Spectral stability of monotone traveling fronts for reaction diffusion-degenerate Nagumo equations

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Reaction diffusion-degenerate equations

Simplest model: scalar reaction-diffusion equation with degenerate diffusion coefficient:

$$u_t = (D(u)u_x)_x + f(u),$$

$u = u(x,t) \in \mathbb{R}$, $x \in \mathbb{R}$, $t > 0$.

- $D = D(u)$ - density-dependent, degenerate, nonlinear diffusion coefficient;
- $f = f(u)$ - nonlinear reaction function

Reaction functions (i)

• **Fisher-KPP, monostable, logistic** type, $f \in C^2([0,1];\mathbb{R})$ has one stable ($u=1$) and one unstable ($u=0$) equilibrium points in $[0,1]$,

$$f(0)=f(1)=0, \qquad f'(0)>0,\ f'(1)<0,$$
$$f(u)>0,\ \text{for all } u\in(0,1).$$

Typical example:

- Logistic function (dynamics of a population with limited resources):

$$f(u)=u(1-u)$$

Reaction functions (ii)

- **Nagumo** (a.k.a. **Bistable, Allen-Cahn, Chafee-Infante**) type:
  $f \in C^2([0,1];\mathbb{R})$ has two stable equilibria ($u = 1, 0$) and one unstable ($u = \alpha$) equilibrium point in $[0,1]$

  $$f(0) = f(\alpha) = f(1) = 0, \qquad f'(0), f'(1) < 0, \quad f'(\alpha) > 0,$$
  $$f(u) > 0 \text{ for all } u \in (\alpha, 1), \qquad f(u) < 0 \text{ for all } u \in (0, \alpha),$$

  for some $\alpha \in (0,1)$.

Typical example:

- Cubic reaction (dynamics of a population with limited resources and cooperation, Allee effect):

  $$f(u) = u(1-u)(u-\alpha)$$

Density-dependent degenerate diffusions (i)

In physics and engineering:

- Mullins diffusion for thermal grooving (surface groove profiles on a heated polycrystal by the mechanism of evaporation-condensation); **Mullins (1957)**, **Broabridge (1989)** (non-degenerate).
- Matano boundary methods in the Allen-Cahn equation for metal binary alloys; **Wagner (1952), Allen, Cahn (1972)** (non-degenerate).
- Porous medium equation, $u_t = \Delta(u^m)$ (with $D(u) = mu^{m-1}$) **Vazquez (2007)** (degenerate)
- Anisotropic diffusivities in binary alloys; **Elliot, Garcke (1996)**, **Taylor, Cahn (1994)** (degenerate).

Density-dependent degenerate diffusions (ii)

In biology:

- Populations' dynamics models, 'motility' depends on density:
  - mammals, **Myers, Krebs (1974)**, **Shigesada et al. (1979)**.
  - ecology, **Gurtin, McCamy (1977)**
  - eukaryotic cell biology, **Sengers et al. (2007)**
- Degenerate diffusions ($D = 0$ in some regions) appear in bacterial aggregation models; **Kawasaki et al. (1997)**, **Leyva et al. (2013)**
- Degenerate diffusions to model sharp tumor invasion fronts: **McGillen et al. (2014)**.

Degenerate diffusion

Density-dependent and degenerate diffusion coefficient:

- $D \in C^2([0,1];\mathbb{R})$
- $D(0) = 0$
- $D(u) > 0$ for all $u \in (0,1]$,
- $D'(u) > 0$ for all $u \in [0,1]$ $(*)$

Examples:

- $D(u) = u^2 + bu, \quad b > 0$; **Shigesada et al. (1979)**. Models dispersive effects of mutual interference between individuals of a population.
- Porous medium type of diffusion, $D(u) = mu^{m-1}$

Mathematical features

Rich mathematical consequences:

- Degenerate diffusion equations may possess finite speed of propagation of initial disturbances; **Gilding, Kersner (1996)**.
- Existence of traveling fronts of sharp type; **Sánchez-Garduño, Maini (1995, 1997)**.
- Loss of hyperbolicity of the associated ODE at degenerate point.

Traveling fronts

Traveling wave solution:

$$u(x,t) = \varphi(x - ct), \quad \varphi : \mathbb{R} \to \mathbb{R},$$

$c \in \mathbb{R}$ - wave speed. Upon substitution:

$$(D(\varphi)\varphi_\xi)_\xi + c\varphi_\xi + f(\varphi) = 0,$$

where $\xi = x - ct$ denotes the translation (Galilean) variable. Asymptotic limits:

$$u_\pm := \varphi(\pm\infty) = \lim_{\xi \to \pm\omega} \varphi(\xi), \qquad \omega = \xi_0, \infty$$

$u_\pm$ is an equilibrium point of the reaction: $u_\pm \in \{0, 1\}$ (Fisher-KPP), $u_\pm \in \{0, 1, \alpha\}$ (bistable).

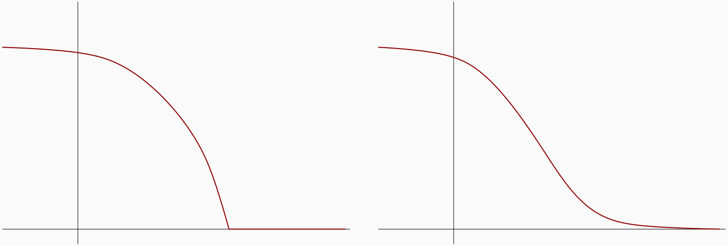
Existence theory of degenerate traveling fronts

Some references:

- **Aronson (1980)**: Fisher-KPP with diffusion of porous medium type.
- **Sanchez-Garduño, Maini, (1994, 1995)**: Fisher-KPP fronts.
- **Sanchez-Garduño, Maini, (1997)**: Nagumo (bistable) fronts.
- **Sanchez-Garduño, Maini, Kappos (1996)**, **El-Adnani, Talibi-Alaoui (2010)** (Conley index techniques).
- **Gilding, Kersner (2005)** ($D(u) = au^k$).
- **Malaguti, Marcelli (2003)** (doubly degenerate diffusions $D(u) = u(1-u)$).
- (Abridged list...)

Examples (i)

In the Fisher-KPP case, the theory predicts the existence of sharp fronts with critical speed $c = c_*$, and monotone smooth fronts for $c > c_*$.

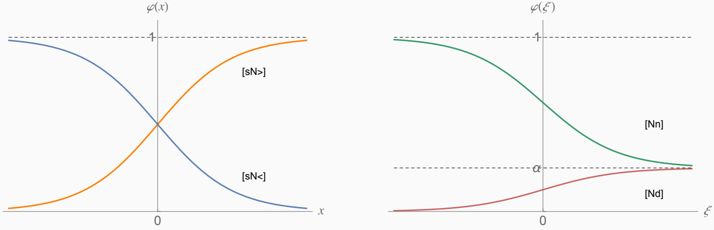


(a) $c = c_*$ (sharp) (b) $c > c_*$ (smooth)

Figure 1: Profile $\varphi = \varphi(\xi)$ for (a) $c = c_*$; (b) $c > c_*$.

Examples (ii)

In the Nagumo case, there are many fronts. Sharp fronts connecting to degenerate equilibria with (unique) critical speed $c = c_* \in (0, \overline{c}(\alpha))$, $\overline{c}(\alpha) := 2\sqrt{D(\alpha)f'(\alpha)}$; smooth monotone fronts for $c > \overline{c}(\alpha)$ or $c = 0$, and even oscillatory profiles.



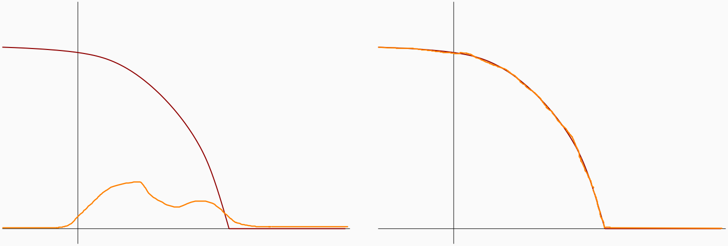
(a) $c = 0$ (b) $c > c(\alpha)$ (smooth)

Figure 2: Profile $\varphi = \varphi(\xi)$ for (a) $c = 0$; (b) $c > c(\alpha)$.

Previous work (i)

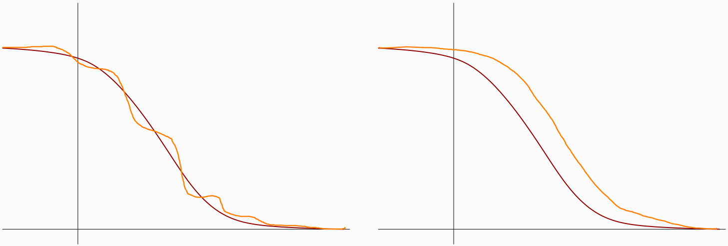
Works on long-time behaviour of solutions to reaction-diffusion degenerate equations:

- **Sherratt-Marchant (1996)**: Fisher-KPP case, numerical study with $D(u) = u$.
- **Biró (2002)**, **Medvedev et al. (2003)**: Fisher-KPP, diffusion porous medium type, compactly supported initial data evolve towards sharp front with $c = c_*$.
- **Kamin, Rosenau (2004)**: Extension to $f(u) = u(1 - u^m)$, same porous medium type diff., fast decaying initial data.
- (Abridged list...)



(a) $t = 0$ (b) $t \to \infty$

Figure 3: (a) initial condition $u_0(x) \in C_0^\infty(\mathbb{R})$; (b) u evolves into the sharp front.



(a) perturbation, $t = 0$ (b) translated, $t \to \infty$

Figure 4: Perturbation of the smooth profile $\varphi = \varphi(x)$: (a) initial condition $u_0(x) = \varphi(x) + v_0(x)$; (b) $u(x,t) \to \varphi(x + \delta(t) - ct)$.

Previous work (ii)

Works on stability of diffusion-degenerate fronts:

- **Hosono (1986)**: Nagumo reaction, diffusion of porous medium type: $D(u) = mu^{m-1}$. Comparison principle techniques: initial data close to sharp front, then asymptotic convergence to a translated (sharp) front.
- **Dalibard, Lopez-Ruiz, Perrin (2021)**: Preprint, arXiv:2108.10563. Porous medium with generalized Fisher-KPP reaction. Nonlinear stability in L^2 weighted energy spaces of smooth fronts.

Spectral stability of reaction diffusion-degenerate fronts

References:

- **Leyva, P. (2020)**, J. Dynam. Differ. Equ. 32. Fisher-KPP reaction, smooth fronts.
- **Leyva, López-Ríos, P. (2021)**, Indiana Univ. Math. J., in press. Nagumo reaction, smooth fronts.

Features:

- Analysis focuses on spectral stability of smooth fronts.
- Techniques related to spectral theory of operators (Kato).
- Follows general program (a) spectral $\Rightarrow$ (b) linear $\Rightarrow$. (c) non-linear stability analyses. Main references: **Alexander, Gardner, Jones (1990)**, **Sandstede (2002)**, **Kapitula, Promislow (2013)**.
- Some ideas could be extrapolated to the case of systems.

The spectral problem and main results

Linearized operator around the front

Abuse of notation $x \to x - ct$. Linearizing around the front yields,

$$u_t = (D(\varphi)u)_{xx} + cu_x + f'(\varphi)u.$$

Specialize to solutions of form $e^{\lambda t}u(x)$, with $\lambda \in \mathbb{C}$, $u \in X$, Banach space. Spectral problem:

$$\begin{aligned} \mathscr{L} u &= \lambda u, \\ \mathscr{L} &: \mathscr{D}(\mathscr{L}) \subset X \to X, \\ \mathscr{L} u &= (D(\varphi)u)_{xx} + cu_x + f'(\varphi)u. \end{aligned}$$

$\mathscr{D}(\mathscr{L})$ is dense in X; $\mathscr{L}$ is the closed, densely defined, linearized operator around the front. (e.g. $X = L^2$, $\mathscr{D} = H^2$, localized perturbations)

Resolvent and spectra

Definition

Let $\mathscr{L} \in \mathscr{C}(X,Y)$ be a closed, densely defined operator from X to Y, Banach. Its resolvent $\rho(\mathscr{L})$ is defined as the set of all complex numbers $\lambda \in \mathbb{C}$ such that $\mathscr{L} - \lambda$ is injective, $\mathscr{R}(\mathscr{L} - \lambda) = Y$ and $(\mathscr{L} - \lambda)^{-1}$ is bounded. Its spectrum is defined as $\sigma(\mathscr{L}) = \mathbb{C} \backslash \rho(\mathscr{L})$.

Definition

We say the traveling front φ is X-spectrally stable if

$$\sigma(\mathscr{L}) \subset \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \leq 0\}.$$

Partition of spectrum

Lemma

For any closed, densely defined linear operator $\mathscr{L} : \mathscr{D} \subset X \to Y$,

$$\sigma(\mathscr{L}) = \sigma_{\mathrm{pt}}(\mathscr{L}) \cup \sigma_{\delta}(\mathscr{L}) \cup \sigma_{\pi}(\mathscr{L}),$$

where

$$\begin{aligned}
\sigma_{\mathrm{pt}}(\mathscr{L}) &:= \{\lambda \in \mathbb{C} : \mathscr{L} - \lambda \text{ is not injective}\}, \\
\sigma_{\delta}(\mathscr{L}) &:= \{\lambda \in \mathbb{C} : \mathscr{L} - \lambda \text{ is injective, } \mathscr{R}(\mathscr{L} - \lambda) \text{ is closed,} \\
&\qquad \text{and } \mathscr{R}(\mathscr{L} - \lambda) \neq Y\}, \\
\sigma_{\pi}(\mathscr{L}) &:= \{\lambda \in \mathbb{C} : \mathscr{L} - \lambda \text{ is injective, and } \mathscr{R}(\mathscr{L} - \lambda) \\
&\qquad \text{is not closed}\}.
\end{aligned}$$

Observations (i)

- In the theory of stability of waves, cf. **Kapitula, Promislow (2013)**, the standard partition is Weyl's partition:

$$\sigma_{\text{ess}}(\mathscr{L}) := \{\lambda \in \mathbb{C} : \mathscr{L} - \lambda \text{ is either not Fredholm,}$$
$$\text{or has index different from zero}\}.,$$
$$\widetilde{\sigma}_{\text{pt}}(\mathscr{L}) := \{\lambda \in \mathbb{C} : \mathscr{L} - \lambda \text{ is Fredholm with index zero}$$
$$\text{and has a non-trivial kernel}\}.$$

  Notice that $\widetilde{\sigma}_{\text{pt}} \subset \sigma_{\text{pt}}$. $\widetilde{\sigma}_{\text{pt}}$ is the set of isolated eigenvalues with finite multiplicity.
- $\sigma_{\text{pt}}(\mathscr{L})$ is called the extended point spectrum; its elements, eigenvalues. $\lambda \in \sigma_{\text{pt}}(\mathscr{L})$ if and only if there exists $u \in \mathscr{D}(\mathscr{L})$, $u \neq 0$, such that $\mathscr{L} u = \lambda u$
- $\lambda = 0$ always belongs to the L^2 - $\sigma_{\text{pt}}(\mathscr{L})$ (translation eigenvalue), as

$$\mathscr{L} \varphi_x = \partial_x \big((D(\varphi)\varphi_x)_x + c\varphi_x + f(\varphi) \big) = 0$$

  in view of the profile equation and $\varphi_x \in H^2(\mathbb{R};\mathbb{C})$ (eigenfunction).

- $\sigma_\pi(\mathscr{L})$ is contained in the approximate spectrum, defined as

$$\sigma_\pi(\mathscr{L}) \subset \sigma_{\text{app}}(\mathscr{L}) := \{\lambda \in \mathbb{C} : \text{there exists } u_n \in \mathscr{D}(\mathscr{L}) \text{ with } \|u_n\| = 1 \\ \text{such that } (\mathscr{L} - \lambda)u_n \to 0 \text{ in } Y \text{ as } n \to \infty\}.$$

  This holds because $\mathscr{R}(\mathscr{L} - \lambda)$ not closed $\Rightarrow$ there exists a Weyl's sequence: $u_n \in \mathscr{D}(\mathscr{L})$, $\|u_n\| = 1$ such that $(\mathscr{L} - \lambda)u_n \to 0$, which contains no convergent subsequence.

- $\sigma_\delta(\mathscr{L})$ is clearly contained in what is often called the compression spectrum:

$$\sigma_\delta(\mathscr{L}) \subset \sigma_{\text{com}}(\mathscr{L}) := \{\lambda \in \mathbb{C} : \mathscr{L} - \lambda \text{ is injective, and } \overline{\mathscr{R}(\mathscr{L} - \lambda)} \neq Y\}.$$

Why this partition?

- Designed to overcome difficulties associated to degeneracy at $\varphi = 0$.
- Spectral problem recast as first order system, $\mathbf{w}_x = \mathbf{A}(x, \lambda)\mathbf{w}$, $\mathbf{w} = (u, u_x)$ (cf. **Alexander, Gardner, Jones (1990)**).
- In the strictly parabolic setting, $D \geq \delta > 0$, and $\lambda \in \Omega \subset \mathbb{C}$, large connected set, $\mathbf{A}_{\pm}(\lambda)$ are strictly hyperbolic, their spectral equations determine Fredholm curves that bound Weyl's essential spectrum. In the degenerate case, hyperbolicity is lost.

Example: constant diffusivity

Constant diffusion problem, $D(u) \equiv D > 0$. Spectral problem reads

$$\lambda u = Du_{xx} + cu_x + f'(\varphi)u.$$

Recast as a first order system:

$$W_x = \mathbf{A}(x,\lambda)W,$$

$$W = \begin{pmatrix} u \\ u_x \end{pmatrix} \in H^1(\mathbb{R};\mathbb{C}^2)$$

$$\mathbf{A}(x,\lambda) = \begin{pmatrix} 0 & 1 \\ (\lambda - f'(\varphi))/D & -c/D \end{pmatrix}.$$

Asymptotic limits:

$$\mathbf{A}_\pm(\lambda) = \lim_{x \to \pm\infty} \mathbf{A}(x,\lambda) = \begin{pmatrix} 0 & 1 \\ (\lambda - f'(u_\pm))/D & -c/D \end{pmatrix}.$$

Fact of life: The Fredholm properties of $\mathscr{L} - \lambda$ are the same as the operators $\mathscr{T}(\lambda) := d/dx - \mathbf{A}(x,\lambda)$. (There is a one-to-one and onto correspondence between the kernels and Jordan chains, with same srtucture and length.) (cf. **Kapitula, Promislow (2013)**.)

How to locate $\sigma_{\rm ess}(\mathscr{L})$? The Fredholm curves $\lambda^\pm = \lambda^\pm(k)$, $k \in \mathbb{R}$ (solutions to $\det(\mathbf{A}_\pm(\lambda) - ikI) = 0$, dispersion relation) determine the boundaries of the open regions in the complex plane on which the operator $\mathscr{T}(\lambda)$ (or $\mathscr{L} - \lambda$) is Fredholm.

- Idea: take a parabolic regularization (add $\varepsilon d^2/dx^2$), compute σ_{ess} and take the limit as $\varepsilon \to 0$.
- As a consequence, we control some component of the essential spectrum, σ_δ, precluding the behaviour of approximate spectra.
- The set σ_π is controlled with the use of Weyl sequences.
- The stability analysis of the point spectrum is based on weighted energy estimates.

Exponentially weighted spaces

For any $m \in \mathbb{Z}$, $m \geq 0$, $a \in \mathbb{R}$,

$$H_a^m(\mathbb{R};\mathbb{C}) = \{v : e^{ax}v(x) \in H^m(\mathbb{R};\mathbb{C})\},$$

Hilbert spaces with inner product and norm,

$$\langle u, v\rangle_{H_a^m} := \langle e^{ax}u, e^{ax}v\rangle_{H^m}, \qquad \|v\|_{H_a^m}^2 := \|e^{ax}v\|_{H^m}^2 = \langle v, v\rangle_{H_a^m}.$$

Custom: $L_a^2(\mathbb{R};\mathbb{C}) = H_a^0(\mathbb{R};\mathbb{C})$.

Facts of life: (cf. **Kapitula, Promislow (2013)**)

- The spectrum of $\mathscr{L} \in \mathscr{C}(L_a^2; L_a^2)$ is equivalent to the spectrum of a conjugated operator, $\mathscr{L}_a \in \mathscr{C}(L^2; L^2)$:

$$\mathscr{L}_a := e^{ax}\mathscr{L}e^{-ax} : \mathscr{D} = H^2(\mathbb{R};\mathbb{C}) \subset L^2(\mathbb{R};\mathbb{C}) \to L^2(\mathbb{R};\mathbb{C}),$$

- The point spectrum is invariant under conjugation $\sigma_{\mathrm{pt}}(\mathscr{L}_a)_{|L^2} = \sigma_{\mathrm{pt}}(\mathscr{L})_{|L_a^2}$.

Main results (i)

Theorem (Leyva, P. (2020))

For any monotone traveling front for Fisher-KPP reaction diffusion-degenerate equations, under hypotheses for $D = D(\cdot)$ and f, and traveling with speed $c \in \mathbb{R}$ satisfying the condition

$$c > \max\left\{ c_*, \ \frac{f'(0)\sqrt{D(1)}}{\sqrt{f'(0) - f'(1)}} \right\} > 0,$$

there exists an exponentially weighted space $L^2_a(\mathbb{R})$, with $a \in \mathbb{R}$, such that the front is L^2_a-spectrally stable. $c_* > 0$ is the minimum threshold speed (the velocity of the sharp wave).

Main results (ii)

Theorem (Leyva, López-Ríos, P. (2021))

Under our hypotheses, the family of all monotone diffusion-degenerate Nagumo fronts connecting the equilibrium states $u = \alpha$ *with* $u = 0$ *and traveling with speed* $c > \overline{c}(\alpha) = 2\sqrt{D(\alpha)f'(\alpha)}$ *are spectrally stable in an exponentially weighted energy space* $L^2_a = \{e^{ax}u \in L^2\}$. *More precisely, there exists* $a > 0$ *such that*

$$\sigma(\mathscr{L})_{|L^2_a} \subset \{\lambda \in \mathbb{C} : Re\,\lambda \leq 0\},$$

where $\mathscr{L} : L^2_a \to L^2_a$ *denotes the linearized operator around the traveling front and* $\sigma(\mathscr{L})_{|L^2_a}$ *denotes its spectrum computed with respect to the energy space* L^2_a.

Method of proof (overview)

Strategy of proof

(A) Calculation of σ_δ (parabolic regularization technique; choice of the weight $a \in \mathbb{R}$).

(B) Control of σ_π (use of Weyl sequences).

(C) Control of σ_{pt} (weighted energy estimates; trick to overcome degeneracy).

(A) Parabolic regularization technique

- For any $\varepsilon > 0$, let
$$D^\varepsilon(\varphi) := D(\varphi) + \varepsilon.$$
$D^\varepsilon(\varphi) > 0$ for all $x \in \mathbb{R}$.
- Regularized conjugated operator:
$$\mathscr{L}_a^\varepsilon : \mathscr{D} = H^2 \subset L^2 \to L^2,$$
$$\mathscr{L}_a^\varepsilon u := e^{ax} \mathscr{L}^\varepsilon e^{-ax} = D^\varepsilon(\varphi) u_{xx} + \Big(2D^\varepsilon(\varphi)_x - 2aD^\varepsilon(\varphi) + c\Big) u_x +$$
$$+ \Big(a^2 D^\varepsilon(\varphi) - 2aD^\varepsilon(\varphi)_x - ac + D^\varepsilon(\varphi)_{xx} + f'(\varphi)\Big) u$$
$a \in \mathbb{R}$ is to be chosen.
- Region of consistent splitting:
$$\Omega(a,\varepsilon) := \big\{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda > \max\{D^\varepsilon(u_+)a^2 - ac + f'(u_+), D^\varepsilon(u_-)a^2 - ac + f'(u_-)\}\big\}$$

- **Lemma 1**: *For all $\varepsilon > 0$, $a \in \mathbb{R}$, and for each $\lambda \in \Omega(a, \varepsilon)$, the operator $\mathscr{L}_a^\varepsilon - \lambda$ is Fredholm with index zero.*
  (Note: hyperbolicity of end points is fundamental: Weyl's essential spectrum theorem + exponential dichotomies).
- **Lemma 2**: *For each fixed $\lambda \in \mathbb{C}$, the operators $\mathscr{L}^\varepsilon - \lambda$ converge in generalized sense to $\mathscr{L} - \lambda$ as $\varepsilon \to 0^+$*
  ($d(G(\mathscr{L}^\varepsilon - \lambda), G(\mathscr{L} - \lambda)) \to 0$).
- Apply Kato's stability theorem (**Kato, 1980**) to locate $\sigma_\delta(\mathscr{L}_a)$.
  Lemma 3: *Suppose that $\mathscr{L}_a - \lambda$ is semi-Fredholm, for $a \in \mathbb{R}$, $\lambda \in \mathbb{C}$. Then for each $0 < \varepsilon \ll 1$ sufficiently small $\mathscr{L}_a^\varepsilon - \lambda$ is semi-Fredholm and $\operatorname{ind}(\mathscr{L}_a^\varepsilon - \lambda) = \operatorname{ind}(\mathscr{L}_a - \lambda)$.*

- **Corollary**: $\sigma_\delta(\mathscr{L}_a) \subset \mathbb{C} \backslash \Omega(a,0)$.
- Choose $a \in \mathbb{R}$ appropriately to stabilize σ_δ: $\mathbb{C} \backslash \Omega(a,0) \subset \{\mathrm{Re}\,\lambda < 0\}$.
- Example: in the Fisher-KPP case it suffices to set

$$0 < \frac{f'(0)}{c} < a < (2D(1))^{-1}\big(c + \sqrt{c^2 - 4D(1)f'(1)}\big).$$

- Consequence: $\sigma_\delta(\mathscr{L}_a)_{|L^2} = \sigma_\delta(\mathscr{L})_{|L^2_a} \subset \{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda < 0\}$.

(B) Location of σ_π: Weyl sequences

- Conjugated operator $\mathscr{L}_a = b_2(x)\partial_x^2 + b_1(x)\partial_x + b_0(x)\mathrm{Id}$.
- Fix $\lambda \in \sigma_\pi(\mathscr{L}_a)_{|L^2}$. Then $\mathscr{R}(\mathscr{L}_a - \lambda)$ is not closed and there exists a singular sequence $u_n \in \mathscr{D}(\mathscr{L}_a) = H^2$ with $\|u_n\|_{L^2} = 1$, for all $n \in \mathbb{N}$, such that $(\mathscr{L}_a - \lambda)u_n \to 0$ in L^2 as $n \to \infty$ and which has no convergent subsequence.
- L^2 is a reflexive space $\Rightarrow u_n \rightharpoonup 0$ in L^2.
- **Lemma 4**: *There exists a subsequence, u_n, such that $u_n \to 0$ in L^2_{loc} as $n \to \infty$.*

- For each $\varepsilon > 0$ we can choose $R > 0$ sufficiently large such that

$$|b_0(x) - \tfrac{1}{2}\partial_x b_1(x) - (a^2 D(u_\pm) - ac + f'(u_\pm))| < \varepsilon, \quad \text{for } |x| \geq R.$$

- From $b_2(x) = D(\varphi) \geq 0$:

$$\begin{aligned}
\operatorname{Re}\lambda &\leq |\langle f_n, u_n\rangle_{L^2}| + \int_{-R}^{R} (b_0(x) - \tfrac{1}{2}\partial_x b_1(x))|u_n|^2\,dx + \int_{|x|\geq R} (b_0(x) - \tfrac{1}{2}\partial_x b_1(x))|u_n|^2\,dx \\
&\leq \|(\mathscr{L} - \lambda)u_n\|_{L^2} + C_1\|u_n\|_{L^2(-R,R)} + C_2\varepsilon\|u_n\|^2_{L^2(|x|\geq R)} + (a^2 D(u_\pm) - ac + f'(u_\pm))\|u_n\|^2_{L^2(} \\
&= \underbrace{\|(\mathscr{L} - \lambda)u_n\|_{L^2} + C_1\|u_n\|_{L^2(-R,R)}}_{\to 0,\ \text{as}\ n\to\infty} + C_2\varepsilon + a^2 D(u_\pm) - ac + f'(u_\pm).
\end{aligned}$$

- Thus, $\operatorname{Re}\lambda \leq a^2 D(u_\pm) - ac + f'(u_\pm)$, or

$$\sigma_\pi(\mathscr{L}_a)_{|L^2} = \sigma_\pi(\mathscr{L})_{|L^2_a} \subset \mathbb{C}\backslash\Omega(a,0) \subset \{\operatorname{Re}\lambda < 0\}.$$

(C) Point spectral stability

- For fixed $\lambda \in \sigma_{\mathrm{pt}}(\mathscr{L}_a)$, there is solution $u \in \mathscr{D}(\mathscr{L}_a) = H^2$ to $(\mathscr{L}_a - \lambda)u = 0$ (eigenfunction).
- Spectral transformation: $w = \Theta(x)u$, with

$$\Theta(x) = \exp\left(\frac{c}{2}\int_{x_0}^{x} \frac{ds}{D(\varphi(s))} - a(x - x_0)\right).$$

- **Lemma 5**: *For the appropriate $a \in \mathbb{R}$ and for any $\lambda \in \Omega(a,0)$, if $u \in H^2$ solves $(\mathscr{L}_a - \lambda)u = 0$ then $w \in H^2$ and solves*

$$(D(\varphi)^2 w_x)_x + D(\varphi)G(x)w - \lambda D(\varphi)w = 0.$$

- Note: one needs detailed information about the decay structure of eigenfunctions and of the traveling fronts.

- Lemma applies also to the translation eigenvalue, $\lambda = 0 \in \sigma_{\rm pt}(\mathscr{L}_a) \cap \Omega(a,0)$: eigenfunction $e^{ax}\varphi_x$ is transformed into $\psi = \Theta(x)e^{ax}\varphi_x$, which solves

$$(D(\varphi)^2\psi_x)_x + D(\varphi)G(x)\psi = 0.$$

- Combine energy estimates on both equations and use monotonicity of the front:

$$\lambda\langle D(\varphi)w, w\rangle_{L^2} = -\|D(\varphi)(w/\psi)_x\psi\|_{L^2}^2.$$

- If $\lambda\sigma_{\rm pt}(\mathscr{L}_a)\cap\Omega(a,0)$ then $\operatorname{Re}\lambda \leq 0$. If $\lambda \in \sigma_{\rm pt}(\mathscr{L}_a)$ and $\lambda \notin \Omega(a,0)$ the automatically $\operatorname{Re}\lambda < 0$. We conclude point spectral stability.
- Note: the weighted L^2 norm $\|u\| = \|\sqrt{D(\varphi)}u\|_{L^2}$ encodes the degeneracy of the front (see also **Dalibard et al. (2021)**).

Thanks...!