The large diffusion limit for the heat equation with a dynamical boundary condition

Tatsuki Kawakami

Ryukoku University

BIRS-CMO Workshop
New Trends in Nonlinear Diffusion:
a Bridge between PDEs, Analysis and Geometry

September 7, 2021 (Online)

(joint work with M. Fila, K. Ishige and J. Lankeit.)

1. Introduction

Consider the heat eq. with a dynamical boundary condition

$$(H) \quad \begin{cases} \varepsilon \partial_t u - \Delta u = 0, & x \in \Omega \subset \mathbb{R}^N, \quad t > 0, \\ \partial_t u + \partial_\nu u = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = \varphi(x), & x \in \Omega, \\ u(x,0) = \varphi_b(x), & x \in \partial\Omega, \end{cases}$$

where $N \geq 2$, $\varepsilon \in (0,1)$ and

$$\varphi : \text{ smooth func. in } \Omega, \qquad \varphi_b : \text{ smooth func. on } \partial\Omega.$$

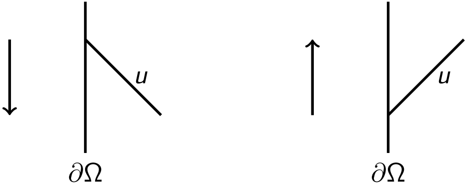
1. Introduction

Consider the heat eq. with a dynamical boundary condition

$$(H) \quad \begin{cases} \varepsilon\partial_t u - \Delta u = 0, & x \in \Omega \subset \mathbb{R}^N, \quad t > 0, \\ \partial_t u + \partial_\nu u = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = \varphi(x), & x \in \Omega, \\ u(x,0) = \varphi_b(x), & x \in \partial\Omega, \end{cases}$$

where $N \geq 2$, $\varepsilon \in (0,1)$ and

φ : smooth func. in Ω, $\qquad \varphi_b$: smooth func. on $\partial\Omega$.



1. Introduction

Consider the heat eq. with a dynamical boundary condition

$$(H)\quad \begin{cases} \varepsilon\partial_t u - \Delta u = 0, & x \in \Omega \subset \mathbb{R}^N, \quad t > 0, \\ \partial_t u + \partial_\nu u = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = \varphi(x), & x \in \Omega, \\ u(x,0) = \varphi_b(x), & x \in \partial\Omega, \end{cases}$$

where $N \geq 2$, $\varepsilon \in (0,1)$ and

$$\varphi : \text{smooth func. in } \Omega, \qquad \varphi_b : \text{smooth func. on } \partial\Omega.$$

Dynamical boundary condition (DBC) $\partial_t u + \partial_\nu u = 0$

- thermal contact with a perfect conductor
- diffusion of solute from a well-stirred fluid

⋮ (Peddie (1901), Crank ('75) etc.)

1. Introduction

Consider the heat eq. with a dynamical boundary condition

$$(H) \quad \begin{cases} \varepsilon\partial_t u - \Delta u = 0, & x \in \Omega \subset \mathbb{R}^N, \quad t > 0, \\ \partial_t u + \partial_\nu u = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = \varphi(x), & x \in \Omega, \\ u(x,0) = \varphi_b(x), & x \in \partial\Omega, \end{cases}$$

where $N \geq 2$, $\varepsilon \in (0,1)$ and

φ : smooth func. in Ω, φ_b : smooth func. on $\partial\Omega$.

Remark

Vázquez and Vitillaro ('08) : b'dd domain, ill-posed if $N \geq 2$ with $\partial_t u - \partial_\nu u = 0$

Aim

- Construct a global-in-time sol. u_ε of (H).
- $u_\varepsilon \to u$ (in a suitable sense) as $\varepsilon \to 0$.

u : sol. of the Laplace eq. with a DBC, i.e.

$$(L) \qquad \begin{cases} -\Delta u = 0, & x \in \Omega, \quad t > 0, \\ \partial_t u + \partial_\nu u = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = \varphi_b(x), & x \in \partial\Omega. \end{cases}$$

Aim

- Construct a global-in-time sol. u_ε of (H).
- $u_\varepsilon \to u$ (in a suitable sense) as $\varepsilon \to 0$.

u : sol. of the Laplace eq. with a DBC, i.e.

$$(L) \qquad \begin{cases} -\Delta u = 0, & x \in \Omega, \ \ t > 0, \\ \partial_t u + \partial_\nu u = 0, & x \in \partial\Omega, \ \ t > 0, \\ u(x,0) = \varphi_b(x), & x \in \partial\Omega. \end{cases}$$

$\underline{\text{(H)} \xrightarrow{\varepsilon \to 0} \text{(L)}}$

- Eq. "$\varepsilon\partial_t u - \Delta u = 0$" changes to "$-\Delta u = 0$".
- The influence of the initial func. φ is lost.

Remark

- Al-Aidarous, Alzahrani, Ishii and Younas ('14) : Eikonal eq., viscosity sol..
- Giga and Hamamuki ('18) : Fully-nonlinear degenerate parabolic eq., viscosity sol..

Known results ($\varepsilon = 0$)

$$\begin{cases} -\Delta u = f(u), & x \in \Omega \subset \mathbb{R}^N, \quad t > 0, \\ \partial_t u + \partial_\nu u = g(u), & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = \varphi_b(x), & x \in \partial\Omega. \end{cases}$$

$N \geq 1$, f, g : smooth func. in $\mathbb{R}$, φ_b : smooth func. on $\partial\Omega$.

. . . .

Ω : b'dd

- $f \equiv 0$: Global existence, uniqueness, boundedness, and blow-up problem

  Lions ('69), Hintermann ('89), Kirane ('92), Fila-Poláčik ('99), Yin ('03), Vitillaro ('06), Koleva-Vulkov ('07) . . .

- $f \not\equiv 0$: Global existence, uniqueness, and smoothness

  Escher ('92, '94), Gal-Meyries ('14) . . .

Tool Eigen func. or general theory for abstract evolution eq.

Known results ($\varepsilon = 0$)

$$\begin{cases} -\Delta u = f(u), & x \in \Omega \subset \mathbb{R}^N, \ \ t > 0, \\ \partial_t u + \partial_\nu u = g(u), & x \in \partial\Omega, \ \ t > 0, \\ u(x,0) = \varphi_b(x), & x \in \partial\Omega. \end{cases}$$

$N \geq 1$, f, g : smooth func. in $\mathbb{R}$, φ_b : smooth func. on $\partial\Omega$.

Ω : the half space $\mathbb{R}^N_+$

- $f \equiv 0$, $g(u) = u^p$ $(p > 1)$, $\varphi_b \geq 0$:

  Critical exponent for existence of global-in-time sols, blow-up problem, and asymptotic behavior

  Amann-Fila ('97), Kirane-Nabana-Pohozaev ('02, '04), Fila-Ishige-K. ('12)

Tool Mild sol. (fundamental sol. of fractional diffusion eq.)

Known results ($\varepsilon = 0$)

$$\begin{cases} -\Delta u = f(u), & x \in \Omega \subset \mathbb{R}^N, \quad t > 0, \\ \partial_t u + \partial_\nu u = g(u), & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = \varphi_b(x), & x \in \partial\Omega. \end{cases}$$

$N \geq 1$, f, g : smooth func. in $\mathbb{R}$, φ_b : smooth func. on $\partial\Omega$.

. . . .

Ω : the half space $\mathbb{R}^N_+$ or the exterior of the unit ball $\mathbb{R}^N \setminus \overline{B_1}$

- $f(u) = u^p$ $(p > 1)$, $g \equiv 0$, $\varphi_b \geq 0$:

  Critical exponent for existence of sols. (not only global but also local), asymptotic behavior, and relationship between minimal sol. of time-dep. problem and minimal stationary sol.

  Fila-Ishige-K. ('13, '15, '16, '17)

Tool Representation formula and Phragmén-Lindelöf thm

Known results ($\varepsilon = 0$)

$$
\begin{cases}
-\Delta u = f(u), & x \in \Omega \subset \mathbb{R}^N, \quad t > 0, \\
\partial_t u + \partial_\nu u = g(u), & x \in \partial\Omega, \quad t > 0, \\
u(x,0) = \varphi_b(x), & x \in \partial\Omega.
\end{cases}
$$

$N \geq 1$, f, g : smooth func. in $\mathbb{R}$, φ_b : smooth func. on $\partial\Omega$.

Ω : the half space $\mathbb{R}^N_+$ or the exterior of the unit ball $\mathbb{R}^N \setminus \overline{B_1}$

- $f(u) = u^p$ $(p > 1)$, $g \equiv 0$, $\varphi_b \geq 0$:

  Critical exponent for existence of sols. (not only global but also local), asymptotic behavior, and relationship between minimal sol. of time-dep. problem and minimal stationary sol.

  Fila-Ishige-K. ('13, '15, '16, '17)

Tool Representation formula and Phragmén-Lindelöf thm

Q. What happen for $\varepsilon \neq 0$?

2. (H) for the case of the half space $\mathbb{R}^N_+$

Fila-Ishige-K. (Commun. Contemp. Math. ('21)),

Fila-Ishige-K.-Lankeit (Asymptot. Anal. ('19))

Construct a sol.

$$
(H)\quad \begin{cases} \varepsilon\partial_t u_\varepsilon - \Delta u_\varepsilon = 0, & x \in \mathbb{R}^N_+, \ \ t > 0, \\ \partial_t u_\varepsilon + \partial_\nu u_\varepsilon = 0, & x \in \partial\mathbb{R}^N_+, \ \ t > 0, \\ u_\varepsilon(x,0) = \varphi(x), & x \in \mathbb{R}^N_+, \\ u_\varepsilon(x,0) = \varphi_b(x'), & x = (x',0) \in \partial\mathbb{R}^N_+. \end{cases}
$$

where $N \geq 2$ and $\mathbb{R}^N_+ := \mathbb{R}^{N-1} \times \mathbb{R}_+$.

Construct a sol.

$$(H)\quad \begin{cases} \varepsilon\partial_t u_\varepsilon - \Delta u_\varepsilon = 0, & x \in \mathbb{R}^N_+,\ \ t > 0, \\ \partial_t u_\varepsilon + \partial_\nu u_\varepsilon = 0, & x \in \partial\mathbb{R}^N_+,\ \ t > 0, \\ u_\varepsilon(x,0) = \varphi(x), & x \in \mathbb{R}^N_+, \\ u_\varepsilon(x,0) = \varphi_b(x'), & x = (x',0) \in \partial\mathbb{R}^N_+. \end{cases}$$

Laplace eq. ($\varepsilon = 0$)

$$(L)\quad \begin{cases} -\Delta u = 0, & x \in \mathbb{R}^N_+,\ \ t > 0, \\ \partial_t u + \partial_\nu u = 0, & x \in \partial\mathbb{R}^N_+,\ \ t > 0, \\ u(x,0) = \varphi_b(x') \geq 0, & x = (x',0) \in \partial\mathbb{R}^N_+. \end{cases}$$

Construct a sol.

$$(H)\quad \begin{cases} \varepsilon\partial_t u_\varepsilon - \Delta u_\varepsilon = 0, & x \in \mathbb{R}^N_+,\ \ t > 0, \\ \partial_t u_\varepsilon + \partial_\nu u_\varepsilon = 0, & x \in \partial\mathbb{R}^N_+,\ \ t > 0, \\ u_\varepsilon(x,0) = \varphi(x), & x \in \mathbb{R}^N_+, \\ u_\varepsilon(x,0) = \varphi_b(x'), & x = (x',0) \in \partial\mathbb{R}^N_+. \end{cases}$$

Laplace eq. ($\varepsilon = 0$)

$$(L)\quad \begin{cases} -\Delta u = 0, & x \in \mathbb{R}^N_+,\ \ t > 0, \\ \partial_t u - \partial_{x_N} u = 0, & x \in \partial\mathbb{R}^N_+,\ \ t > 0, \\ u(x,0) = \varphi_b(x') \ge 0, & x = (x',0) \in \partial\mathbb{R}^N_+. \end{cases}$$

Construct a sol.

$$(H)\quad \begin{cases} \varepsilon\partial_t u_\varepsilon - \Delta u_\varepsilon = 0, & x \in \mathbb{R}^N_+,\ \ t > 0, \\ \partial_t u_\varepsilon + \partial_\nu u_\varepsilon = 0, & x \in \partial\mathbb{R}^N_+,\ \ t > 0, \\ u_\varepsilon(x,0) = \varphi(x), & x \in \mathbb{R}^N_+, \\ u_\varepsilon(x,0) = \varphi_b(x'), & x = (x',0) \in \partial\mathbb{R}^N_+. \end{cases}$$

Laplace eq. $(\varepsilon = 0)$

$$(L)\quad \begin{cases} -\Delta u = 0, & x \in \mathbb{R}^N_+,\ \ t > 0, \\ \partial_t u - \partial_{x_N} u = 0, & x \in \partial\mathbb{R}^N_+,\ \ t > 0, \\ u(x,0) = \varphi_b(x') \geq 0, & x = (x',0) \in \partial\mathbb{R}^N_+. \end{cases}$$

$$\Rightarrow \quad u(x',0,t) = \int_{\mathbb{R}^{N-1}} P(x'-y',t)\varphi_b(y')\,dy' \quad (=: v(x',t)).$$

$$(\because)\quad \partial_t u - \partial_{x_N} u = 0 \quad \text{on } \partial\mathbb{R}^N_+ \qquad \Leftrightarrow \qquad \partial_t v + (-\Delta_{x'})^{\frac{1}{2}} v = 0 \quad \text{in } \mathbb{R}^{N-1},$$

where $P = P(x',t)$: $N-1$ dim Poisson kernel

$$P(x',t) = c_N t^{1-N}\left(1 + \left|\frac{x'}{t}\right|^2\right)^{-N/2}, \quad x' \in \mathbb{R}^{N-1},\ \ t > 0.$$

Construct a sol.

$$(H)\quad \begin{cases} \varepsilon\partial_t u_\varepsilon - \Delta u_\varepsilon = 0, & x \in \mathbb{R}^N_+, \quad t > 0, \\ \partial_t u_\varepsilon + \partial_\nu u_\varepsilon = 0, & x \in \partial\mathbb{R}^N_+, \quad t > 0, \\ u_\varepsilon(x,0) = \varphi(x), & x \in \mathbb{R}^N_+, \\ u_\varepsilon(x,0) = \varphi_b(x'), & x = (x',0) \in \partial\mathbb{R}^N_+. \end{cases}$$

Laplace eq. ($\varepsilon = 0$)

$$(L)\quad \begin{cases} -\Delta u = 0, & x \in \mathbb{R}^N_+, \quad t > 0, \\ \partial_t u - \partial_{x_N} u = 0, & x \in \partial\mathbb{R}^N_+, \quad t > 0, \\ u(x,0) = \varphi_b(x') \geq 0, & x = (x',0) \in \partial\mathbb{R}^N_+. \end{cases}$$

$$\Rightarrow \quad u(x',0,t) = \int_{\mathbb{R}^{N-1}} P(x'-y',t)\varphi_b(y')\,dy'.$$

u : sol. of (L) $\Leftrightarrow$ the harmonic extension of $u(x',0,t)$

$$u(x,t) = \int_{\mathbb{R}^{N-1}} P(x'-y',x_N)u(y',0,t)\,dy' = \int_{\mathbb{R}^{N-1}} P(x'-y',x_N+t)\varphi_b(y')\,dy'.$$

$$[S(t)\varphi_b](x) := \int_{\mathbb{R}^{N-1}} P(x'-y',x_N+t)\varphi_b(y')\,dy'.$$

Construct a sol.

$$(H)\quad \begin{cases} \varepsilon\partial_t u_\varepsilon - \Delta u_\varepsilon = 0, & x \in \mathbb{R}^N_+, \quad t > 0, \\ \partial_t u_\varepsilon + \partial_\nu u_\varepsilon = 0, & x \in \partial\mathbb{R}^N_+, \quad t > 0, \\ u_\varepsilon(x,0) = \varphi(x), & x \in \mathbb{R}^N_+, \\ u_\varepsilon(x,0) = \varphi_b(x'), & x = (x',0) \in \partial\mathbb{R}^N_+. \end{cases}$$

Consider the following system

$$(S)\quad \begin{cases} \partial_t v = \varepsilon^{-1}\Delta v - \partial_t w, \quad \Delta w = 0, & x \in \mathbb{R}^N_+, \quad t > 0, \\ v = 0, \quad \partial_t w - \partial_{x_N} w = \partial_{x_N} v, & x \in \partial\mathbb{R}^N_+, \quad t > 0, \\ v(x,0) = \Phi(x) := \varphi(x) - [S(0)\varphi_b](x), & x \in \mathbb{R}^N_+, \\ w(x,0) = \varphi_b(x'), & x = (x',0) \in \partial\mathbb{R}^N_+. \end{cases}$$

$u := v + w \Rightarrow u$: sol. of (H). $\quad (\because) \quad v_t = 0,\ x \in \partial\mathbb{R}^N_+,\ t \geq 0.$

Construct a sol.

$$(H)\quad \begin{cases} \varepsilon\partial_t u_\varepsilon - \Delta u_\varepsilon = 0, & x \in \mathbb{R}^N_+,\ \ t > 0, \\ \partial_t u_\varepsilon + \partial_\nu u_\varepsilon = 0, & x \in \partial\mathbb{R}^N_+,\ \ t > 0, \\ u_\varepsilon(x,0) = \varphi(x), & x \in \mathbb{R}^N_+, \\ u_\varepsilon(x,0) = \varphi_b(x'), & x = (x',0) \in \partial\mathbb{R}^N_+. \end{cases}$$

Consider the following system

$$(S)\quad \begin{cases} \partial_t v = \varepsilon^{-1}\Delta v - \partial_t w, \quad \Delta w = 0, & x \in \mathbb{R}^N_+,\ \ t > 0, \\ v = 0, \quad \partial_t w - \partial_{x_N} w = \partial_{x_N} v, & x \in \partial\mathbb{R}^N_+,\ \ t > 0, \\ v(x,0) = \Phi(x) := \varphi(x) - [S(0)\varphi_b](x), & x \in \mathbb{R}^N_+, \\ w(x,0) = \varphi_b(x'), & x = (x',0) \in \partial\mathbb{R}^N_+. \end{cases}$$

$$\Rightarrow \qquad w(x', x_N, t) = \underline{[S(t)\varphi_b](x) + \int_0^t [S(t-s)\partial_{x_N} v(s)](x)\, ds}.$$

Construct a sol.

$$(H)\quad \begin{cases} \varepsilon\partial_t u_\varepsilon - \Delta u_\varepsilon = 0, & x\in\mathbb{R}^N_+,\ \ t>0,\\ \partial_t u_\varepsilon + \partial_\nu u_\varepsilon = 0, & x\in\partial\mathbb{R}^N_+,\ \ t>0,\\ u_\varepsilon(x,0)=\varphi(x), & x\in\mathbb{R}^N_+,\\ u_\varepsilon(x,0)=\varphi_b(x'), & x=(x',0)\in\partial\mathbb{R}^N_+. \end{cases}$$

Consider the following system

$$(S)\quad \begin{cases} \partial_t v = \varepsilon^{-1}\Delta v - F_1[\varphi_b] - F_2[v], \quad \Delta w = 0, & x\in\mathbb{R}^N_+,\ \ t>0,\\ v=0, \quad \partial_t w - \partial_{x_N} w = \partial_{x_N} v, & x\in\partial\mathbb{R}^N_+,\ \ t>0,\\ v(x,0)=\Phi(x), & x\in\mathbb{R}^N_+,\\ w(x,0)=\varphi_b(x'), & x=(x',0)\in\partial\mathbb{R}^N_+. \end{cases}$$

Here

$$F_1[\varphi_b](x,t) := \partial_t[S(t)\varphi_b](x),$$

$$F_2[v](x,t) := \partial_t\left(\int_0^t\int_{\mathbb{R}^{N-1}} P(x'-y', x_N+t-s)\partial_{x_N} v(y',0,s)\,dy'\,ds\right),$$

$$\Phi(x) := \varphi(x) - [S(0)\varphi_b](x).$$

$$(S) \quad \begin{cases} \partial_t v = \varepsilon^{-1}\Delta v - F_1[\varphi_b] - F_2[v], \ \ \Delta w = 0, & x \in \mathbb{R}^N_+, \ \ t > 0, \\ v = 0, \quad \partial_t w - \partial_{x_N} w = \partial_{x_N} v, & x \in \partial\mathbb{R}^N_+, \ \ t > 0, \\ v(x,0) = \Phi(x), & x \in \mathbb{R}^N_+, \\ w(x,0) = \varphi_b(x'), & x = (x',0) \in \partial\mathbb{R}^N_+. \end{cases}$$

Def. 1

φ : mesurable func. in $\mathbb{R}^N_+$, φ_b : mesurable func. in $\mathbb{R}^{N-1}$, $0 < T \le \infty$.

(v, w) : sol. of (S) in $\mathbb{R}^N_+ \times (0, T)$ $\Leftrightarrow$

(i) $v, \ \partial_{x_N} v, \ w \in C(\overline{\mathbb{R}^N_+} \times (0, T))$;

(ii) v and w satisfy

$$v(x,t) = [e^{\varepsilon^{-1} t \Delta_D}\Phi](x) - \int_0^t [e^{\varepsilon^{-1}(t-s)\Delta_D}(F_1[\varphi_b](s) + F_2[v](s))](x)\, ds,$$

$$w(x,t) = [S(t)\varphi_b](x) + \int_0^t [S(t-s)\partial_{x_N} v(s)](x)\, ds,$$

for $x \in \overline{\mathbb{R}^N_+}$ and $t \in (0, T)$.

Furthermore, we say that $u_\varepsilon := v + w$ is sol. of (H) in $\mathbb{R}^N_+ \times (0, T)$.

Main results 1

$$(S)\quad \begin{cases} \partial_t v = \varepsilon^{-1}\Delta v - F_1[\varphi_b] - F_2[v], \quad \Delta w = 0, & x \in \mathbb{R}^N_+, \quad t > 0, \\ v = 0, \quad \partial_t w - \partial_{x_N} w = \partial_{x_N} v, & x \in \partial\mathbb{R}^N_+, \quad t > 0, \\ v(x,0) = \Phi(x), & x \in \mathbb{R}^N_+, \\ w(x,0) = \varphi_b(x'), & x = (x',0) \in \partial\mathbb{R}^N_+. \end{cases}$$

Thm. 1

$N \geq 2$, $\varepsilon \in (0,1)$, $\varphi \in L^\infty(\mathbb{R}^N_+)$, $\varphi_b \in L^\infty(\mathbb{R}^{N-1})$.

$\Rightarrow$ $\exists!(v_\varepsilon, w_\varepsilon)$: global-in-time sol. of (S) s.t., for $^\forall T > 0$,

$$\sup_{0<t<T} \left[\|v_\varepsilon(t)\|_{L^\infty} + (\varepsilon^{-1}t)^{\frac{1}{2}} \|\partial_{x_N} v_\varepsilon(t)\|_{L^\infty} + \|w_\varepsilon(t)\|_{L^\infty} \right] < \infty.$$

Main results 1

$$(S)\quad \begin{cases} \partial_t v = \varepsilon^{-1}\Delta v - F_1[\varphi_b] - F_2[v], \quad \Delta w = 0, & x \in \mathbb{R}^N_+, \quad t > 0, \\ v = 0, \quad \partial_t w - \partial_{x_N} w = \partial_{x_N} v, & x \in \partial\mathbb{R}^N_+, \quad t > 0, \\ v(x,0) = \Phi(x), & x \in \mathbb{R}^N_+, \\ w(x,0) = \varphi_b(x'), & x = (x',0) \in \partial\mathbb{R}^N_+. \end{cases}$$

Thm. 1

$N \geq 2$, $\varepsilon \in (0,1)$, $\varphi \in L^\infty(\mathbb{R}^N_+)$, $\varphi_b \in L^\infty(\mathbb{R}^{N-1})$.

$\Rightarrow \quad \exists!(v_\varepsilon, w_\varepsilon)$: global-in-time sol. of (S) s.t., for $^\forall T > 0$,

$$\sup_{0<t<T} \left[\|v_\varepsilon(t)\|_{L^\infty} + (\varepsilon^{-1}t)^{\frac{1}{2}} \|\partial_{x_N} v_\varepsilon(t)\|_{L^\infty} + \|w_\varepsilon(t)\|_{L^\infty} \right] < \infty.$$

Furthermore, for $^\forall \tau > 0$, the following holds:

(a) $\displaystyle \lim_{\varepsilon\to 0} \sup_{0<t<\tau} t^{\frac{1}{2}} \|v_\varepsilon(t)\|_{L^\infty(\mathbb{R}^{N-1}\times(0,L))} = 0$ for any $L > 0$;

(b) $\displaystyle \lim_{\varepsilon\to 0} \sup_{0<t<\tau} \|w_\varepsilon(t) - S(t)\varphi_b\|_{L^\infty} = 0.$

Main results 1

$$(H) \quad \begin{cases} \varepsilon\partial_t u_\varepsilon - \Delta u_\varepsilon = 0, & x \in \mathbb{R}^N_+, \ \ t > 0, \\ \partial_t u_\varepsilon + \partial_\nu u_\varepsilon = 0, & x \in \partial\mathbb{R}^N_+, \ \ t > 0, \\ u_\varepsilon(x,0) = \varphi(x), & x \in \mathbb{R}^N_+, \\ u_\varepsilon(x,0) = \varphi_b(x'), & x = (x',0) \in \partial\mathbb{R}^N_+. \end{cases}$$

Cor. 1

$(v_\varepsilon, w_\varepsilon)$: sol. of (S) given in Thm. 1.

$\Rightarrow$ $u_\varepsilon := v_\varepsilon + w_\varepsilon$: classical sol. of (H) and it satisfies

$$\sup_{\tau_1<t<\tau_2} \|u_\varepsilon(t) - S(t)\varphi_b\|_{L^\infty(K)} \le C\varepsilon^{\frac{1}{2}}, \qquad {}^\forall\varepsilon \in (0,1),$$

for any compact set $K \subset \overline{\mathbb{R}^N_+}$ and $0 < \tau_1 < \tau_2 < \infty$. Furthermore, this rate is optimal.

Main results 1

$$(H)\quad \begin{cases} \varepsilon\partial_t u_\varepsilon - \Delta u_\varepsilon = 0, & x\in\mathbb{R}^N_+,\ \ t>0,\\ \partial_t u_\varepsilon + \partial_\nu u_\varepsilon = 0, & x\in\partial\mathbb{R}^N_+,\ \ t>0,\\ u_\varepsilon(x,0)=\varphi(x), & x\in\mathbb{R}^N_+,\\ u_\varepsilon(x,0)=\varphi_b(x'), & x=(x',0)\in\partial\mathbb{R}^N_+. \end{cases}$$

Cor. 1

$(v_\varepsilon, w_\varepsilon)$: sol. of (S) given in Thm. 1.

$\Rightarrow$ $u_\varepsilon := v_\varepsilon + w_\varepsilon$: classical sol. of (H) and it satisfies

$$\sup_{\tau_1<t<\tau_2} \|u_\varepsilon(t) - S(t)\varphi_b\|_{L^\infty(K)} \le C\varepsilon^{\frac{1}{2}}, \qquad {}^\forall\varepsilon\in(0,1),$$

for any compact set $K\subset\overline{\mathbb{R}^N_+}$ and $0<\tau_1<\tau_2<\infty$. Furthermore, this rate is optimal.

- $u(x,t) := [S(t)\varphi_b](x)$: classical sol. of

$$(L)\quad \begin{cases} -\Delta u = 0 & \text{in } \mathbb{R}^N_+\times(0,\infty),\\ \partial_t u + \partial_\nu u = 0 & \text{on } \partial\mathbb{R}^N_+\times(0,\infty), \end{cases} \quad \text{with} \quad u(x,0)=\varphi_b(x')\ge 0 \text{ on } \partial\mathbb{R}^N_+.$$

$\therefore$ u_ε : sol. of (H) $\to$ u : sol. of (L) in $L^\infty(K)$ as $\varepsilon\to 0$.

Outline of proofs

- Contraction mapping thm related to the function v

Let $T > 0$ and $\varepsilon \in (0,1)$. Set

$$X_T := \left\{ v, \partial_{x_N} v \in C(\overline{\mathbb{R}^N_+} \times (0,T)) \,:\, \|v\|_{X_T} < \infty \right\}, \quad \|v\|_{X_T} := \sup_{0<t<T} E[v](t),$$

where

$$E[v](t) := \|v(t)\|_{L^\infty} + (\varepsilon^{-1} t)^{\frac{1}{2}} \|\partial_{x_N} v(t)\|_{L^\infty}.$$

Then X_T is a Banach space equipped with the norm $\|\cdot\|_{X_T}$.

Outline of proofs

- Contraction mapping thm related to the function v

Let $T > 0$ and $\varepsilon \in (0,1)$. Set

$$X_T := \left\{ v, \partial_{x_N} v \in C(\overline{\mathbb{R}^N_+} \times (0,T)) \,:\, \|v\|_{X_T} < \infty \right\}, \quad \|v\|_{X_T} := \sup_{0<t<T} E[v](t),$$

where

$$E[v](t) := \|v(t)\|_{L^\infty} + (\varepsilon^{-1} t)^{\frac{1}{2}} \|\partial_{x_N} v(t)\|_{L^\infty}.$$

Then X_T is a Banach space equipped with the norm $\|\cdot\|_{X_T}$.

Applying the Banach contraction mapping principle in X_T, we find a fixed point of

$$Q[v](t) := e^{\varepsilon^{-1} t \Delta_D} \Phi - D_\varepsilon[\varphi_b](t) - \tilde{D}_\varepsilon[v](t)$$

where

$$D_\varepsilon[\varphi_b](x,t) := \int_0^t [e^{\varepsilon^{-1}(t-s)\Delta_D} F_1[\varphi_b](s)](x)\, ds,$$

$$\tilde{D}_\varepsilon[v](x,t) := \int_0^t [e^{\varepsilon^{-1}(t-s)\Delta_D} F_2[v](s)](x)\, ds.$$

Lem. 1

Let $\phi \in L^\infty(\mathbb{R}^N_+)$. Then

$$\sup_{t>0} \|e^{t\Delta_D}\phi\|_{L^\infty} + \sup_{t>0} t^{\frac{1}{2}} \|\partial_{x_N}[e^{t\Delta_D}\phi]\|_{L^\infty} \le 2\|\phi\|_{L^\infty}$$

In addition, for any $L > 0$,

$$\sup_{t>0} t^{\frac{1}{2}} \|e^{\varepsilon^{-1}t\Delta_D}\phi\|_{L^\infty(\mathbb{R}^{N-1}\times(0,L))} \le C\varepsilon^{\frac{1}{2}}.$$

Lem. 1

Let $\phi \in L^\infty(\mathbb{R}^N_+)$. Then

$$\sup_{t>0} \|e^{t\Delta_D}\phi\|_{L^\infty} + \sup_{t>0} t^{\frac{1}{2}} \|\partial_{x_N}[e^{t\Delta_D}\phi]\|_{L^\infty} \le 2\|\phi\|_{L^\infty}$$

In addition, for any $L > 0$,

$$\sup_{t>0} t^{\frac{1}{2}} \|e^{\varepsilon^{-1}t\Delta_D}\phi\|_{L^\infty(\mathbb{R}^{N-1}\times(0,L))} \le C\varepsilon^{\frac{1}{2}}.$$

Key points

- $\displaystyle\int_{\mathbb{R}^N_+} |\partial^j_{x_N}\Gamma_D(x,y,t)|\, dy \le t^{-\frac{j}{2}}, \qquad x \in \overline{\mathbb{R}^N_+}, \quad t > 0, \quad j = 0, 1.$
- For any $L > 0$,

$$\int_{\mathbb{R}^N_+} \Gamma_D(x,y,\varepsilon^{-1}t)\, dy = 2(4\pi\varepsilon^{-1}t)^{-\frac{1}{2}} \int_0^{x_N} \exp\left(-\frac{\varepsilon\eta^2}{4t}\right) d\eta$$
$$\le 2(4\pi\varepsilon^{-1}t)^{-\frac{1}{2}} L \le C(\varepsilon^{-1}t)^{-\frac{1}{2}}$$

for $x \in \mathbb{R}^{N-1} \times (0, L)$, $t > 0$ and $\varepsilon > 0$.

Lem. 2

Let $\psi \in L^\infty(\mathbb{R}^{N-1})$. Then ${}^\exists C > 0$ s.t.

$$\|D_\varepsilon[\psi]\|_{X_{T_*}} \le C T_*^{\frac{1}{4}}(1 + T_*^{\frac{3}{4}})|\psi|_{L^\infty}$$

for any $T_* > 0$. In addition,

$$\lim_{\varepsilon \to 0} \sup_{t \in (0, T_1)} \|D_\varepsilon[\psi](t)\|_{L^\infty(\mathbb{R}^{N-1} \times (0,L))} = 0$$

for any $T_1 > 0$ and $L > 0$. Furthermore, $D_\varepsilon[\psi]$ and $\partial_{x_N} D_\varepsilon[\psi]$ are bounded smooth in $\overline{\mathbb{R}^N_+} \times (T, \infty)$ for any $T > 0$.

Lem. 3

${}^\exists T_* = T_*(N) > 0$ s.t.

$$\|\tilde{D}_\varepsilon[v]\|_{X_{T_*}} \le \frac{1}{4}\|v\|_{X_{T_*}}$$

for $v \in X_{T_*}$ and $\varepsilon \in (0,1)$. Furthermore, $\tilde{D}_\varepsilon[v]$ and $\partial_{x_N} \tilde{D}_\varepsilon[v]$ are bounded smooth in $\overline{\mathbb{R}^N_+} \times (\tau, T_*)$ for any $0 < \tau < T_*$.

3. The exterior of the unit ball

Fila-Ishige-K.-Lankeit (Asymptot. Anal. ('19)),

Fila-Ishige-K.-Lankeit (Discrete Contin. Dyn. Syst. ('20))

Construct a sol.

$$(H) \quad \begin{cases} \varepsilon\partial_t u_\varepsilon - \Delta u_\varepsilon = 0, & x \in \Omega, \quad t > 0, \\ \partial_t u_\varepsilon + \partial_\nu u_\varepsilon = 0, & x \in \partial\Omega, \quad t > 0, \\ u_\varepsilon(x,0) = \varphi(x), & x \in \Omega, \\ u_\varepsilon(x,0) = \varphi_b(x), & x \in \partial\Omega. \end{cases}$$

Here $N > 2$, $p > 1$ and $\Omega := \{x \in \mathbb{R}^N : |x| > 1\}$.

Construct a sol.

$$(H)\quad \begin{cases} \varepsilon\partial_t u_\varepsilon - \Delta u_\varepsilon = 0, & x \in \Omega, \quad t > 0, \\ \partial_t u_\varepsilon + \partial_\nu u_\varepsilon = 0, & x \in \partial\Omega, \quad t > 0, \\ u_\varepsilon(x,0) = \varphi(x), & x \in \Omega, \\ u_\varepsilon(x,0) = \varphi_b(x), & x \in \partial\Omega. \end{cases}$$

Laplace eq. ($\varepsilon = 0$)

$$(L)\quad \begin{cases} -\Delta u = 0, & x \in \Omega, \quad t > 0, \\ \partial_t u + \partial_\nu u = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = \varphi_b(x), & x \in \partial\Omega. \end{cases}$$

Construct a sol.

$$(H)\quad \begin{cases} \varepsilon\partial_t u_\varepsilon - \Delta u_\varepsilon = 0, & x \in \Omega,\quad t > 0, \\ \partial_t u_\varepsilon + \partial_\nu u_\varepsilon = 0, & x \in \partial\Omega,\quad t > 0, \\ u_\varepsilon(x,0) = \varphi(x), & x \in \Omega, \\ u_\varepsilon(x,0) = \varphi_b(x), & x \in \partial\Omega. \end{cases}$$

Laplace eq. $(\varepsilon = 0)$

$$(L)\quad \begin{cases} -\Delta u = 0, & x \in \Omega,\quad t > 0, \\ \partial_t u + \partial_\nu u = 0, & x \in \partial\Omega,\quad t > 0, \\ u(x,0) = \varphi_b(x), & x \in \partial\Omega. \end{cases}$$

$P = P(x,y)$: Poisson kernel on B_1,

$K = K(x,y)$: Kelvin transform of P

$$K(x,y) := |x|^{-(N-2)} P\left(\frac{x}{|x|^2}, y\right), \qquad x \in \overline{\Omega}, \quad y \in \partial\Omega.$$

$\Rightarrow \quad K$: Poisson kernel on Ω.

Construct a sol.

$$(H)\quad \begin{cases} \varepsilon\partial_t u_\varepsilon - \Delta u_\varepsilon = 0, & x\in\Omega,\quad t>0,\\ \partial_t u_\varepsilon + \partial_\nu u_\varepsilon = 0, & x\in\partial\Omega,\quad t>0,\\ u_\varepsilon(x,0)=\varphi(x), & x\in\Omega,\\ u_\varepsilon(x,0)=\varphi_b(x), & x\in\partial\Omega. \end{cases}$$

Laplace eq. ($\varepsilon = 0$)

$$(L)\quad \begin{cases} -\Delta u = 0, & x\in\Omega,\quad t>0,\\ \partial_t u + \partial_\nu u = 0, & x\in\partial\Omega,\quad t>0,\\ u(x,0)=\varphi_b(x), & x\in\partial\Omega. \end{cases}$$

$K = K(x,y)$: Poisson kenrel on Ω

$$K(x,y) := |x|^{-(N-2)} P\left(\frac{x}{|x|^2}, y\right), \qquad x\in\overline{\Omega},\quad y\in\partial\Omega.$$

$\Rightarrow$ $\mathcal{K}(x,y,t) := K(e^t x, y)$: fundamental sol. of (L).

Construct a sol.

$$(H) \quad \begin{cases} \varepsilon\partial_t u_\varepsilon - \Delta u_\varepsilon = 0, & x \in \Omega, \quad t > 0, \\ \partial_t u_\varepsilon + \partial_\nu u_\varepsilon = 0, & x \in \partial\Omega, \quad t > 0, \\ u_\varepsilon(x, 0) = \varphi(x), & x \in \Omega, \\ u_\varepsilon(x, 0) = \varphi_b(x), & x \in \partial\Omega. \end{cases}$$

Laplace eq. ($\varepsilon = 0$)

$$(L) \quad \begin{cases} -\Delta u = 0, & x \in \Omega, \quad t > 0, \\ \partial_t u + \partial_\nu u = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x, 0) = \varphi_b(x), & x \in \partial\Omega. \end{cases}$$

$K = K(x, y)$: Poisson kenrel on Ω

$$K(x, y) := |x|^{-(N-2)} P\left(\frac{x}{|x|^2}, y\right), \qquad x \in \overline{\Omega}, \quad y \in \partial\Omega.$$

$\Rightarrow$ $\mathcal{K}(x, y, t) := K(e^t x, y)$: fundamental sol. of (L).

φ : nonnegative measurable func. on $\partial\Omega$

$$[S(t)\varphi_b](x) := \int_{\partial\Omega} \mathcal{K}(x, y, t)\varphi_b(y)\, d\sigma_y \equiv \int_{\partial\Omega} K(e^t x, y)\varphi_b(y)\, d\sigma_y, \quad x \in \overline{\Omega}.$$

$$(S)\quad \begin{cases} \partial_t v = \varepsilon^{-1}\Delta v - F_1[\varphi_b] - F_2[v], \quad \Delta w = 0, & x \in \Omega, \ \ t > 0, \\ v = 0, \quad \partial_t w + \partial_\nu w = -\partial_\nu v, & x \in \partial\Omega, \ \ t > 0, \\ v(x,0) = \Phi(x), & x \in \Omega, \\ w(x,0) = \varphi_b(x), & x \in \partial\Omega. \end{cases}$$

Def. 2

φ : mesurable func. in Ω, φ_b : mesurable func. on $\partial\Omega$, $0 < T \le \infty$.

(v, w) : sol. of (S) in $\Omega \times (0, T)$ $\Leftrightarrow$

(i) $v,\ \nabla v,\ w \in C(\overline{\Omega} \times (0, T))$;

(ii) v and w satisfy

$$v(x,t) = [e^{\varepsilon^{-1} t \Delta_D}\Phi](x) - \int_0^t [e^{\varepsilon^{-1}(t-s)\Delta_D}(F_1[\varphi_b](s) + F_2[v](s))](x)\, ds,$$

$$w(x,t) = [S(t)\varphi_b](x) - \int_0^t [S(t-s)\partial_\nu v(s)](x)\, ds.$$

for $x \in \overline{\Omega}$ and $t \in (0, T)$.

Furthermore, we say that $u_\varepsilon := v + w$ is sol. of (H) in $\Omega \times (0, T)$.

Main results 2

$$(S)\quad \begin{cases} \partial_t v = \varepsilon^{-1}\Delta v - F_1[\varphi_b] - F_2[v], \ \ \Delta w = 0, & x \in \Omega, \ \ t > 0, \\ v = 0, \quad \partial_t w + \partial_\nu w = -\partial_\nu v, & x \in \partial\Omega, \ \ t > 0, \\ v(x,0) = \Phi(x), & x \in \Omega, \\ w(x,0) = \varphi_b(x), & x \in \partial\Omega. \end{cases}$$

Thm. 2

$N \geq 3$, $\varepsilon \in (0,1)$, $\sup_{x\in\Omega}(1+|x|)^{N-2}|\varphi(x)| < \infty$, $\varphi_b \in C^{1,\theta}(\partial\Omega)$ $(\theta \in (0,1))$.

$\Rightarrow$ $\exists!(v_\varepsilon, w_\varepsilon)$: global-in-time sol. of (S) s.t., for ${}^\forall T > 0$,

$$\sup_{0<t<T}\left[\|v_\varepsilon(t)\|_{L^\infty} + (\varepsilon^{-1}t)^{\frac{1}{2}}\|\partial_{x_N} v_\varepsilon(t)\|_{L^\infty} + \|w_\varepsilon(t)\|_{L^\infty}\right] < \infty.$$

Main results 2

$$(S)\quad \begin{cases} \partial_t v = \varepsilon^{-1}\Delta v - F_1[\varphi_b] - F_2[v], \ \ \Delta w = 0, & x \in \Omega, \ \ t > 0, \\ v = 0, \quad \partial_t w + \partial_\nu w = -\partial_\nu v, & x \in \partial\Omega, \ \ t > 0, \\ v(x,0) = \Phi(x), & x \in \Omega, \\ w(x,0) = \varphi_b(x), & x \in \partial\Omega. \end{cases}$$

Thm. 2

$N \geq 3$, $\varepsilon \in (0,1)$, $\sup_{x\in\Omega}(1+|x|)^{N-2}|\varphi(x)| < \infty$, $\varphi_b \in C^{1,\theta}(\partial\Omega)$ $(\theta \in (0,1))$.

$\Rightarrow$ $\exists!(v_\varepsilon, w_\varepsilon)$: global-in-time sol. of (S) s.t., for ${}^\forall T > 0$,

$$\sup_{0<t<T}\left[\|v_\varepsilon(t)\|_{L^\infty} + (\varepsilon^{-1}t)^{\frac{1}{2}}\|\partial_{x_N} v_\varepsilon(t)\|_{L^\infty} + \|w_\varepsilon(t)\|_{L^\infty}\right] < \infty.$$

Furthermore, for ${}^\forall\tau > 0$, the following holds:

$$(a)\ \sup_{\tau<t<T}\|v_\varepsilon(t)\|_{L^\infty} \leq C\varepsilon^{\frac{\alpha}{2}}; \qquad (b)\ \sup_{0<t<\tau}\|w_\varepsilon(t) - S(t)\varphi_b\|_{L^\infty} \leq C\varepsilon^{\frac{\alpha}{2}}.$$

Here

$$\alpha = 1 \quad \text{if} \quad N = 3, \qquad \alpha \in (1,2) \quad \text{if} \quad N \geq 4.$$

Main results 2

$$
(S)\quad \begin{cases} \partial_t v = \varepsilon^{-1}\Delta v - F_1[\varphi_b] - F_2[v], \ \ \Delta w = 0, & x \in \Omega, \ \ t > 0, \\ v = 0, \quad \partial_t w + \partial_\nu w = -\partial_\nu v, & x \in \partial\Omega, \ \ t > 0, \\ v(x,0) = \Phi(x), & x \in \Omega, \\ w(x,0) = \varphi_b(x), & x \in \partial\Omega. \end{cases}
$$

Cor. 2

$(v_\varepsilon, w_\varepsilon)$: sol. of (S) given in Thm. 2.

$\Rightarrow$ $u_\varepsilon := v_\varepsilon + w_\varepsilon$: classical sol. of (H) and it satisfies

$$
\sup_{\tau_1 < t < \tau_2} \|u_\varepsilon(t) - S(t)\varphi_b\|_{L^\infty} \le C\varepsilon^{\frac{\alpha}{2}}, \qquad {}^\forall\varepsilon \in (0,1),
$$

for any $0 < \tau_1 < \tau_2 < \infty$.

Main results 2

$$(S)\quad \begin{cases} \partial_t v = \varepsilon^{-1}\Delta v - F_1[\varphi_b] - F_2[v], \ \ \Delta w = 0, & x \in \Omega, \ \ t > 0, \\ v = 0, \quad \partial_t w + \partial_\nu w = -\partial_\nu v, & x \in \partial\Omega, \ \ t > 0, \\ v(x,0) = \Phi(x), & x \in \Omega, \\ w(x,0) = \varphi_b(x), & x \in \partial\Omega. \end{cases}$$

Cor. 2

$(v_\varepsilon, w_\varepsilon)$: sol. of (S) given in Thm. 2.

$\Rightarrow$ $u_\varepsilon := v_\varepsilon + w_\varepsilon$: classical sol. of (H) and it satisfies

$$\sup_{\tau_1 < t < \tau_2} \|u_\varepsilon(t) - S(t)\varphi_b\|_{L^\infty} \le C\varepsilon^{\frac{\alpha}{2}}, \qquad {}^\forall\varepsilon \in (0,1),$$

for any $0 < \tau_1 < \tau_2 < \infty$.

Remark

$\exists\varphi \in L^\infty$ s.t. u_ε : sol. of (S) with $\varphi_b \equiv 0$ satisfies

$$u_\varepsilon(x,t) \ge C\varepsilon^{\frac{N}{2}-1}, \qquad {}^\forall(x,t) \in K, \quad {}^\forall\varepsilon \in (0,1),$$

where K : compact set of $\Omega \times (0,\infty)$.

Summary

- Large diffusion limit for the heat eq. with DBC

$$(H)\ \begin{cases} \varepsilon\partial_t u_\varepsilon - \Delta u_\varepsilon = 0 & \text{in } \Omega\times(0,\infty), \\ \partial_t u_\varepsilon + \partial_\nu u_\varepsilon = 0 & \text{on } \partial\Omega\times(0,\infty), \end{cases} \quad \text{with} \quad \begin{cases} u_\varepsilon(x,0) = \varphi(x) & \text{in } \Omega, \\ u_\varepsilon(x,0) = \varphi_b(x) & \text{on } \partial\Omega. \end{cases}$$

$$(L)\ \begin{cases} \Delta u = 0 & \text{in } \Omega\times(0,\infty), \\ \partial_t u + \partial_\nu u = 0 & \text{on } \partial\Omega\times(0,\infty), \end{cases} \quad \text{with} \quad u(x,0) = \varphi_b(x') \text{ on } \partial\Omega.$$

u_ε : sol. of (H) $\to$ u : sol. of (L) in $L^\infty(K)$ (or $L^\infty(\Omega)$) as $\varepsilon \to 0$.

Summary

- Large diffusion limit for the heat eq. with DBC

$$(H)\ \begin{cases} \varepsilon\partial_t u_\varepsilon - \Delta u_\varepsilon = 0 & \text{in } \Omega\times(0,\infty), \\ \partial_t u_\varepsilon + \partial_\nu u_\varepsilon = 0 & \text{on } \partial\Omega\times(0,\infty), \end{cases} \quad \text{with} \quad \begin{cases} u_\varepsilon(x,0) = \varphi(x) & \text{in } \Omega, \\ u_\varepsilon(x,0) = \varphi_b(x) & \text{on } \partial\Omega. \end{cases}$$

$$(L)\ \begin{cases} \Delta u = 0 & \text{in } \Omega\times(0,\infty), \\ \partial_t u + \partial_\nu u = 0 & \text{on } \partial\Omega\times(0,\infty), \end{cases} \quad \text{with} \quad u(x,0) = \varphi_b(x') \text{ on } \partial\Omega.$$

u_ε : sol. of (H) $\to$ u : sol. of (L) in $L^\infty(K)$ (or $L^\infty(\Omega)$) as $\varepsilon\to 0$.

Future work

- Construct the fundamental sol. of the heat eq. with DBC and its application for nonlinear problems.
- Large diffusion limit for the boundary reaction.

Summary

- Large diffusion limit for the heat eq. with DBC

$$(H)\ \begin{cases} \varepsilon\partial_t u_\varepsilon - \Delta u_\varepsilon = 0 & \text{in } \Omega\times(0,\infty), \\ \partial_t u_\varepsilon + \partial_\nu u_\varepsilon = 0 & \text{on } \partial\Omega\times(0,\infty), \end{cases} \quad \text{with} \quad \begin{cases} u_\varepsilon(x,0) = \varphi(x) & \text{in } \Omega, \\ u_\varepsilon(x,0) = \varphi_b(x) & \text{on } \partial\Omega. \end{cases}$$

$$(L)\ \begin{cases} \Delta u = 0 & \text{in } \Omega\times(0,\infty), \\ \partial_t u + \partial_\nu u = 0 & \text{on } \partial\Omega\times(0,\infty), \end{cases} \quad \text{with} \quad u(x,0) = \varphi_b(x') \text{ on } \partial\Omega.$$

u_ε : sol. of (H) $\to$ u : sol. of (L) in $L^\infty(K)$ (or $L^\infty(\Omega)$) as $\varepsilon \to 0$.

Future work

- Construct the fundamental sol. of the heat eq. with DBC and its application for nonlinear problems.
- Large diffusion limit for the boundary reaction.

Thank you for your attention!

Outline of proof of Lem. 2

$$D_\varepsilon[\psi](x,t) = \int_0^t [e^{\varepsilon^{-1}(t-s)\Delta_D} F_1[\psi](s)](x)\, ds,$$

$$F_1[\psi](x,t) = \int_{\mathbb{R}^{N-1}} \partial_t P(x'-y', x_N + t)\psi(y')\, dy'.$$

Outline of proof of Lem. 2

$$D_\varepsilon[\psi](x,t) = \int_0^t [e^{\varepsilon^{-1}(t-s)\Delta_D} F_1[\psi](s)](x)\,ds,$$
$$F_1[\psi](x,t) = \int_{\mathbb{R}^{N-1}} \partial_t P(x'-y', x_N+t)\psi(y')\,dy'.$$

Since

$$|\partial_t P(x', x_N+t)| \le C(x_N+t)^{-1} P(x', x_N+t), \qquad x \in \overline{\mathbb{R}^N_+}, \quad t>0,$$

we have

$$\begin{aligned} |F_1[\psi](y_N,s)|_{L^\infty} &\le C(y_N+s)^{-1}|S_2(s+y_N)\psi|_{L^\infty} \\ &\le C|\psi|_{L^\infty} \begin{cases} y_N^{-\frac{3}{4}} s^{-\frac{1}{4}} & \text{for} \quad 0 \le y_N \le 1, \\ 1 & \text{for} \quad y_N > 1, \end{cases} \end{aligned}$$

for $y_N \in [0,\infty)$ and $s>0$.

Then it follows from

$$\Gamma_D(x,y,t) = \Gamma_{N-1}(x'-y',t)\Big(\Gamma_1(x_N-y_N,t) - \Gamma_1(x_N+y_N,t)\Big)$$

that

$$\begin{aligned}
|D_\varepsilon[\psi](x,t)| &\le \int_0^t \int_{\mathbb{R}^N_+} \Gamma_D(x,y,\varepsilon^{-1}(t-s))|F_1[\psi](y,s)|\,dy\,ds \\
&\le C\int_0^t \int_0^\infty \Gamma_1(x_N-y_N,\varepsilon^{-1}(t-s))|F_1[\psi](y_N,s)|_{L^\infty}\,dy_N\,ds \\
&\le C|\psi|_{L^\infty} \int_0^t \int_0^\infty (\varepsilon^{-1}(t-s))^{-\frac{1}{2}} \exp\left(-\frac{\varepsilon(x_N-y_N)^2}{4(t-s)}\right)(y_N+s)^{-1}\,dy_N\,ds \\
&\le C|\psi|_{L^\infty}\left\{\varepsilon^{\frac{1}{2}}\int_0^t (t-s)^{-\frac{1}{2}}s^{-\frac{1}{4}}\int_0^1 y_N^{-\frac{3}{4}}\,dy_N\,ds + \int_0^t ds\right\} \\
&\le C|\psi|_{L^\infty} t^{\frac{1}{4}}(\varepsilon^{\frac{1}{2}} + t^{\frac{3}{4}})
\end{aligned}$$

for $x \in \overline{\mathbb{R}^N_+}$, $t > 0$ and $\varepsilon > 0$. Here Γ_d is the d-dim Gauss kernel.

Furthermore, applying similar arguments as above with the weighted estimates

$$\int_0^\infty \frac{|x \pm y|}{t}\Gamma_1(x \pm y, t) y^{-\alpha}\, dy \le Ct^{-\frac{\alpha+1}{2}}, \qquad 0 \le \alpha < 1,$$

for $x \ge 0$ and $t > 0$, we see that

$$\begin{aligned}
&|\partial_{x_N} D_\varepsilon[\psi](x,t)| \\
&\le C\int_0^t \int_0^\infty \left(\frac{|x_N - y_N|}{\tau_\varepsilon}\Gamma_1(x_N - y_N, \tau_\varepsilon) + \frac{x_N + y_N}{\tau_\varepsilon}\Gamma_1(x_N + y_N, \tau_\varepsilon)\right) \\
&\qquad\qquad \times |F_1[\psi](y_N, s)|_{L^\infty}\, dy_N\, ds \\
&\le C|\psi|_{L^\infty} \int_0^t s^{-\frac{1}{2}} \int_0^\infty \left(\frac{|x_N - y_N|}{\tau_\varepsilon}\Gamma_1(x_N - y_N, \tau_\varepsilon) + \frac{x_N + y_N}{\tau_\varepsilon}\Gamma_1(x_N + y_N, \tau_\varepsilon)\right) \\
&\qquad\qquad \times y_N^{-\frac{1}{2}}\, dy_N\, ds \\
&\le C|\psi|_{L^\infty} \int_0^t s^{-\frac{1}{2}} \tau_\varepsilon^{-\frac{3}{4}}\, ds \le C\varepsilon^{\frac{3}{4}} t^{-\frac{1}{4}} |\psi|_{L^\infty}
\end{aligned}$$

for $x \in \overline{\mathbb{R}^N_+}$, $t > 0$ and $\varepsilon > 0$, where $\tau_\varepsilon = \varepsilon^{-1}(t-s)$. Thus Lem. 2 holds.

Outline of proof of Thm. 1

Put

$$m = 16\max\{\|\varphi\|_{L^\infty}, |\varphi_b|_{L^\infty}\}.$$

Let $T_* > 0$ be as in Lem. 3 and $v \in X_{T_*}$ with $\|v\|_{X_{T_*}} \le m$. Then, since

$$\|\Phi\|_{L^\infty} \le \|\varphi\|_{L^\infty} + \|S(0)\varphi_b\|_{L^\infty} \le \|\varphi\|_{L^\infty} + |\varphi_b|_{L^\infty},$$

by Lems. 1, 2 and 3, we see that $^{\exists!} v_\varepsilon \in X_{T_*}$ with $\|v_\varepsilon\|_{X_{T_*}} \le m$ s.t.

$$v_\varepsilon = Q[v_\varepsilon] = e^{\varepsilon^{-1} t\Delta_D}\Phi - D_\varepsilon[\varphi_b](t) - \tilde{D}_\varepsilon[v_\varepsilon](t) \quad \text{in} \quad X_{T_*}.$$

In particular,

$$\|v_\varepsilon\|_{X_{T_*}} \le C(\|\varphi\|_{L^\infty} + |\varphi_b|_{L^\infty}).$$

Furthermore, v_ε is bounded smooth in $\overline{\mathbb{R}^N_+} \times (T_1, T_*)$ for any $0 < T_1 < T_*$.

Set

$$w_\varepsilon(x,t) := [S(t)\varphi_b](x) + \int_0^t [S(t-s)\partial_{x_N} v_\varepsilon(s)](x)\,ds$$

for $x \in \overline{\mathbb{R}^N_+}$ and $t \in (0, T_*)$. Then we have

$$\begin{aligned}\|w_\varepsilon(t)\|_{L^\infty} &\le \|S(t)\varphi_b\|_{L^\infty} + \int_0^t \|S(t-s)\partial_{x_N} v(s)\|_{L^\infty}\,ds \\ &\le C(1+T_*^{\frac{1}{2}})(\|\varphi\|_{L^\infty} + |\varphi_b|_{L^\infty}) < \infty, \qquad 0 < t < T_*.\end{aligned}$$

Furthermore, w_ε is bounded smooth in $\overline{\mathbb{R}^N_+} \times (T_1, T_*)$ for any $0 < T_1 < T_*$.

$\Rightarrow$ $(v_\varepsilon, w_\varepsilon)$ is a sol. of (S) in $\mathbb{R}^N_+ \times (0, T_*)$
and satisfies boundedness for any $0 < \tau < T_*$.

Set

$$w_\varepsilon(x,t) := [S(t)\varphi_b](x) + \int_0^t [S(t-s)\partial_{x_N} v_\varepsilon(s)](x)\, ds$$

for $x \in \overline{\mathbb{R}^N_+}$ and $t \in (0, T_*)$. Then we have

$$\begin{aligned} \|w_\varepsilon(t)\|_{L^\infty} &\le \|S(t)\varphi_b\|_{L^\infty} + \int_0^t \|S(t-s)\partial_{x_N} v(s)\|_{L^\infty}\, ds \\ &\le C(1 + T_*^{\frac{1}{2}})(\|\varphi\|_{L^\infty} + |\varphi_b|_{L^\infty}) < \infty, \qquad 0 < t < T_*. \end{aligned}$$

Furthermore, w_ε is bounded smooth in $\overline{\mathbb{R}^N_+} \times (T_1, T_*)$ for any $0 < T_1 < T_*$.

$\Rightarrow$ $(v_\varepsilon, w_\varepsilon)$ is a sol. of (S) in $\mathbb{R}^N_+ \times (0, T_*)$

and satisfies boundedness for any $0 < \tau < T_*$.

T_* is indep. of m & semigroup properties of $e^{t\Delta_D}$ and $S(t)$,

$\Rightarrow$ $(v_\varepsilon, w_\varepsilon)$ is a global-in-time sol. of (S)

and satisfies boundedness for any $\tau > 0$.

Let $(\tilde{v}_\varepsilon, \tilde{w}_\varepsilon)$ be a global-in-time sol. of (S) satisfying $(*)$. Since

$$v_\varepsilon - \tilde{v}_\varepsilon = Q[v_\varepsilon] - Q[\tilde{v}_\varepsilon] = \tilde{D}_\varepsilon[v_\varepsilon - \tilde{v}_\varepsilon] \quad \text{in} \quad X_{T_*},$$

by Lem. 3 we have

$$\|v_\varepsilon - \tilde{v}_\varepsilon\|_{X_{T_*}} \le \frac{1}{4}\|v_\varepsilon - \tilde{v}_\varepsilon\|_{X_{T_*}}.$$

This implies that $v_\varepsilon = \tilde{v}_\varepsilon$ in X_{T_*}. Repeating this argument, we see that $v_\varepsilon = \tilde{v}_\varepsilon$ in X_T for any $T > 0$.

$\Rightarrow \quad (v_\varepsilon, w_\varepsilon)$ is a unique global-in-time sol. of (S).

Let $(\tilde{v}_\varepsilon, \tilde{w}_\varepsilon)$ be a global-in-time sol. of (S) satisfying $(*)$. Since

$$v_\varepsilon - \tilde{v}_\varepsilon = Q[v_\varepsilon] - Q[\tilde{v}_\varepsilon] = \tilde{D}_\varepsilon[v_\varepsilon - \tilde{v}_\varepsilon] \quad \text{in} \quad X_{T_*},$$

by Lem. 3 we have

$$\|v_\varepsilon - \tilde{v}_\varepsilon\|_{X_{T_*}} \leq \frac{1}{4}\|v_\varepsilon - \tilde{v}_\varepsilon\|_{X_{T_*}}.$$

This implies that $v_\varepsilon = \tilde{v}_\varepsilon$ in X_{T_*}. Repeating this argument, we see that $v_\varepsilon = \tilde{v}_\varepsilon$ in X_T for any $T > 0$.

$\Rightarrow \qquad (v_\varepsilon, w_\varepsilon)$ is a unique global-in-time sol. of (S).

It remains to prove assertions (a) and (b). Let $T' > 0$ and $L > 0$. Then we have

$$\|w_\varepsilon(t) - S(t)\varphi_b\|_{L^\infty} \leq \int_0^t \|S(t-s)\partial_{x_N} v_\varepsilon(s)\|_{L^\infty}\, ds \leq C\|v_\varepsilon\|_{X_{T'}}\varepsilon^{\frac{1}{2}} T'^{\frac{1}{2}}$$

for all $t \in (0, T')$. This implies assertion (b).

Let $(\tilde{v}_\varepsilon, \tilde{w}_\varepsilon)$ be a global-in-time sol. of (S) satisfying $(*)$. Since

$$v_\varepsilon - \tilde{v}_\varepsilon = Q[v_\varepsilon] - Q[\tilde{v}_\varepsilon] = \tilde{D}_\varepsilon[v_\varepsilon - \tilde{v}_\varepsilon] \quad \text{in} \quad X_{T_*},$$

by Lem. 3 we have

$$\|v_\varepsilon - \tilde{v}_\varepsilon\|_{X_{T_*}} \le \frac{1}{4}\|v_\varepsilon - \tilde{v}_\varepsilon\|_{X_{T_*}}.$$

This implies that $v_\varepsilon = \tilde{v}_\varepsilon$ in X_{T_*}. Repeating this argument, we see that $v_\varepsilon = \tilde{v}_\varepsilon$ in X_T for any $T > 0$.

$\Rightarrow$ $(v_\varepsilon, w_\varepsilon)$ is a unique global-in-time sol. of (S).

It remains to prove assertions (a) and (b). Let $T' > 0$ and $L > 0$. Then we have

$$\|w_\varepsilon(t) - S(t)\varphi_b\|_{L^\infty} \le \int_0^t \|S(t-s)\partial_{x_N} v_\varepsilon(s)\|_{L^\infty}\, ds \le C\|v_\varepsilon\|_{X_{T'}}\varepsilon^{\frac{1}{2}} T'^{\frac{1}{2}}$$

for all $t \in (0, T')$. This implies assertion (b).

Furthermore, since $\tilde{D}_\varepsilon[v_\varepsilon]$ is given with $F_1[\psi]$ replaced by $F_2[v_\varepsilon]$, by applying a similar argument as in the proof of Lem. 2 to obtain

$$\lim_{\varepsilon \to 0} \sup_{t \in (0, T')} \|\tilde{D}_\varepsilon[v_\varepsilon](t)\|_{L^\infty(\mathbb{R}^{N-1} \times (0,L))} = 0.$$

Thus assertion-(a) holds.