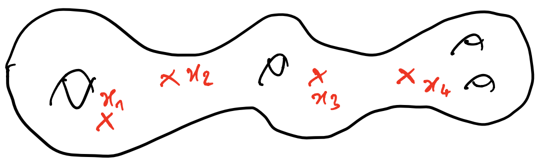
Segal's Axioms and Modular Bootstrap

1) Conformal Field Theory in dim 2

Data: • A Riemann surface



$(M, [g])$

genus $= h$

x_i : marked points
α_i : weights
⇝ "angles of conical singularities"

• Correlations functions:

$$Z_g(x;\alpha) \in \mathbb{C}$$

$x = (x_1, \dots, x_n)$
$\alpha = (\alpha_1, \dots, \alpha_n)$

① Diffeo invariance:

$\psi \in \mathrm{Diff}(M) \Rightarrow$

$$Z_{\psi^* g}(x_1, \dots, x_n; \alpha_1, \dots, \alpha_n) = Z_g(\psi(x_1), \dots, \psi(x_n); \alpha_1, \dots, \alpha_n)$$

② Conformal anomaly

$$\omega \in C^\infty(M) : \quad Z_{e^\omega g}(x,\alpha) = Z_g(x,\alpha)\, e^{c A_g(\omega) - \sum_{i=1}^{n} \Delta_i \omega(x_i)}$$

where $A_g(\omega) = \dfrac{1}{96\pi} \displaystyle\int_M \left(|d\omega|_g^2 + 2 R_g \omega\right) d\mathrm{vol}_g$ <u>Anomaly</u>

R_g = Scalar curvature

c = central charge of the CFT

Δ_i = Function of α_i called conformal weights

<u>Rem</u>: $Z_g(x,\alpha)$ can be viewed as sections of a line bundle over moduli space of Riemann surfaces

<u>Physical meaning</u>:

Correlation functions are expected values of random fields

$\rightarrow$ $\boxed{\text{Feynman integrals}}$

$$Z_g(x,\alpha) = \int_{E(M)} \prod_{i=1}^{n} e^{\alpha_i \varphi(x_i)} \underbrace{e^{-S_g(\varphi)} d\varphi}_{\text{measure on } E}$$

$S_g : C^\infty(M) \rightarrow \mathbb{C}$ action

$e^{\alpha_i \varphi(x_i)}$: primary fields

$E(M)$: space of fields on M

Problem in theoretical physics:

- give an expression for correlation functions
- difficulty: hard to make sense to Feynman integrals

Tools in physics:

- use symmetries of the model
- use representation theory of

Virasoro Algebras

central charge

L_m, $m \in \mathbb{Z}$,

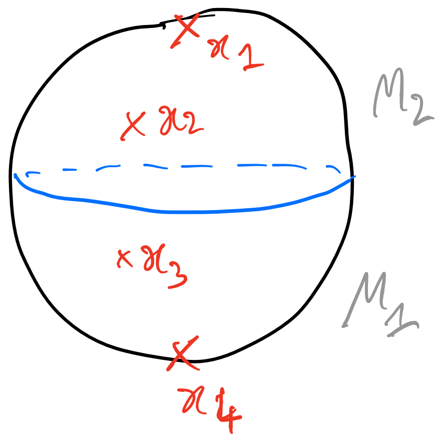
$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3-m)\delta_{m+n}$$

- it is a Central extension of Witt algebra

$$L_n = -z^{n+1}\partial_z \quad , \; n \in \mathbb{Z}$$

Heuristics in physics :

$M = S^2$
with 4 marked
points



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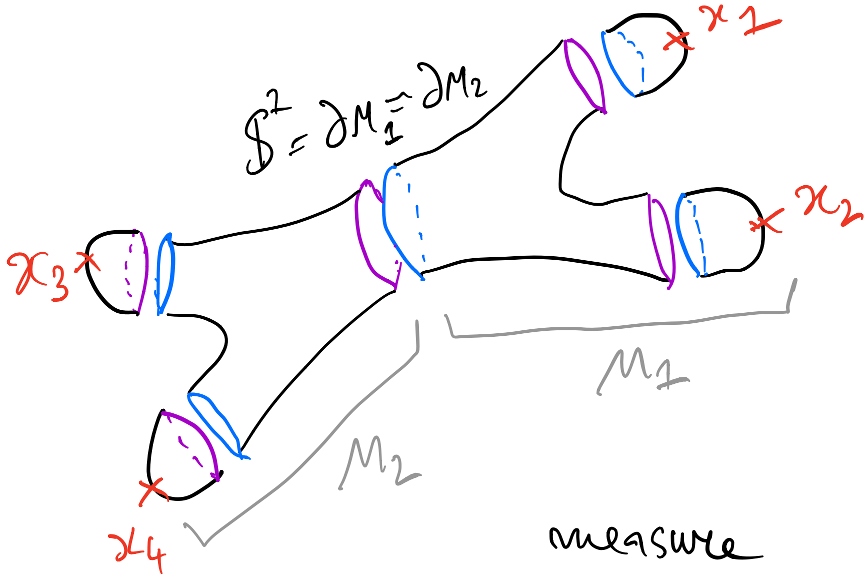


$$Z_{M,g}(x,\alpha) = \int_{E(M)} \prod_{i=1}^{4} e^{\alpha_i \varphi(x_i)} \overbrace{e^{-S_g(\varphi)} d\varphi}^{\text{measure}}$$

can be disintegrated :

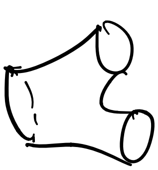
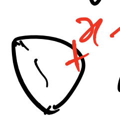
let $$A_{M_1,g}(\varphi_0) := \int_{\substack{\varphi\in E(M_1)\\ \varphi|_{S^1}=\varphi_0}} e^{\alpha_1\varphi(x_1)} e^{\alpha_2\varphi(x_2)} \underbrace{e^{-S_g^{M_1}(\varphi)}\, d\varphi}_{\text{measure}}$$

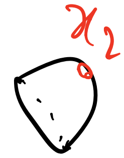
and $$A_{M_2,g}(\varphi_0) := \int_{\substack{\varphi\in E(M_2)\\ \varphi|_{S^1}=\varphi_0}} e^{\alpha_3\varphi(x_3)} e^{\alpha_4\varphi(x_4)} \overbrace{e^{-S_g^{M_2}(\varphi)}}\, d\varphi$$

Conditioning

Then
$$Z_{M,g}(x,\alpha) = \int_{E(S^1)} A_{M_1,g}(\varphi_0)\, A_{M_2,g}(\varphi_0)\, d\varphi_0$$

$A_{M_j,g}$ is called amplitude of M_j

- Similarly $A_{M_1,g}$ can be decomposed into products of amplitudes of  with amplitudes of 

and 

Segal Axioms give a math definition to this procedure

2) Segal Axioms for CFT

View CFT as a functor

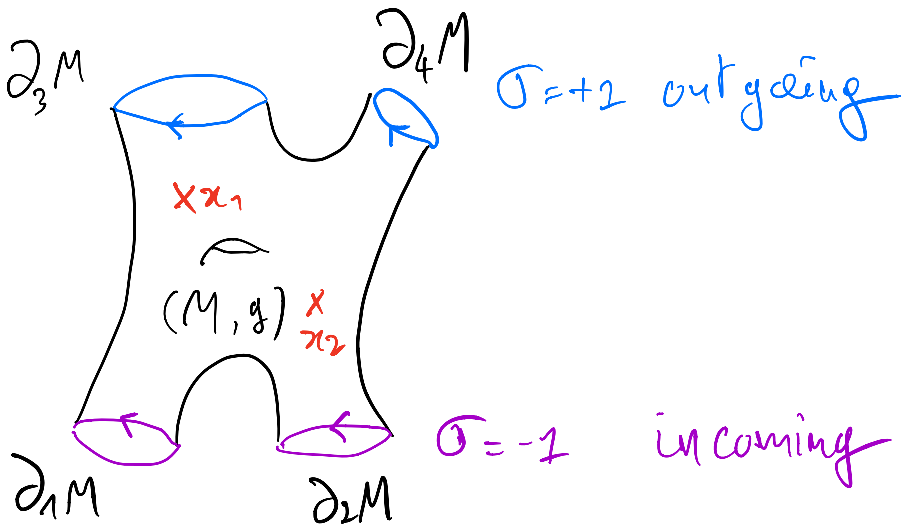
- Objects: $\coprod_{i=1}^{n} S^1$ $\quad n \in \mathbb{N} \cup \{0\}$
- Morphisms: (M, g) oriented Riemannian surface with geodesic bdry

$$\partial M = \coprod_{i=1}^{n} \partial_i M \sqcup \coprod_{j=1}^{m} \partial_j M$$

and an orientation for each bdry component

$\sigma_i = -1$, $\sigma_j = +1$,

and $(x_j; \alpha_j)$ marked points with weights.



CFT Functor

$$\begin{cases} \coprod_{i=1}^{n} S^1 \longrightarrow \bigotimes_{i=1}^{n} \mathcal{H} \\ (M, g) \longrightarrow A_{M,g} : \bigotimes_{i,\sigma_i=-1} \mathcal{H} \longrightarrow \bigotimes_{i,\sigma_i=+1} \mathcal{H} \end{cases}$$

with $\mathcal{H}$ a separable Hilbert Space

with $A_{M,g}$ Hilbert-Schmidt operator

• rem:

$$A_{M,g} : \begin{cases} \mathbb{C} \longrightarrow \bigotimes_{i=1}^{m} \mathcal{H} & \text{if no incoming bdry} \\ \bigotimes_{i=1}^{n} \mathcal{H} \longrightarrow \mathbb{C} & \text{if no outgoing bdry} \end{cases}$$

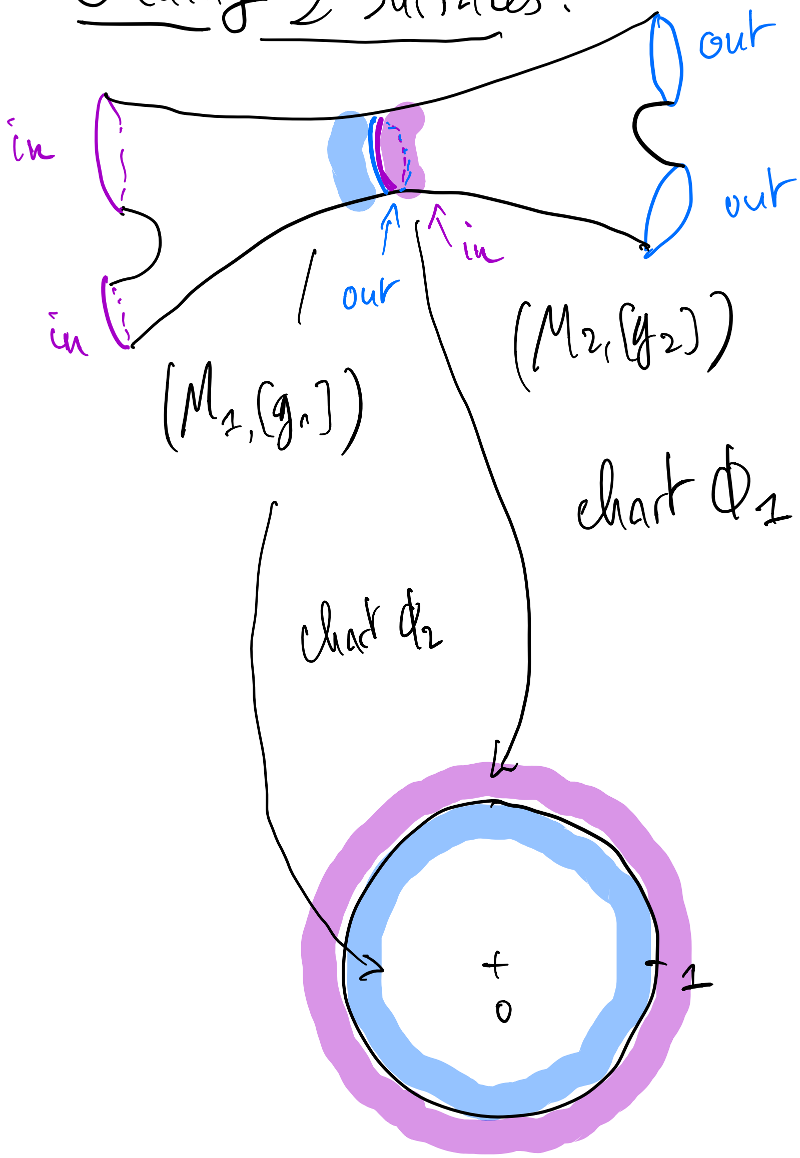
• Moreover, $A_{M,e^{\omega}g} = e^{\frac{c}{96\pi}\int_M (|d\omega|_g^2 + 2R_g\omega)\,dvol_g} \times A_{M,g}$

(Weyl Anomaly)

Def: $A_{M,g}$ called Amplitude of $M, g, (x, \alpha)$

Moreover, Following rules for amplitudes need to hold:

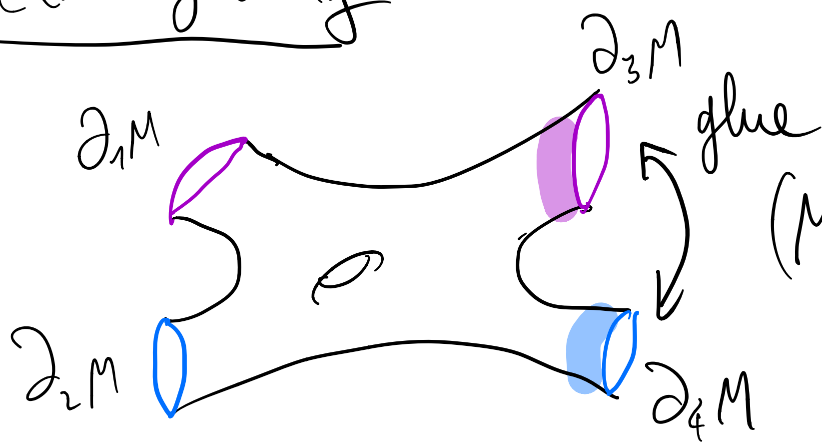
Gluing 2 surfaces:



Amplitudes must satisfy

$$\boxed{A_{(M_1 \# M_2,\, g_1 \# g_2)} = A_{M_2, g_2} \circ A_{M_1, g_1}}$$

Self-gluing :



$$\boxed{A_{\tilde{M}, \tilde{g}} = \mathrm{Tr}_{3,4}\left(A_{M,g}\right)}$$ partial trace

Remark: if $\mathcal{H} = L^2(\Omega, \mu)$ for some measured space (Ω, μ), Amplitude is an element

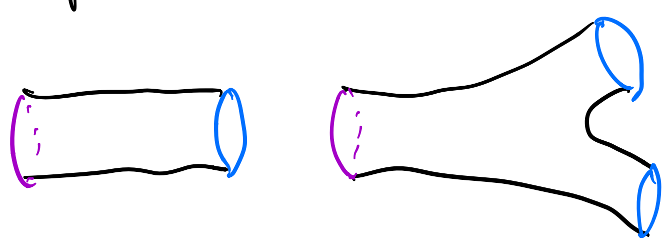
$$A_{M,g}(\varphi_1, \dots, \varphi_m ; \varphi_1', \dots, \varphi_n') \in L^2(\Omega^{m+n}; \mu^{\otimes m+n})$$

if one views it as a Hilbert-Schmidt operator.

Physics: Ω = space of fields on $S^1 = E(S^1)$

exemple: composition of

(M_1, g_1) (M_2, g_2)



$$A_{M_1 \# M_2,\, g_1 \# g_2}(\varphi_1, \varphi_2 ; \varphi_1')$$

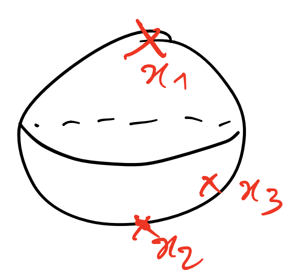
$$= \int_{E(S^1)} A_{M_2, g_2}(\varphi_1, \varphi_2 ; \varphi) A_{M_1, g_1}(\varphi ; \varphi_1')\, d\mu(\varphi)$$

"Theorem" in physics

$$Z_{M,g}(x_1,\dots,x_n;\alpha_1,\dots,\alpha_n) = \sum_{\substack{\Delta=(\Delta_1,\dots,\Delta_N)\\ \in \text{Spectrum}(CFT)}}' P(\Delta,\alpha)\,\left|\mathcal{F}(\Delta,\alpha;q,x)\right|^2$$

where: $N = 3\underbrace{\text{genus}(M)}_{h} - 3 + n = \dim_{\mathbb{C}} \overbrace{\mathcal{M}_{h,n}}^{\text{moduli space}}$

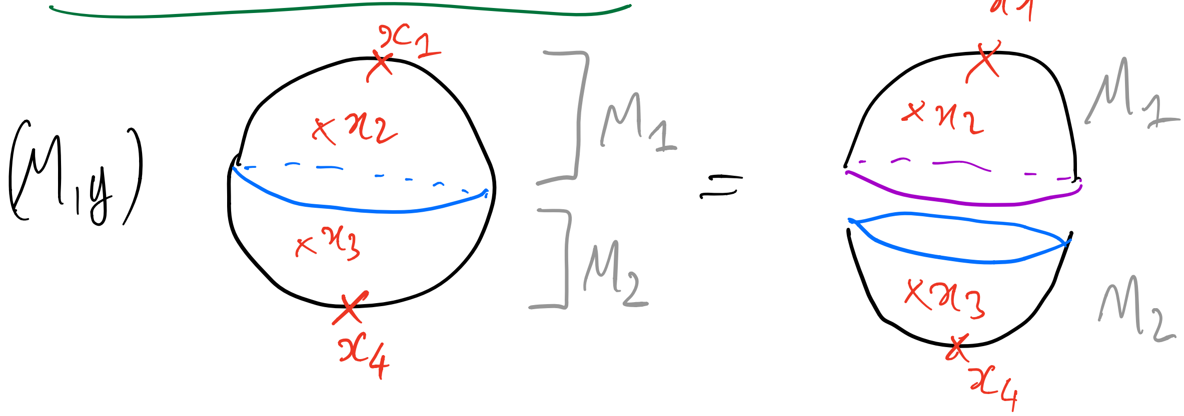
- $P(\Delta,\alpha)$ = product of 3-points correlations functions on S^2



- $\mathcal{F}(\Delta, \alpha; q, x) =$ conformal blocks are holomorphic functions of (q, x) where (q, x) are complex coordinates on moduli space $\mathcal{M}_{h,n}$

- $\Delta \in$ spectrum of $H_0 = L_0 + L_0^*$ where L_0 is a representation of the Virasoro element into operators on $\mathcal{H}$

Idea behind this :



$$Z_{M,g}(x,\alpha) = \int_\Omega A_{M_1,g}(\varphi)\, A_{M_2,g}(\varphi)\, d\varphi$$

$$= \langle A_{M_1,g} , A_{M_2,g} \rangle_{L^2(\Omega)}$$

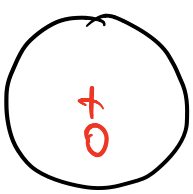
$$= \sum_{\nu,\tilde\nu \in \mathcal{T}} \int_{\Delta \in Sp(H_0)} \langle A_{M_1,g}, \Psi_{\Delta,\nu,\tilde\nu} \rangle_{L^2} \langle \Psi_{\Delta,\nu,\tilde\nu} , A_{M_2,g} \rangle_{L^2} \, d\Delta$$

Young Tableaux $\simeq \mathbb{N}$

with $H_0 \Psi_{\Delta,\nu,\tilde\nu} = (\Delta + |\nu| + |\tilde\nu|)\, \Psi_{\Delta,\nu,\tilde\nu}$

eigenbasis of $\mathcal{H}$ for H_0

But $\Psi_{\Delta,\emptyset,\emptyset}(\varphi) = A_{\mathbb{D},(0,\, i\sqrt{\Delta-\Delta_0})}(\varphi)$

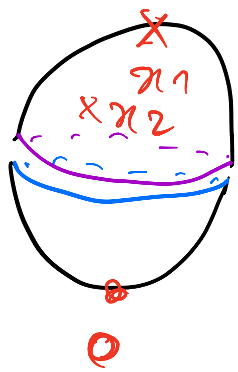


amplitude of the unit disk with marked point at $x = 0$ and complex weight $i\sqrt{\Delta - \Delta_0}$

Δ_0 = bottom of spectrum(H_0)

thus $\langle A_{M_{1,y}}, \Psi_{\Delta,\phi,\phi} \rangle_{L^2(\Omega)}$

= Amplitude of



= 3 points Corel Function with complex weigat

- $\Psi_{\Delta,\nu,\tilde{\nu}} = L_{-\nu_k} L_{-\nu_{k-1}} \cdots L_{-\nu_1} \tilde{L}_{-\tilde{\nu}_j} \cdots \tilde{L}_{-\tilde{\nu}_1} \Psi_{\Delta,\phi,\phi}$

  where $\nu = (\nu_1, \dots, \nu_k)$ $\quad \tilde{\nu} = (\tilde{\nu}_1, \dots, \tilde{\nu}_j)$, with $\nu_1, \dots, \nu_k \in \mathbb{N}$ and $\tilde{\nu}_1, \dots, \tilde{\nu}_j \in \mathbb{N}$

- $L_{-\nu_i}$, $\tilde{L}_{-\tilde{\nu}_j}$ are two representations of Virasoro generators as operators on $\mathcal{H}$

$\Rightarrow$ Similar to creation operators for Harmonic oscillator

- Ward identities :

$$\langle A_{m_1,g},\ L_{-\nu_k}\cdots L_{-\nu_1}\tilde{L}_{-\tilde{\nu}_j}\cdots\tilde{L}_{-\tilde{\nu}_1}\Psi_{\Delta,\phi,\phi}\rangle$$

$$= b_{\nu,\tilde{\nu}}(x)\ \underbrace{w_{\nu,\tilde{\nu}}(\alpha_1,\alpha_2,\Delta)}\ \langle A_{m_1,g},\Psi_{\Delta,\phi,\phi}\rangle$$

"explicit" algebraic factors only related to Virasoro algebra

powers of $x_1, x_2 \in \mathbb{D}^c$

These produce the conformal blocks

Ward: "Vertex operator algebra".

3] Liouville CFT

Action is that of uniformisation of surfaces

$$S_g(\varphi) := \frac{1}{\pi} \int_M \left(|d\varphi|_g^2 + \frac{Q}{4} R_g \varphi + \pi e^{\gamma\varphi} \right) dv_g$$

with $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$, $\gamma \in (0,2)$

$$\boxed{Z_{M,g}(x,\alpha) = \int_{E(M)} \prod_{i=1}^{n} e^{\alpha_i \varphi(x_i)} \, e^{-S_g(\varphi)} \, d\varphi}$$

Theorem 1 : (David, Kupiainen, Rhodes, Vargas 2016)
(Guillarmou, Rhodes, Vargas 2018)

There is a probabilistic definition for correlations functions in all genus if $\sum \alpha_i > Q\chi(M)$

$$Z_{M,g}(x,\alpha) \overset{\text{def}}{:=} \frac{\sqrt{\mathrm{vol}_g(M)}}{\sqrt{\det' \Delta_g}} \times$$

$$\int_{\mathbb{R}} \mathbb{E}\left[\prod_j e^{\alpha_i (c+X(x_i)) - \frac{\alpha_i^2}{2}\mathbb{E}(X^2(x_i))} \, e^{-\frac{Q}{4\pi}\langle c+X, R_g\rangle - \pi\int_M e^{\gamma(c+X) - \frac{\gamma^2}{2}\mathbb{E}(X^2)} dv_g}\right] dc$$

where $X :=$ Gaussian Free-field $\in H^{-\varepsilon}(M)$

$$= \sqrt{2\pi} \sum_{k=1}^{\infty} \omega_k \frac{u_k}{\sqrt{\lambda_k}} \quad , \quad \omega_k \in N(0,1) \text{ i.i.d Gaussian}$$

$$\Delta_g u_k = \lambda_k u_k$$

with covariance

$$\mathbb{E}(X(x)X(x')) = 2\pi G_g(x,x') \quad \text{Green's fct}^{\circ}$$

- $e^{\gamma X - \frac{\gamma^2}{2}\mathbb{E}(X^2)} dv_g$ makes sense as a random measure

<u>Kahane multiplicative Chaos</u>

- field is $\varphi = \underset{\text{constant}}{c} + \underset{\text{random field} \perp \text{to constants}}{X}$

Theorem 2 : (Guillarmou-Kupiainen-Rhodes-Vargas 2020 + 2021)

1. $\exists$ Hilbert space $\mathcal{H} = L^2(\mathbb{R}_c \times \Omega, dc \otimes d\mathbb{P})$ with $L^2(\Omega, \mathbb{P})$ = Fock space

$$\Omega = (\mathbb{R}^2)^{\mathbb{N}^*}, \quad d\mathbb{P} = \bigotimes_{n \geq 1} \frac{1}{2\pi} e^{-\frac{1}{2}(x_n^2 + y_n^2)} dx_n \, dy_n$$

representing a measure on $H^{-s}(\mathbb{S}^1)$ for $s > 0$ via the random field $\varphi(\theta) = \sum_{n \neq 0} \frac{1}{2\sqrt{n}}(x_n + i y_n) e^{in\theta}$ $\to \varphi_n$

2. $\exists$ probabilist definition for amplitude $A_{M,g}$ of Riemann surfaces with geodesic bdries & marked points using conditioning on boundary ∂M

$\Rightarrow$ Use $X = Y + P\varphi - c$

Y = GFF with Dirichlet condition and

$P\varphi$ = harmonic extension of $\varphi = X_{|\partial M}$

c = constant

and integrate away Y variable

③ Segal Axioms for gluing amplitudes holds for Liouville CFT

④ $\exists$ unitary representation of $H_0 = L_0 + L_0^*$ as a self adjoint op. on $\mathcal{H}$

$$H_0 = \tfrac{1}{2}(-\partial_c^2 + Q^2) + 2\underbrace{\sum_{n\neq 0} A_{-n}A_n + \tilde{A}_{-n}\tilde{A}_n}_{P_0} + e^{\gamma c} V(\varphi)$$

$A_n = \frac{i}{2}\partial_{\varphi_n}$, $\tilde{A}_n = \frac{i}{2}\partial_{\varphi_{-n}}$ $\quad P_0 =$ infinite dim harmonic oscillator spec $= \mathbb{N}$

$V \in L^{2/\gamma^2}(\Omega)$ positive "potential"

+ with a full spectral decomposition in terms of scattering eigenstates

$$Sp(H_0) = [\tfrac{Q^2}{2}, \infty) \quad \text{purely continuous}$$

Generalized eigenvectors are analytic extensions in $\alpha \in Q + i\mathbb{R}$ of amplitude of $\mathbb{D}$ with marked pt at $x = 0$ and weight α

5 Correlations Functions can be decomposed using Segal decompositions

$$Z_{M,g}(x,\alpha)$$

$$= \int_{P\in\mathbb{R}^{3h-3+n}} \rho(P,\alpha) \left| \mathcal{F}_{P,\alpha}(q) \right|^2 dP$$

- where $\rho(P,\alpha) = \prod_{j=1}^{3h-3+n} C_j^{DOZZ}(P,\alpha)$

  $C_j(P,\alpha)$ are 3-points correlations function on $(\mathbb{S}^2, 0,1,\infty)$

- $\mathcal{F}_{P,\alpha}(q)$ are the conformal blocks

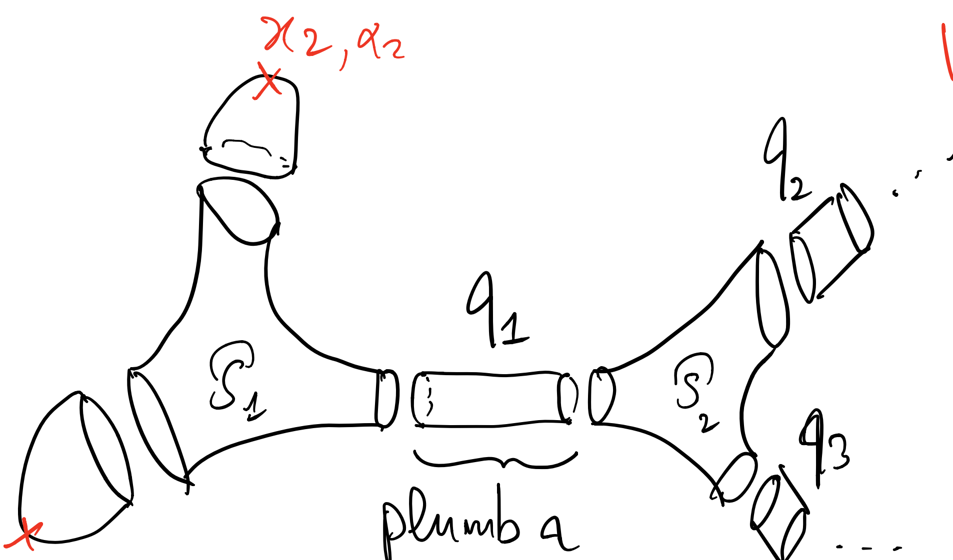
$$= \sum_{N\in\mathbb{N}^{3h-3+n}} \prod_{j=1}^{3h-3+n} q_j^{\frac{Q^2}{4}+\frac{P_j^2}{4}+N_j} \underbrace{\mathrm{Tr}(B_N)}$$

B_N are amplitudes on $\mathcal{T}$, the set of Young tableaux.

$q=(q_1, \dots, q_{3h-3+n}) \in \mathbb{D}(0,1)^{3h-3+n}$

are complex parameters for moduli space $\mathcal{M}_{h,n}$ of genus h and n marked points

"Plumbing parameters" (Marden, Kra, Wolpert)

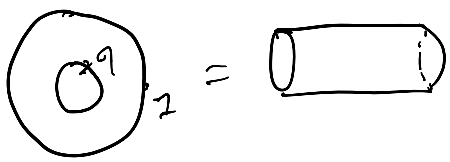


x_1, α_1 Flat annulus $z \in [|q|, 1]$, $g = \frac{|dz|^2}{|z^2|}$

of length $\log|q|$, twist $\arg(q)$

Remark: Result associated to a pant decomposition

Tools : • Amplitude of Annulus



is $q^{L_0}\,\bar{q}^{\tilde{L}_0}$

$$= e^{-tH_0 + i\theta\Pi}$$

if $q = e^{-t+i\theta}$ $\boxed{\text{propagator}}$

$$\Pi := \frac{L_0 - L_0^*}{i} \qquad H_0 := L_0 + L_0^*$$

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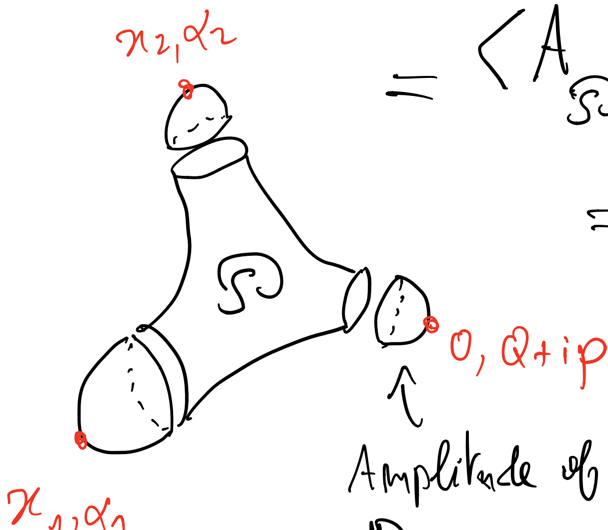


$$= \langle A_{\mathcal{S}} , \Psi_{Q+ip,\phi,\phi} \otimes \Psi_{\alpha_1,\phi,\phi} \otimes \Psi_{\alpha_2,\phi,\phi} \rangle$$

= amplitude of S^2 with 3 marked point

Amplitude of $\mathbb{D}, 0, Q+ip$ $= \Psi_{Q+ip,\phi,\phi}$

Theorem (Kupiainen - Rhodes-Vargas 2019)

$$Z_{\mathbb{S}^2, \mathrm{can}}(0,1,\infty; \alpha_1,\alpha_2,\alpha_3) = C^{DOZZ}(\alpha_1,\alpha_2,\alpha_3)$$

is an explicit constant in terms of special function

$\Rightarrow$ Dorn-Otto-Zamolodchikov-Zamolodchikov Formula in physics

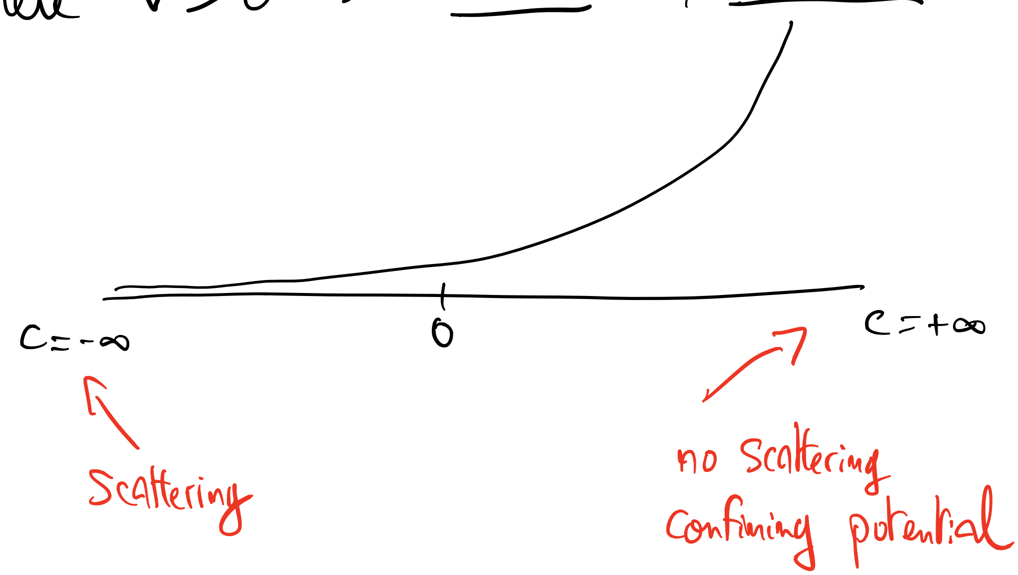
Remark for b-community:

$$H_0 = -\frac{1}{2}\partial_c^2 + \frac{Q^2}{2} + P_0 + e^{\gamma c}V$$

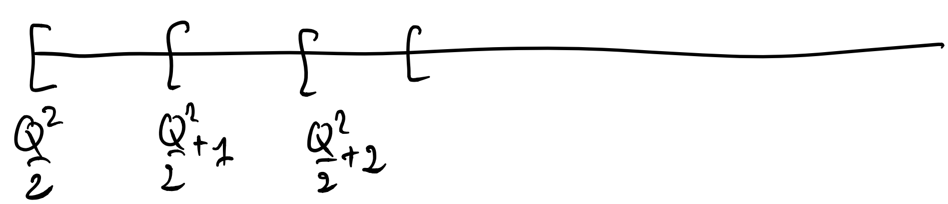
∞ dim harmonic oscillator
has discrete spectrum on $L^2(\Omega)$
Fock space $\rightsquigarrow$ Young Tableaux

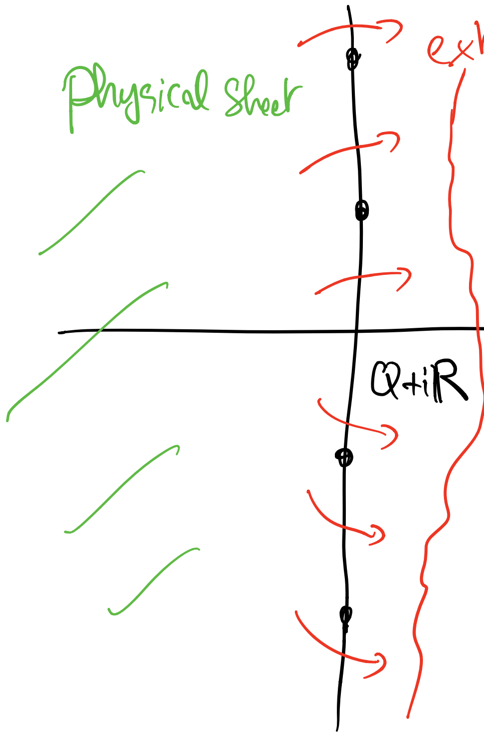
$\Leftrightarrow$ Thus H_0 look formally like a
b-Schrödinger operator with potential $e^{\gamma c}V$

here $V>0$ but unbounded, not bounded below



Spectrum





extension of resolvent in neighb of continuous sp on a Riemann surface

(like b-Laplacian