Convergence in relative error for the fast diffusion equation

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Our motivation

Stability in Gagliardo-Nirenberg-Sobolev inequalities
Flows, regularity and the entropy method
by M. Bonforte, J. Dolbeault, B. Nazaret, and N. S.
Preprint https://arxiv.org/abs/2007.03674

Stability of the following family of functional inequalities

$$\mathcal{C}_{\mathrm{GNS}}(p)\,\|f\|_{\mathrm{L}^{2p}(\mathbb{R}^d)} \le \|\nabla f\|^{\theta}_{\mathrm{L}^2(\mathbb{R}^d)}\,\|f\|^{1-\theta}_{\mathrm{L}^{p+1}(\mathbb{R}^d)} \qquad \text{for any} \quad f \in C^\infty_c(\mathbb{R}^d)\,,$$

where $d \ge 3, \quad p \in \left(1, \frac{d}{d-2}\right]$.

- ▷ Our main result is the stability

$$\|\nabla f\|^{\vartheta}_{L^2(\mathbb{R}^d)}\,\|f\|^{1-\vartheta}_{L^{p+1}(\mathbb{R}^d)} - \|f\|_{L^{2p}(\mathbb{R}^d)}\,\mathcal{C}_{\mathrm{GNS}}(p) \ge \mathcal{C}\,R[f]$$

where $R[f]$ is a distance to the manifold of optimal functions.

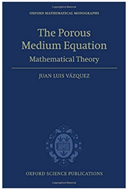
- ▷ Strategy: based on the entropy method for the fast diffusion flow
- ▷ Advantage: the proof is constructive !

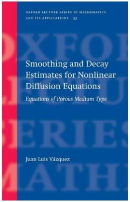
Porous Medium and Fast Diffusion Equations

$$u_t = \Delta u^m = \nabla.(m\, u^{m-1} \nabla u) \quad \text{where} \quad m > 0$$

- $m > 1$, slow diffusion, Porous Medium Equation
- $m = 1$, Heat Equation
- $0 < m < 1$, Fast Diffusion Equation

Main references: two monographs of **J. L. Vázquez**





Fast Diffusion Equation

Nonnegative, integrable solutions to the Cauchy problem

$$\text{(CP)} \qquad \begin{cases} \partial_t u = \Delta u^m & \text{in } (0,\infty)\times\mathbb{R}^d, \\ u(0,x) = u_0(x) & \text{in } \mathbb{R}^d, \end{cases}$$

Parameters and main features:

- m in the range $$m_1 := \frac{d-1}{d} \le m < 1\,, \quad \text{with} \quad d \ge 3\,.$$
- Initial data in $$u_0 \in \mathrm{L}^1_+(\mathbb{R}^d) = \{u_0 : \mathbb{R}^d \to \mathbb{R} : u_0 \ge 0\,, \int_{\mathbb{R}^d} u_0 \,\mathrm{d}x < \infty\}\,.$$
- Existence and uniqueness in $\mathrm{L}^1_{\mathrm{loc}}$ are settled, solutions are C^∞, see **Herrero-Pierre** '85.
- **Mass** is conserved, namely for all $t > 0$, $$\int_{\mathbb{R}^d} u(t,x)\,\mathrm{d}x = \int_{\mathbb{R}^d} u_0(x)\,\mathrm{d}x$$

Fast Diffusion Equation

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$$

- ▷ (CP) admits the self-similar solution (called **Barenblatt**)

$$
\mathcal{B}_M(t,x) = \frac{t^{\frac{1}{1-m}}}{\left[b_0 \frac{t^{2\vartheta}}{M^{2\vartheta(1-m)}} + b_1 |x|^2\right]^{\frac{1}{1-m}}} = t^{-d\vartheta}\,\mathsf{B}_M(x\,t^{-\vartheta}),
$$

where $\vartheta^{-1} = 2 - d(1-m) > 0$, and

$$
\mathsf{B}_M(x) = \left[\frac{b_0}{M^{2\vartheta(1-m)}} + b_1 |x|^2\right]^{\frac{1}{m-1}}
$$

- ▷ Asymptotic behaviour as $t \to \infty$

$$
\|u(t) - \mathcal{B}_M(t)\|_{\mathrm{L}^1(\mathbb{R}^d)} \to 0 \qquad \text{and} \qquad t^{d\vartheta}\,\|u(t) - \mathcal{B}_M(t)\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \to 0
$$

Refined asymptotic behaviour: questions

We consider the *uniform relative error*, for any $t > 0$

$$\left\| \frac{u(t)}{\mathcal{B}_M(t)} - 1 \right\|_{\mathrm{L}^\infty(\mathbb{R}^d)}$$

Q_1) For which initial datum $u_0 \in \mathrm{L}^1_+(\mathbb{R}^d)$ the solution $u(t,x)$ to (CP) converges to $\mathcal{B}_M(t,x)$ in *uniform relative error*, i.e.

$$\lim_{t\to\infty} \left\| \frac{u(t)}{\mathcal{B}_M(t)} - 1 \right\|_{\mathrm{L}^\infty(\mathbb{R}^d)} = 0$$

Q_2) When Q_1) has a positive answer, fix $\varepsilon > 0$ can we give a *constructive estimate* of $T(\varepsilon)$ for which

$$\left\| \frac{u(t)}{\mathcal{B}_M(t)} - 1 \right\|_{\mathrm{L}^\infty(\mathbb{R}^d)} < \varepsilon \quad \forall t \geq T(\varepsilon)$$

Answer to Q1: it is a matter of tails!

The **relative error**

$$\left|\frac{u(t,x)-\mathcal{B}_M(t,x)}{\mathcal{B}_M(t,x)}\right|$$

is not always uniformly bounded in $\mathbb{R}^d$

- ▷ Fast Diffusion : any solution develop a **fat tail** $u(t,x) \gtrsim |x|^{-\frac{2}{1-m}}$ for $|x|$ large
- ▷ However, let the initial datum be

$$u_0(x) = \frac{1}{(1+|x|^2)^{\frac{m}{1-m}}} > \mathcal{B}_M(t,x)\,,$$

then the solution $u(t,x)$ to (CP) with initial data u_0 satisfies

$$\mathcal{B}_M(t,x) < \frac{1}{\left[(c\,t+1)^{\frac{1}{1-m}} + |x|^2\right]^{\frac{m}{1-m}}} \le u(t,x) \le \frac{(1+t)^{\frac{m}{1-m}}}{(1+t+|x|^2)^{\frac{m}{1-m}}}\,,$$

Recall that $\mathcal{B}_M(t,x) \sim |x|^{-\frac{2}{1-m}}$

Answer to Q1: the path to the Global Harnack Principle

We can reformulate the problem as an inequality for $x \in \mathbb{R}^d$ and t large of the form

$$\mathcal{B}_{M_1}(t-\tau_1, x) \le u(t,x) \le \mathcal{B}_{M_2}(t+\tau_2, x) \qquad \textbf{(GHP)}$$

The **GHP** holds if $u_0 \lesssim |x|^{-\frac{2}{1-m}}$, **Vázquez 2003/ Bonforte - Vázquez 2006**

Let us define

$$|f|_{\mathcal{X}} := \sup_{R>0} R^{\frac{2}{1-m}-d} \int_{\mathbb{R}^d \setminus B_R(0)} |f(x)| \, \mathrm{d}x < \infty \,,$$

and the space

$$\mathcal{X} := \{ f \in \mathrm{L}^1_+(\mathbb{R}^d) : |f|_{\mathcal{X}} < +\infty \}.$$

Theorem [M. Bonforte, N.S. - 2020]

Under the running assumptions, **GHP** holds, i.e.,

$$\mathcal{B}_{M_1}(t-\tau_1, x) \le u(t,x) \le \mathcal{B}_{M_2}(t+\tau_2, x) \,,$$

if and only if the initial data $u_0 \in \mathcal{X} \setminus \{0\}$.

Our contribution: we found the maximal set of initial data for which **GHP** holds!
However: see **Vázquez 2003** where a similar condition is introduced.

Answer to Q1: convergence in uniform relative error

$$
\text{(CP)} \quad \begin{cases} \partial_t u = \Delta u^m & \text{in } (0,\infty)\times\mathbb{R}^d, \\ u(0,x) = u_0(x) & \text{in } \mathbb{R}^d, \end{cases} \qquad \frac{d-1}{d} \le m < 1\,, \quad \text{with} \quad d \ge 3\,.
$$

Theorem-1 [M. Bonforte, N.S. - 2020]

Under the running assumption, a solution $u(t,x)$ to (CP) converges to $\mathcal{B}_M(t,x)$ in *uniform relative error*, i.e.

$$
\lim_{t\to\infty} \left\| \frac{u(t)-\mathcal{B}_M(t)}{\mathcal{B}_M(t)} \right\|_{\mathrm{L}^\infty(\mathbb{R}^d)} = 0
$$

if and only if

$$
u_0 \in \mathcal{X} \setminus \{0\} \quad \text{and} \quad M = \|u_0\|_{\mathrm{L}^1(\mathbb{R}^d)}
$$

where

$$
\mathcal{B}_M(t,x) = \frac{t^{\frac{1}{1-m}}}{\left[b_0 \frac{t^{2\vartheta}}{M^{2\vartheta(1-m)}} + b_1|x|^2\right]^{\frac{1}{1-m}}} \text{ and } \qquad |f|_{\mathcal{X}} := \sup_{R>0} R^{\frac{2}{1-m}-d} \int_{\mathbb{R}^d\setminus B_R(0)} |f(x)|\,\mathrm{d}x
$$

Answer to Q2: computation of the asymptotic time layer

Working (a lot) more, we prove that

Theorem-2 [M. Bonforte, J. Dolbeault, B. Nazaret, N.S. - 2021]

Under the running assumption, let $\varepsilon > 0$ and u_0 be a *"centred"* initial datum such that

$$\|u_0\|_{\mathcal{X}} = \sup_{R>0} R^{\frac{2}{1-m}-d} \int_{\mathbb{R}^d \setminus B_R(0)} |u_0(x)| \, \mathrm{d}x < \infty$$

then there exists $T > 0$ such that

$$\left\| \frac{u(t) - \mathcal{B}_{M_\star}(t)}{\mathcal{B}_{M_\star}(t)} \right\|_{\mathrm{L}^\infty(\mathbb{R}^d)} < \varepsilon \quad \forall t \geq T$$

where

$$T = T(\varepsilon, \|u_0\|_{\mathcal{X}}, \mathcal{F}[u_0]) = \mathsf{c}_\star \, \frac{1 + \|u_0\|_{\mathcal{X}}^{1-m} + \mathcal{F}[u_0]^{\frac{2-d(1-m)}{2}}}{\varepsilon^{\mathsf{a}}}$$

where $\mathsf{c}_\star$ and $\mathsf{a} > 1$ are constants depending on m, d, and $\mathcal{F}[\cdot]$ is the entropy functional.

The entropy functional

$$\mathcal{F}[u] := \frac{1}{m-1} \int_{\mathbb{R}^d} \left[u^m - \mathcal{B}_{M_\star}^m - \mathcal{B}_{M_\star}^{m-1} \, (u - \mathcal{B}_{M_\star}) \right] \mathrm{d}x$$

Sketch of the proof of Theorem-2: the role of the GHP

We fix $\varepsilon > 0$ and look at the uniform relative error in the two "cylinders"

$$\{|x| \geq Ct^{\vartheta}\} \quad \text{and} \quad \{|x| \leq Ct^{\vartheta}\}$$

In the first case we observe, as in **Vázquez 2003** and **Carillo-Vázquez 2003**, that

$$\mathcal{B}_M(t \pm \tau, x) = \frac{(t \pm \tau)^{\frac{1}{1-m}}}{\left[b_0 \frac{(t\pm\tau)^{2\vartheta}}{M^{2\vartheta(1-m)}} + b_1|x|^2\right]^{\frac{1}{1-m}}} \sim \frac{(t \pm \tau)^{\frac{1}{1-m}}}{b_1\,|x|^{\frac{2}{1-m}}}\,, \quad \text{as } |x| \to \infty\,.$$

Thus as $|xt^{-\vartheta}| \to \infty$ we have:

$$\left(\frac{t-\tau_1}{t}\right)^{\frac{1}{1-m}} \leq \lim_{|xt^{-\vartheta}| \to \infty} \frac{u(t,x)}{\mathcal{B}_M(t,x)} \leq \left(\frac{t+\tau_2}{t}\right)^{\frac{1}{1-m}}$$

We conclude that there exist $C'_\varepsilon, t'_\varepsilon > 0$ such that

$$1 - \varepsilon \leq \frac{u(t,x)}{\mathcal{B}_M(t,x)} \leq 1 + \varepsilon\,, \quad \text{for all } t \geq t'_\varepsilon\,, \quad \text{and } x \in \{|x| \geq C'_\varepsilon\, t^{\vartheta}\}\,.$$

The dependence on $\|u_0\|_{\mathcal{X}}$ comes from τ_2!

Sketch of the proof of Theorem-2: change of variables

In the cylinder $\{|x| \leq Ct^{\vartheta}\}$ it is useful to introduce a time-dependent rescaling

$$u(t,x) = \frac{1}{R^d}\, v\left(\tau, \frac{x}{R}\right) \quad \text{where} \quad \frac{dR}{dt} = R^{1-\frac{1}{\vartheta}}\,, \quad \tau(t) := \tfrac{1}{2}\, \log\left(\frac{R(t)}{R_0}\right)$$

Then the function v solves a Fokker-Planck type equation

$$\frac{\partial v}{\partial \tau} + \nabla \cdot \left[v \left(\nabla v^{m-1} - 2\,x \right) \right] = 0$$

with *same initial datum* $v_0 = u_0$ if $R_0 = R(0) = 1$
In such a way the self-similar solution

$$\mathcal{B}_M(t,x) \quad \text{is mapped to} \quad \mathsf{B}_M(x) = \left[\frac{b_0}{M^{2\vartheta(1-m)}} + b_1 |x|^2 \right]^{\frac{1}{m-1}}$$

The cylinder $\{|x| \leq Ct^{\vartheta}\}$ it mapped to $\{|x| \leq C\}$

Sketch of the proof of Theorem-2: uniform regularity estimates

On the cylinder $\{|x| \leq C\}$ we have

$$\left|\frac{v(\tau,x) - \mathsf{B}_M(x)}{\mathsf{B}_M(x)}\right| \leq \left[\frac{b_0}{M^{2\vartheta(1-m)}} + b_1\, C^2\right]^{\frac{1}{1-m}} \|v(\tau) - \mathsf{B}_M\|_{\mathrm{L}^\infty(\mathbb{R}^d)}$$

We can interpolate as

$$\|v(\tau) - \mathsf{B}_M\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \leq \|v(\tau) - \mathsf{B}_M\|_{C^\alpha(\mathbb{R}^d)}^{\frac{d}{d+\alpha}} \|v(\tau) - \mathsf{B}_M\|_{\mathrm{L}^1(\mathbb{R}^d)}^{\frac{\alpha}{d+\alpha}}$$

- ▷ $\|v(\tau) - \mathsf{B}_M\|_{\mathrm{L}^1(\mathbb{R}^d)} \lesssim \mathcal{F}[v_0]\, e^{-4\,\tau}$
- ▷ How to uniformly bound $\|v(\tau) - \mathsf{B}_M\|_{C^\alpha(\mathbb{R}^d)}$ for which $0 < \alpha < 1$ which does not dependent on the solution itself?
- ▷ Delicate pointwise estimates for solutions to $u_t = \Delta u^m$ (based on Moser iteration and other arguments)
- ▷ Delicate and explicit regularity estimates for solutions to $v_t = \mathrm{div}\,(A(t,x)\,\nabla v)$ (again based on Moser iteration and Harnack inequalities)

Thank you for your attention!

Generalized Global Harnack principle

What happens for if the initial data $u_0 \notin \mathcal{X}$?

If the initial data

$$\frac{1}{(A+|x|)^\alpha} \leq u_0(x) \leq \frac{1}{(B+|x|)^\alpha} \quad \text{where } d < \alpha < \frac{2}{1-m},$$

then the solution

$$u(t,x) \asymp \frac{1}{|x|^\alpha} \qquad \text{for large |x|.}$$

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On the heat equation and fractional heat equation

For solutions to $u_t = \Delta u$ uniform convergence in relative error does **not** hold,

$$\sup_{\mathbb{R}^d} \left| \frac{u(t,x)}{e^{\frac{-|x|^2}{4t}}} \right| = +\infty\,.$$

Take for instance $(4\pi)^{\frac{d}{2}} u(1,x) = e^{\frac{-|x+x_0|^2}{4t}}$. For the **fractional** heat equation

$$u_t + (-\Delta)^s u = 0\,, \quad 0 < s < 1\,, \tag{F}$$

we have

Theorem (J.L. Vázquez, 2018)

Let $u(t,x)$ be a solution to (F) with initial datum $u_0 \in \mathrm{L}^1(\mathbb{R}^d)$ and compactedly supported. Then

$$\sup_{\mathbb{R}^d} \left| \frac{u(t,x) - M\,P_t(x)}{M\,P_t(x)} \right| \le C\,M\,R\,t^{-2s}$$

where M is the mass of u_0, P_t is the fundamental solutions to (F), t large and u_0 is supporeted in the ball of radius R.

p-Laplace evolution equation –> Talk of Diana Stan

(Almost) everything holds for the problem

$$(\mathrm{p-CP}) \quad \begin{cases} \partial_t u = \Delta_p(u) & \text{in } (0,\infty)\times\mathbb{R}^d, \\ u\,(0,x) = u_0(x) & \text{in } \mathbb{R}^d\,. \end{cases}$$

Recall that $\Delta_p(u) = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$, we focus on the range $d \geq 3$, $\frac{2d}{d+1} < p < 2$.
Let us define

$$|f|_{\mathcal{X}_p} := \sup_{R>0} R^{\frac{p}{2-p}-d} \int_{B_R^c(0)} |f(x)|\,\mathrm{d}x < \infty\,,$$

and the space

$$\mathcal{X}_p := \{u \in \mathrm{L}^1_+(\mathbb{R}^d) : |u|_{\mathcal{X}_p} < +\infty\}.$$

Theorem [M. Bonforte, N.S., D. Stan]

The **GHP** holds if and only if $u_0 \in \mathcal{X}_p \setminus \{0\}$.

Convergence of the **relative error** holds if and only if $u_0 \in \mathcal{X}_p \setminus \{0\}$.

On the space $\mathcal{X}$

In 2003, **Vázquez** also introduced the following condition for which a form of GHP holds

$$\int_{B_{\frac{|x|}{2}}(x)} u_0(y)\,\mathrm{d}y = O\left(|x|^{d-\frac{2}{1-m}}\right)$$

which is “a posteriori” equivalent to

$$|f|_{\mathcal{X}} := \sup_{R>0} R^{\frac{2}{1-m}-d} \int_{B_R^c(0)} |f(x)|\,\mathrm{d}x < \infty\,,$$

The proof of the equivalence uses the GHP!

Convergence of the relative error-1

The **relative error**

$$\left|\frac{u(t,x)-\mathcal{B}_M(t,x)}{\mathcal{B}_M(t,x)}\right|$$

is not always uniformly bounded in $\mathbb{R}^d$ (recall the solution $w(t,x)$).

However, for initial data in $\mathcal{X}$ it is!

Carrillo and Vázquez, proved for **radial** solution whose initial data satisfy $u_0(|x|) \lesssim |x|^{-\frac{2}{1-m}}$

$$\left\|\frac{u(t,x)-\mathcal{B}_M(t,x)}{\mathcal{B}_M(t,x)}\right\|_{L^\infty(\mathbb{R}^d)} \leq \frac{C}{t}\,.$$

Later Kim and McCann get rid of the **radial** assumption, but no result are available for the whole space $\mathcal{X}$.

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Convergence of the relative error-2

Theorem [M. Bonforte, N.S.]

Under the running assumption, a solution $u(t,x)$ to (CP) converges to $\mathcal{B}_M(t,x)$ in *uniform relative error*, i.e.

$$\lim_{t\to\infty}\left\|\frac{u(t,x)-\mathcal{B}_M(t,x)}{\mathcal{B}_M(t,x)}\right\|_{\mathrm{L}^\infty(\mathbb{R}^d)}=0$$

if and only if

$$u_0\in\mathcal{X}\setminus\{0\}$$

In the case of radial initial data we find the estimate of Carrillo and Vázquez for the whole $\mathcal{X}$

$$\left\|\frac{u(t,x)-\mathcal{B}_M(t,x)}{\mathcal{B}_M(t,x)}\right\|_{\mathrm{L}^\infty(\mathbb{R}^d)}\le\frac{C}{t}\,.$$

Convergence of the relative error-2

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