10 that

$$\left\|\phi^{q-1} - v^{q-1}(s)\right\|_{L^2(\Omega;\phi^{2-q}\mathrm{d}x)} \le \int_s^\infty \left\|\partial_s\left(v^{q-1}\right)(\sigma)\right\|_{L^2(\Omega;\phi^{2-q}\mathrm{d}x)} \mathrm{d}\sigma$$
$$\le C\left[J(v(s)) - J(\phi)\right]^{1/2} \lesssim e^{-\frac{\lambda}{2}s}.$$

Step 3. Convergence of Sobolev norm with rate

On the other hand, we observe that

$$\int_\Omega |v(\cdot,s)-\phi|^2\phi^{q-2}\,\mathrm{d}x$$
$$\le \int_\Omega \left|v(\cdot,s)^{q-1}-\phi^{q-1}\right|^2\phi^{2-q}\,\mathrm{d}x \lesssim \mathrm{e}^{-\lambda s}.$$

Furthermore, a simple calculation yields

$$J(v(s))-J(\phi)$$
$$=\frac{1}{2}\|\nabla(v(s)-\phi)\|^2_{L^2(\Omega)}-\frac{\lambda_q}{2}(q-1)\int_\Omega |v-\phi|^2\phi^{q-2}\,\mathrm{d}x$$
$$+o\left(\|v(s)-\phi\|^2_{H^1_0(\Omega)}\right).$$

Thus the desired conclusion follows from Lemma 10 and the above. □

Step 4. "Sharp" rate of convergence

Now, we are ready to prove Theorem 5. For simplicity, assume $q \geq 3$ and then recall that

$$\begin{aligned} H(s) &:= J(v(s)) - J(\phi) \\ &\leq -\left(\frac{q-1}{2\nu_k} + \varepsilon(s)\right)(1+\delta(s))^{q-2}\frac{\mathrm{d}}{\mathrm{d}s}J(v(s)). \end{aligned}$$

By Lemma 11, we observe

$$\varepsilon(s) = \frac{o\left(\|v(s)-\phi\|_{H_0^1}^2\right)}{\|v(s)-\phi\|_{H_0^1}^2} \lesssim \|v(s)-\phi\|_{H_0^1} \lesssim \mathrm{e}^{-\frac{\lambda}{2}s}.$$

Moreover, thanks to Lemma 11 with [Theorem 4.1, BF '21], we can prove

$$\delta(s) = \left\|\frac{v(s)}{\phi} - 1\right\|_{L^\infty} \lesssim \mathrm{e}^{-bs} \quad \text{for} \quad s \gg 1$$

for some $b > 0$.

Step 4. "Sharp" rate of convergence

Lemma 12 ([Theorem 4.1, Bonforte-Figalli '21])

There exist positive constants C, L, s_* such that

$$\left\| \frac{v(s)}{\phi} - 1 \right\|_{L^\infty} \leq C \frac{e^{L(s-s_0)}}{s - s_0} \sup_{\sigma \in [s_0, s]} \left(\int_\Omega |v(\sigma) - \phi|^2 \phi^{q-2} \, dx \right)^{\frac{1}{4N}} + C(s - s_0) e^{L(s-s_0)} \quad \text{for } s > s_0 \geq s_*.$$

Proof. Let $s > 0$ and set $s_0 = s - e^{-as}$, where a is a positive number to be determined later. Then

$$\left\| \frac{v(s)}{\phi} - 1 \right\|_{L^\infty(\Omega)} \leq C \frac{e^{Le^{-as}}}{e^{-as}} \sup_{\sigma \in [s - e^{-as}, s]} \left(\int_\Omega |v(\sigma) - \phi|^2 \phi^{q-2} \, dx \right)^{\frac{1}{4N}} + C e^{-as} e^{Le^{-as}}.$$

Step 4. "Sharp" rate of convergence

Thus Lemma 11 yields

$$\delta(s) = \left\| \frac{v(s)}{\phi} - 1 \right\|_{L^\infty(\Omega)} \le C e^{L} e^{as} e^{-\frac{\lambda}{4N}(s-1)} + C e^{-as} e^{L}.$$

Hence it suffices to choose $0 < a < \lambda/(4N)$. □

Step 4. "Sharp" rate of convergence

Therefore we have

$$\frac{\mathrm{d}H}{\mathrm{d}s}(s) + \lambda_0 H(s) \le C\mathrm{e}^{-cs} H(s) \quad \text{for} \quad s > s_*$$

for some $s_* > 0$. Then there exists $C > 0$ such that

$$H(s) \le C H(s_*) \mathrm{e}^{-\lambda_0 (s - s_0)} \quad \text{for} \quad s \ge s_*.$$

Consequently, we obtain

(20) $$0 \le J(v(s)) - J(\phi) \le \kappa_1 \mathrm{e}^{-\lambda_0 s} \quad \text{for} \quad s \ge 0$$

for some $\kappa_1 > 0$. This completes the proof of Theorem 5. □

<u>Proof of Corollaries</u>. Combine (20) with Lemma 11 (see Step 3). □