Rooted Clusters for Graph LP Algebras

"Bases for Cluster Algebras" at Casa Mathemática Oaxaca

(joint work with Esther Banaian, Sunita Chepuri, and Sylvester W Zhang)

Snake Graphs

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There are many types of algebras that arise as generalizations of cluster algebras.[1]

LP algebras[2] are one such generalization, where the hallmark binomial exchange relations of ordinary cluster algebras are replaced by irreducible polynomials.

[1] "Cluster Algebras I-IV", Fomin & Zelevinsky

[2] "Laurent phenomenon algebras", Lam & Pylyavskyy

Why study LP algebras?

- Some interesting combinatorial recurrences occur as exchange relations of LP algebras, including the Gale-Robinson sequence, the Somos sequences ($n \leq 7$), and the cube recurrence.

- Some LP algebras arise naturally as coordinate rings of electrical Lie groups.[1]

- Other generalizations of cluster algebras, like Chekhov-Shapiro algebras, appear as special cases.

- Some of the structural features of cluster algebras extend to LP algebras, including the Laurent Phenomenon.

> A natural question: What can we say about positivity?

[1] an example appears in "Laurent Phenomenon algebras"

Graph LP algebras[1] are a subclass of LP algebras[2] with a nice combinatorial definition in terms of graphs.

If we want to eventually think about positivity for LP algebras, we might start by studying Graph LP algebras where we can take advantage of this more concrete structure.

To define Graph LP algebras, we'll need descriptions of:

- cluster variables
- clusters
- exchange relations

[1] "Linear Laurent Phenomenon Algebras", Lam & Pylyavskyy

[2] "Laurent Phenomenon Algebras", Lam & Pylyavskyy

Given a simple graph G with vertices labeled by $[n] = \{1, \dots, n\}$, we can define the cluster variables in terms of nested collections of vertex subsets.

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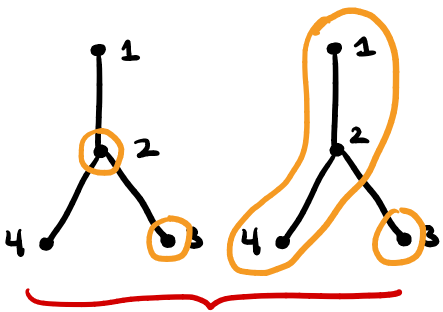
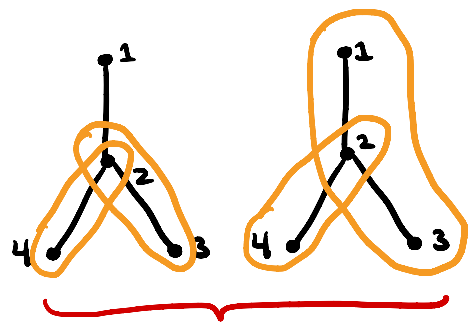
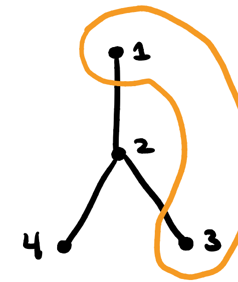
__Non-Examples:__



no pairs of subsets I_i, I_j that contain adjacent vertices, unless $I_i \subseteq I_j$ or $I_j \subseteq I_i$



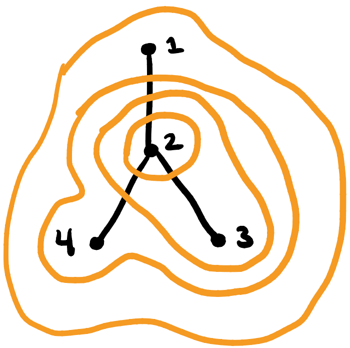
no pairs of subsets I_i, I_j with non-trivial symmetric difference



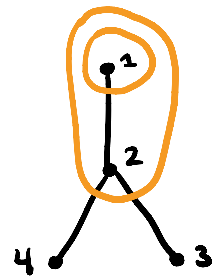
no disconnected subsets

Given a simple graph G with vertices labeled by $[n] = \{1, \dots, n\}$, we can define the cluster variables in terms of nested collections of vertex subsets.

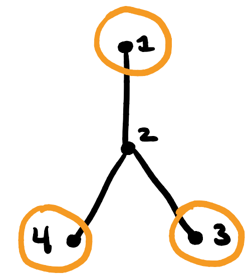
Examples:



$\{1\}, \{1,3\}, \{1,3,4\}, \{1,2,3,4\}$



$\{2\}, \{1,2\}$



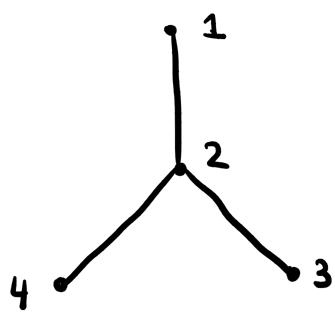
$\{2\}, \{3\}, \{4\}$

✓ all of these are valid nested collections

Given a graph Γ with vertices labeled by $[n]$, each connected vertex subset $I \subseteq V(\Gamma)$ corresponds to a cluster variable Y_I.

For each $i \in [n]$, there is an additional cluster variable X_i.

EX 11



has cluster variables: $X_1, X_2, X_3, X_4,$

$Y_1, Y_2, Y_3, Y_4,$

Y_{12}, Y_{23}, Y_{24}

$Y_{123}, Y_{124}, Y_{234}$

Y_{1234}

Given a graph Γ with vertices labeled by $[n]$, each connected vertex subset $I \subseteq V(\Gamma)$ corresponds to a cluster variable Y_I.

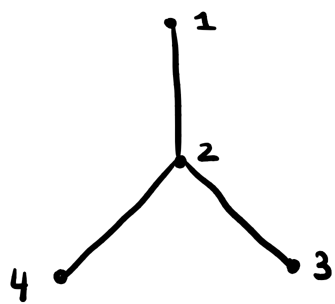
For each $i \in [n]$, there is an additional cluster variable X_i.

Note: For disconnected I, we'll define Y_I as

$$Y_I = Y_{I_1} \cdots Y_{I_k}$$

where $I_1, \ldots, I_k$ are the connected components of I

EX11



In general, the collection of all cluster variables of Γ is

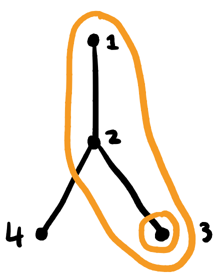
$$\{X_i\}_{i \in [n]} \cup \{Y_I : I \subseteq n \text{ is connected}\}$$

A graph Γ with n vertices has clusters of size n of the form

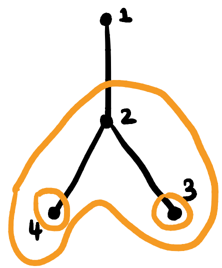
$$\{X_{i_1}, \dots, X_{i_k}\} \cup \{Y_S : S \in \mathcal{S}\}$$

$\mathcal{S}$ is a maximal nested collection of subsets on $[n] \setminus \{i_1, \dots, i_k\}$

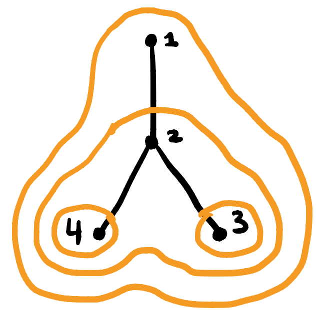
Examples & Non-Examples of maximal nested collections:



✗



✓ maximal on $\{2,3,4\}$

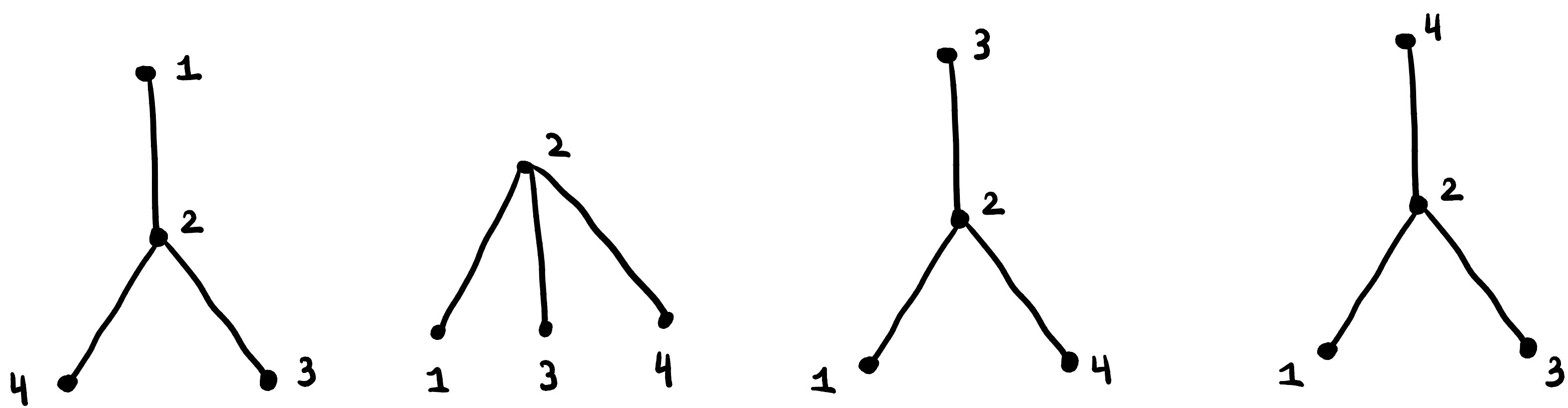


✓ maximal on $\{1,2,3,4\}$

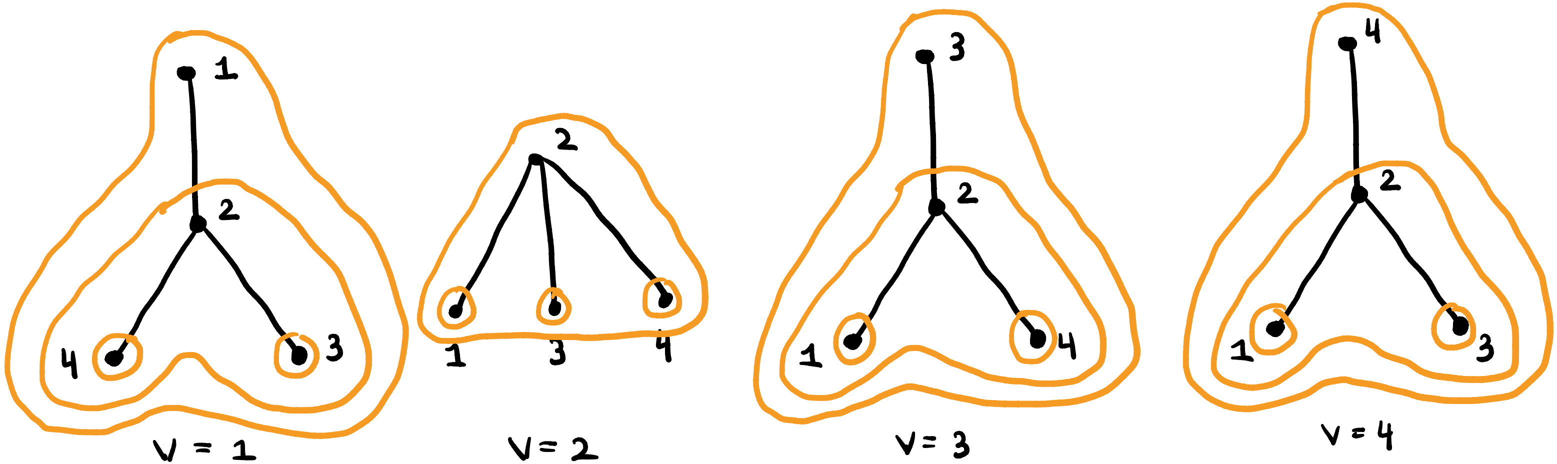
Rooted Clusters

Given a tree Γ, choose a root v.

We can think of Γ as a poset, with v as the maximal element and cover relations given by the edges of Γ.



$v = 1$ $\quad$ $v = 2$ $\quad$ $v = 3$ $\quad$ $v = 4$



Let $I^{v}_{\leq x} = \{\text{elements} \leq x \text{ in the poset with maximal element } v\}$

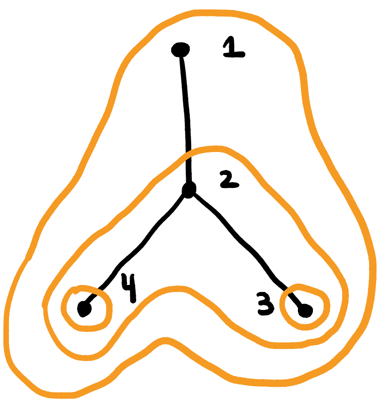
Then each v corresponds to a unique rooted cluster $e_v = \{I^{v}_{\leq x}\}_{x \in [n]}$

We'll often abbreviate $I^{v}_{\leq x}$ as just I_x.

Rooted Sets

A set S is **rooted** with respect to a maximal nested collection $\mathcal{I}$ if there exists some $v \in S$ such that for all $i, j \in S$,

$$I_i \subseteq I_j \iff \text{the path from } i \text{ to } v \text{ passes through } j$$



All sets are rooted with respect to a rooted cluster.

(just take v to be the root!)

Rooted Sets

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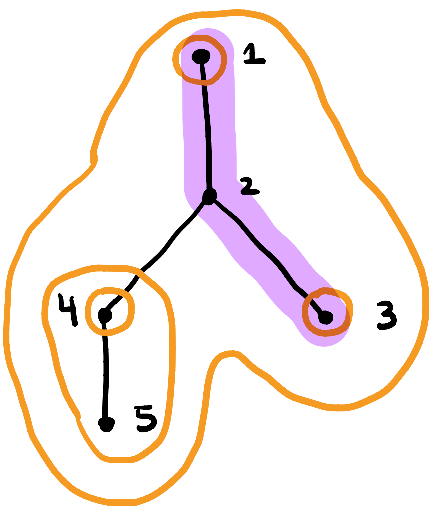


Other clusters may have some sets that are rooted with respect to them and some that are not.

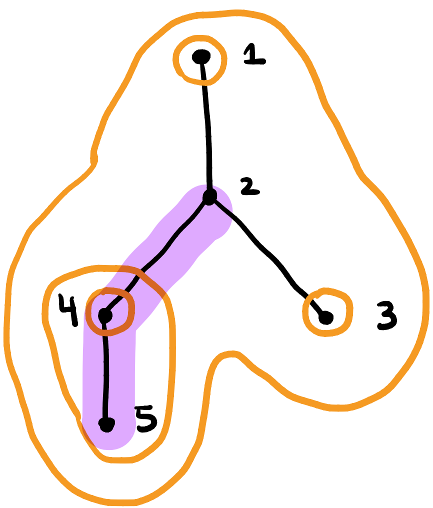
$\{1, 2, 3\}$ is rooted with respect to e.

(take $v = 2$)

Rooted Sets

A set S is rooted with respect to a maximal nested collection $\mathcal{I}$ if there exists some $v \in S$ such that for all $i, j \in S$,

$$I_i \subseteq I_j \iff \text{the path from } i \text{ to } v \text{ passes through } j$$



Other clusters may have some sets that are rooted with respect to them and some that are not.

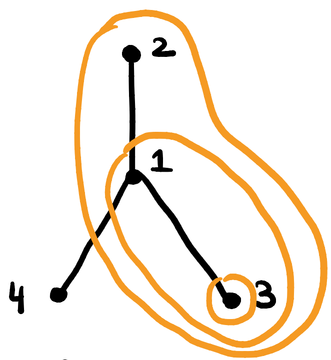
$\{2, 4, 5\}$ is <u>not</u> rooted with respect to $\mathcal{e}$.

To define a Graph LP algebra, we also need some definition of an exchange relation.

In fact, there are two types of exchange relations.

1. We can exchange X_i with $\underbrace{Y_{S\oplus i}}$ for $i \notin S$ if $S \oplus i$ is compatible with $\mathcal{C}$

this is the connected component of $S \cup \{i\}$ that contains i.



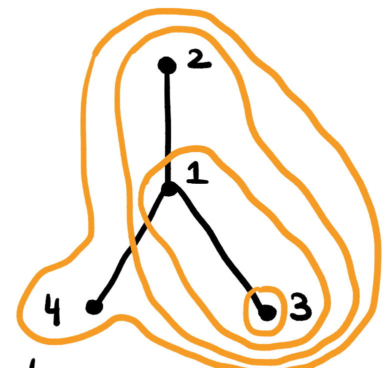
$$\mathcal{C} = \{Y_3, Y_{13}, Y_{123}, X_4\}$$

$$\longleftrightarrow$$

$i = 4$

$S = \{1,2,3\}$

$S \oplus i = \{1,2,3,4\}$

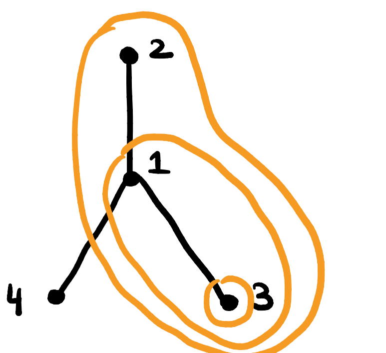


$$\mathcal{C}' = \{Y_3, Y_{13}, Y_{123}, Y_{1234}\}$$

To define a Graph LP algebra, we also need some definition of an exchange relation.

In fact, there are two types of exchange relations.

2. We can exchange $Y_{S\oplus i}$ with $Y_{S\oplus j}$ for $i,j \notin S$ and $i \neq j$, if $S \oplus i$ is compatible with $\mathcal{E} \setminus \{S \oplus j\}$.

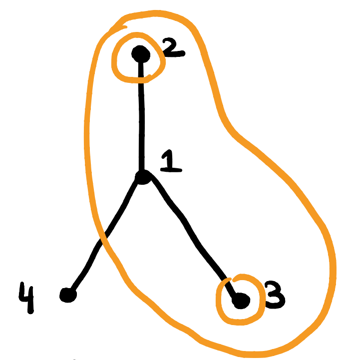


$\mathcal{E} = \{Y_3, Y_{13}, Y_{123}, X_4\}$

$i = 2$
$j = 1$
$S = \{3\}$

$\longleftrightarrow$

$S \oplus i = \{2\}$
$S \oplus j = \{1,3\}$



$\mathcal{E} = \{Y_3, Y_2, Y_{123}, X_4\}$

The precise exchange relations require some notation that we're not going to define.

However, I will include them just in case you're curious:

<u>Lemma</u>: (Lam-Pylyavskyy, 2016)

$$X_i Y_{S \oplus i} = \frac{\sum_{j \notin S_i} P_S^{ij} X_j + \sum_{j \in S_i} P_S^{ij} A_j}{Y_{S \ominus i}} \quad \text{for } i \notin S$$

$$Y_{S \oplus i} Y_{S \oplus j} = \frac{Y_{Sij} Y_S + (P_S^{ij})^2}{Y_{S \ominus i} Y_{S \ominus j}} \quad \text{for } i, j \notin S \text{ and } i \neq j$$

The Story so far...

Given a simple graph Γ with vertices labeled by $[n]$, the associated Graph LP algebra is defined over the ground ring $\mathbb{Z}[A_1, \ldots, A_n]$ and we have the dictionary:

Vertex i of Γ	$\longleftrightarrow$	cluster variable X_i
connected subset $S \subseteq V(\Gamma)$	$\longleftrightarrow$	Y_S
a nested collection of vertex subsets	$\longleftrightarrow$	a cluster $\{X_{i_1}, \ldots, X_{i_k}\} \cup \{Y_S : S \in \mathcal{S}\}$
exchange relations	$\longleftrightarrow$	X_i for $Y_{S \oplus i}$ $Y_{S \oplus j}$ for $Y_{S \oplus i}$ (as described)

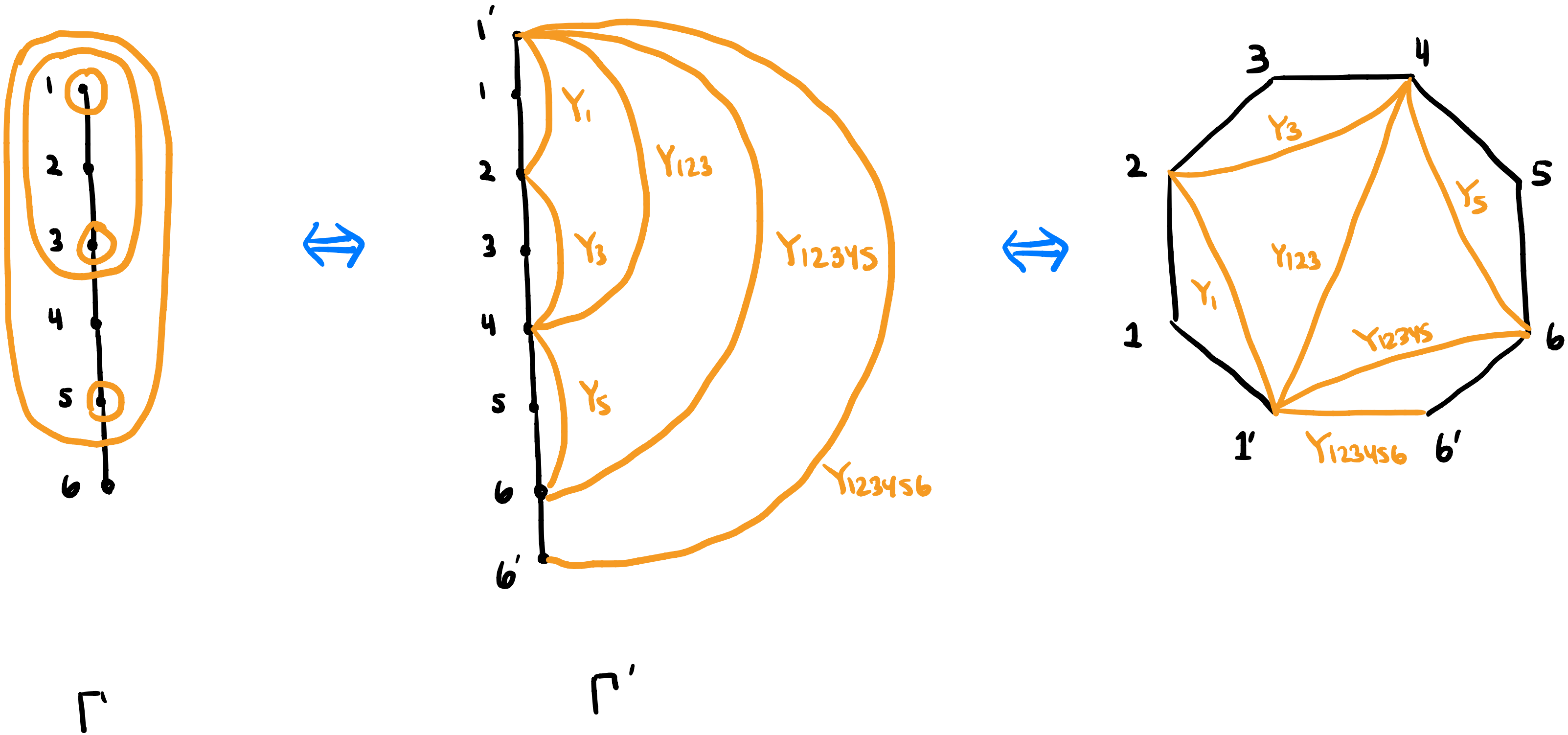
<u>Conjecture</u>: (Lam-Pylyvaskyy, 2016)

Every cluster variable of $\mathcal{A}_\Gamma$ can be written as a Laurent polynomial with positive coefficients in terms of any cluster.

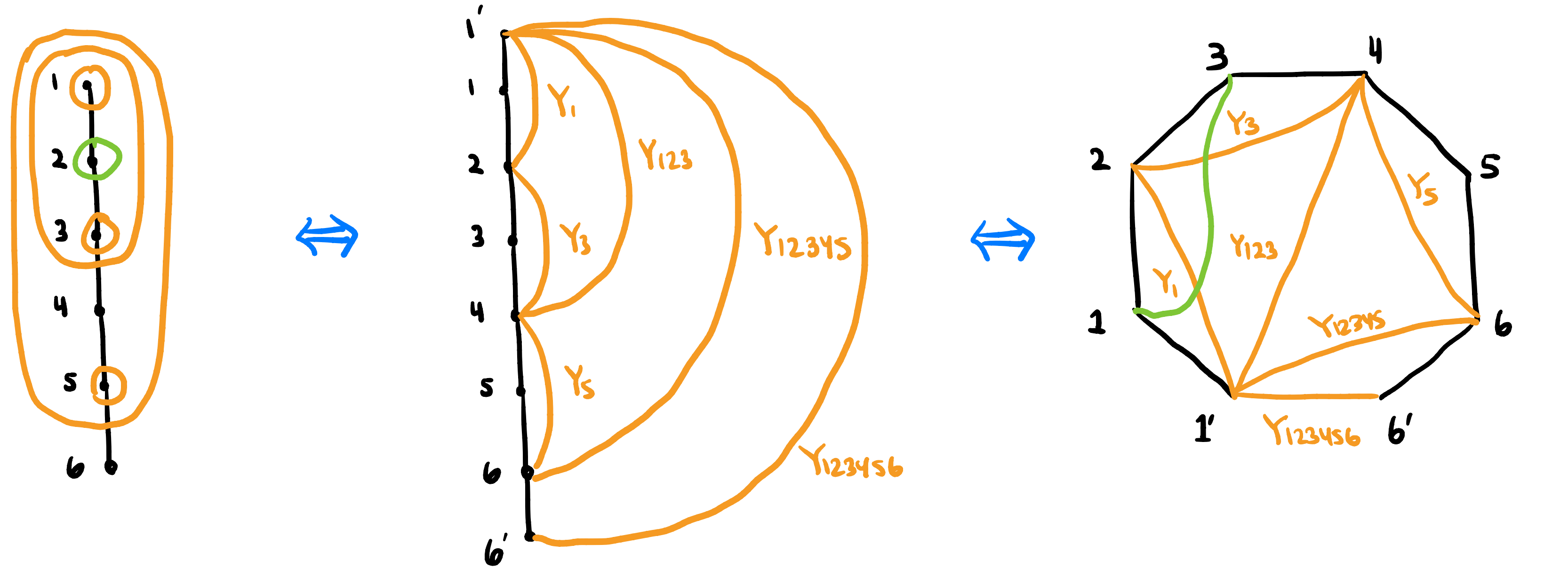
Natural Question: What tools could we use to approach this conjecture?

One idea - maybe we can draw on the observation, made by Lam and Pylyvaskyy, that $\mathcal{A}_{P_n}$ can be identified with the ordinary cluster algebra of type A_{n-1}.

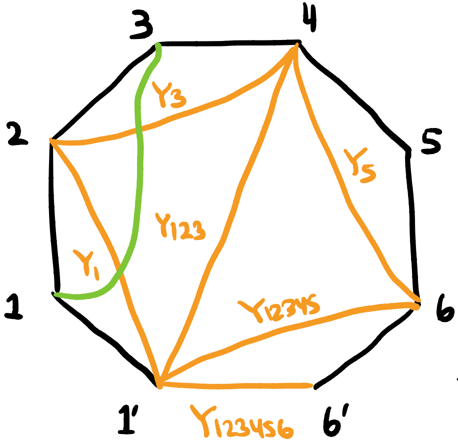
Here's the basic idea:

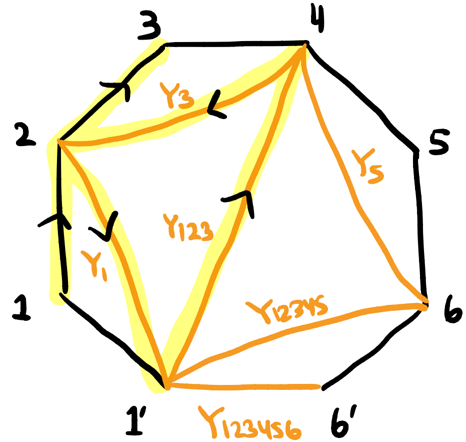


Observe: Arcs on the surface that aren't compatible with the triangulation correspond to incompatible vertex subsets of Γ.



For cluster algebras of type A_n, there are many combinatorial gadgets that can be used to prove positivity.

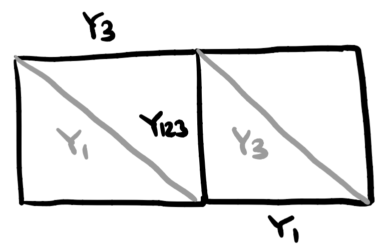




" T-paths "

(Schiffler)

Hyper T-paths for Rooted Clusters:
arXiv 2107.14785



" Snake Graphs "

(Musiker, Schiffler, Williams)

Snake Graphs for Graph LP Algebras: In progress

The only thing I'm going to say about T-paths is that we proved:

Theorem: (Banaian - Chepuri - K. - Zhang, 2021)

Let Γ be a tree and $\mathcal{C}$ be a rooted cluster on Γ.

If $S \subseteq V(\Gamma)$ is connected, then

$$Y_S = \sum{}' \, wt(\alpha)$$

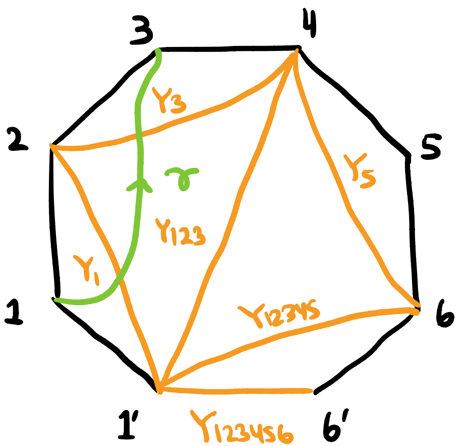
where the sum is over complete hyper T-paths α for S.

Corollary: (Banaian - Chepuri - K. - Zhang, 2021)

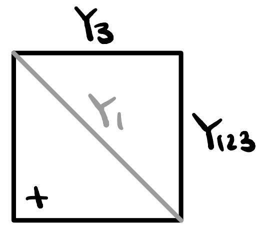
For any $S \subseteq V(\Gamma)$, Y_S can be written as a Laurent polynomial with positive coefficients in terms of any rooted cluster $\mathcal{C}$.

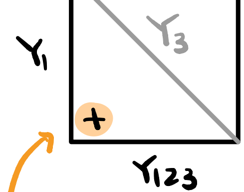
and if you'd like to know more, you can look at arXiv 2107.14785!

Snake Graphs[1]



Each crossing of γ is encoded by a square tile:

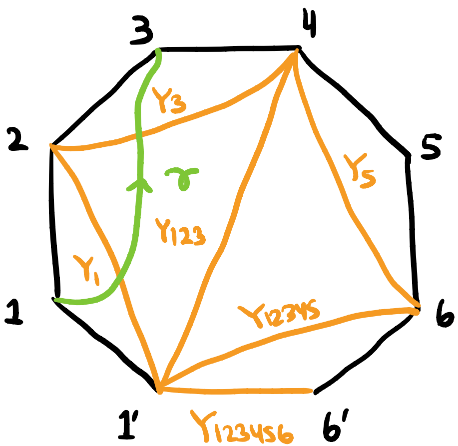




the + indicates that the tile orientation matches the surface. We can also draw it with the opposite orientation.

[1] "Positivity for Cluster Algebras from Surfaces": Musiker, Schiffler, & Williams

Snake Graphs[1]



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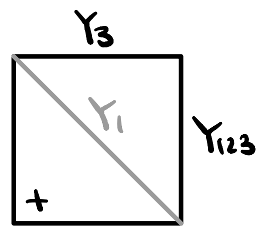
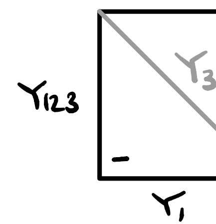
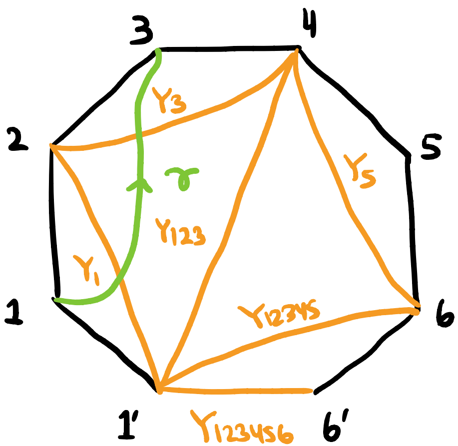




[1] "Positivity for Cluster Algebras from Surfaces": Musiker, Schiffler, & Williams

Snake Graphs[1]



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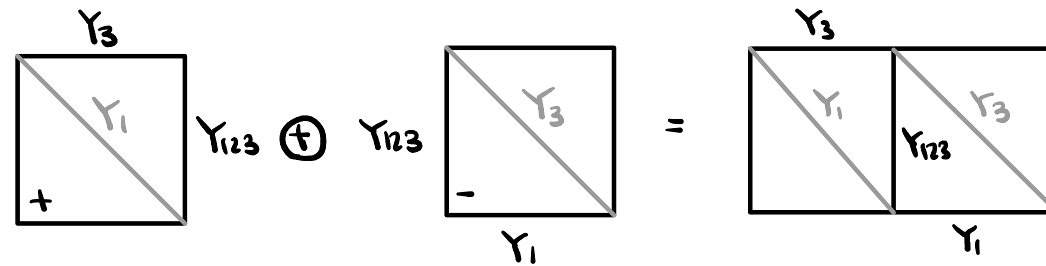


We then glue tiles together, using alternating tile orientations, in the order of the crossings of γ.

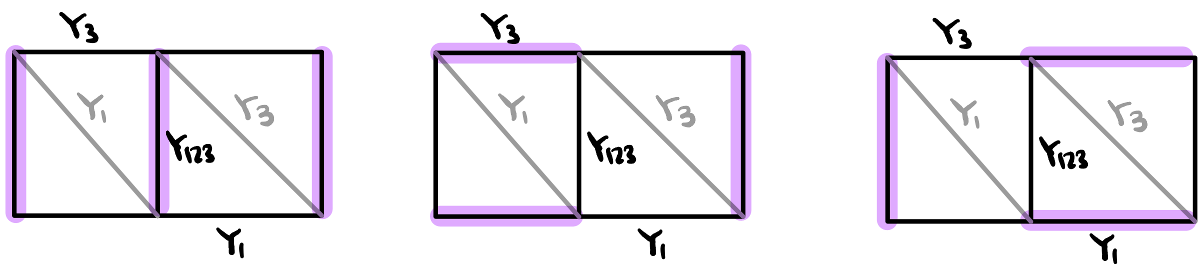
[1] "Positivity for Cluster Algebras from Surfaces": Musiker, Schiffler, & Williams

Theorem: (Musiker - Schiffler - Williams, 2011)

Consider a surface with triangulation T and $\gamma \notin T$. Let $G_{\gamma, T}$ be the snake graph corresponding to γ. Then

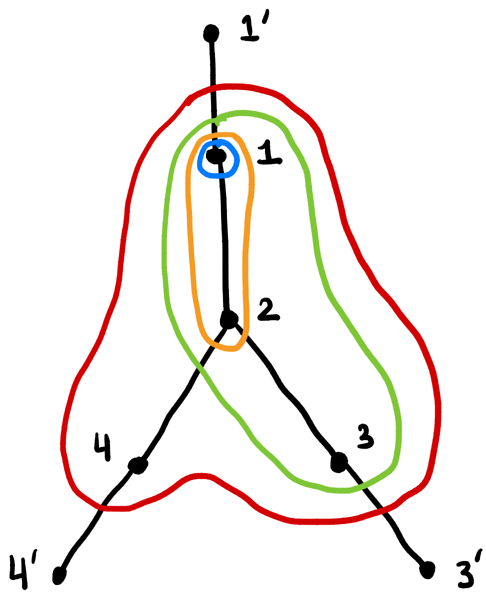
$$x_\gamma = \frac{1}{\text{cross}(T, \gamma)} \sum_P x(P)$$

where the sum runs over perfect matchings of $G_{\gamma, T}$.



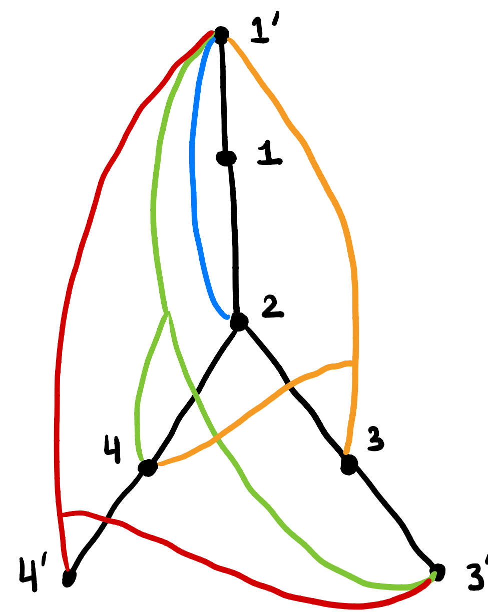
$$Y_2 = \frac{1}{Y_1 Y_3}\left[Y_{123} + Y_3 + Y_1 \right]$$

For the rest of the talk, we'll consider trees. Now, vertex subsets may have more than two neighbors and we can have hyperedges in Γ'.



Γ

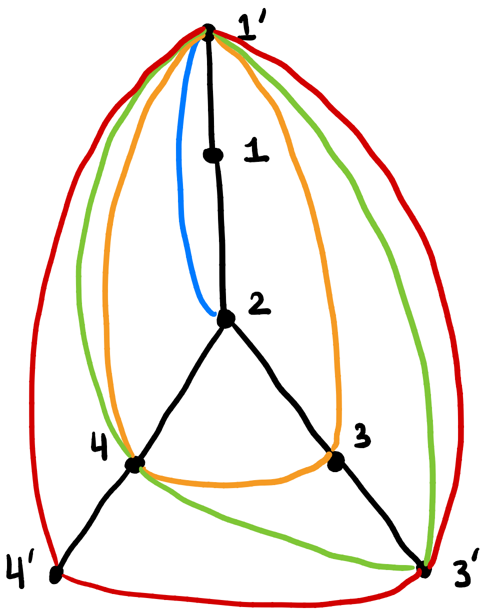
$\Rightarrow$



Γ'

Why trees?

We can take some inspiration from the path graph...



Let's construct G_4 for this cluster.

snake graph for $\{4\}$

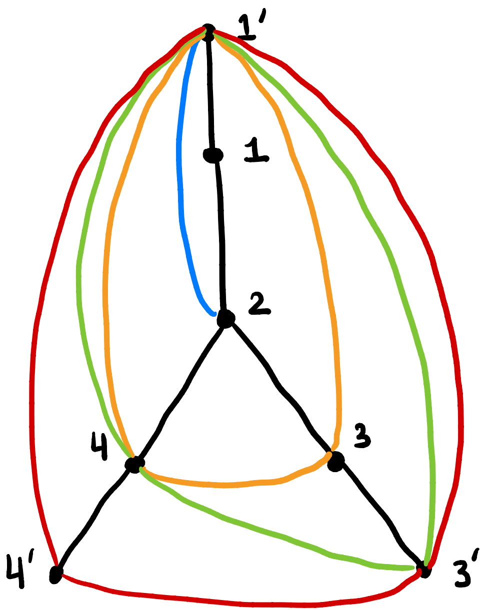
We choose an incompatible neighboring set, Y_{12}, and a vertex 2 in that set that's adjacent to 4.

Let

$$B := \left\{ \begin{array}{l} \text{leaves } i \text{ of } \Gamma' \text{ such that } i \to 4 \\ \text{passes through } 2 \end{array} \right\} = \{1', 3'\}$$

$$E := \{ \text{all other leaves of } \Gamma' \} = \{4'\}$$

Then look at the collection of path graphs $P_{i,j}$ where $i \in B = \{1', 3'\}$ and $j \in E = \{4'\}$ and construct corresponding triangulated polygons $T_{i,j}$.



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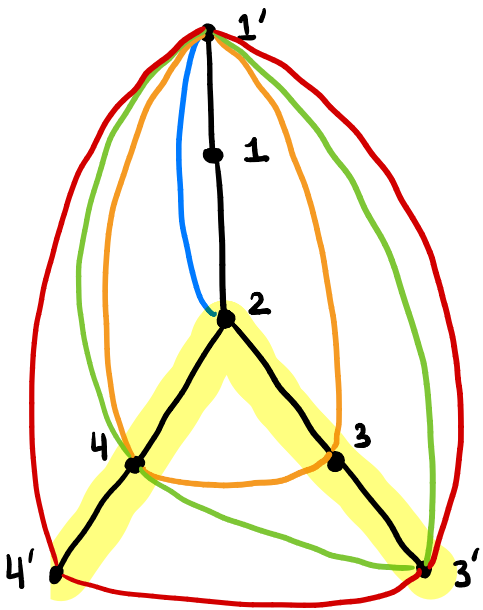

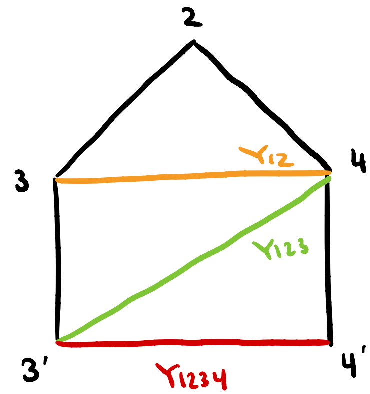

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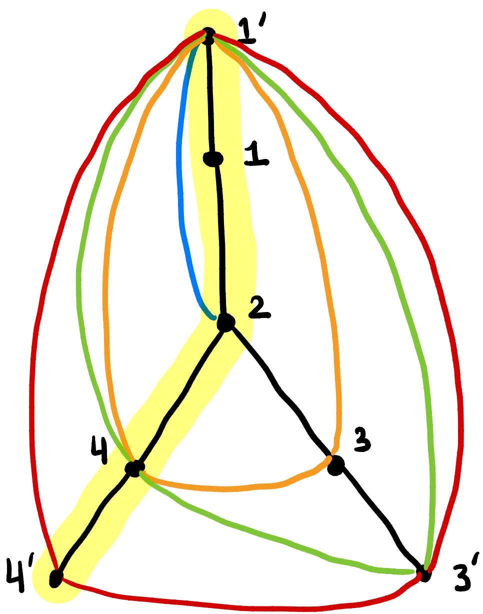
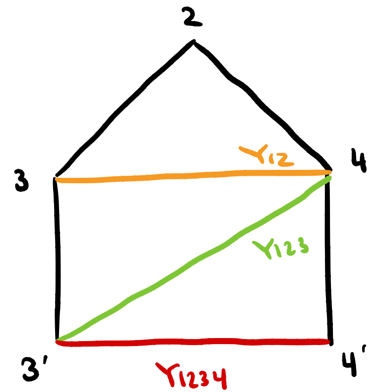
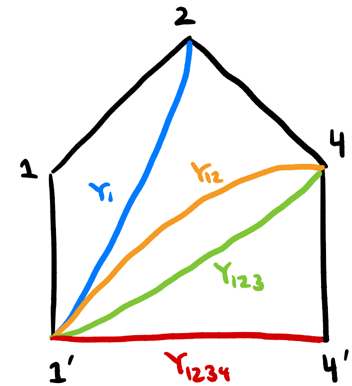
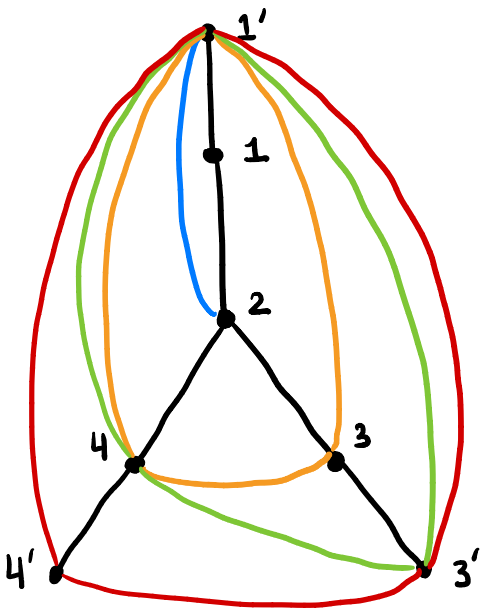
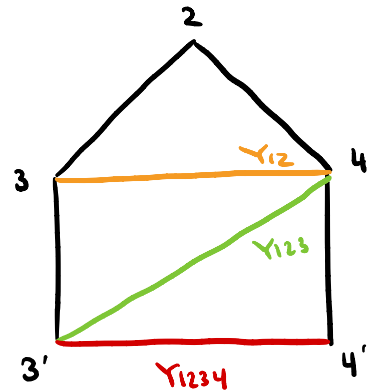


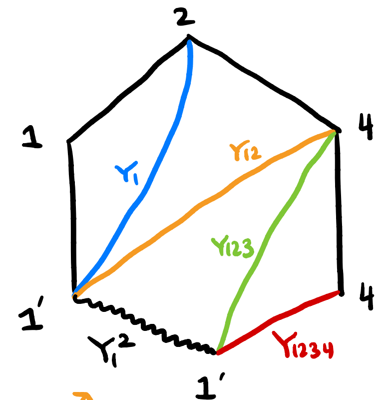




Then look at the collection of path graphs $P_{i,j}$ where $i \in B = \{1', 3'\}$ and $j \in E = \{4'\}$ and construct corresponding triangulated polygons $T_{i,j}$.







The weight on this virtual edge is determined by a condition based on the labels of the separated arcs

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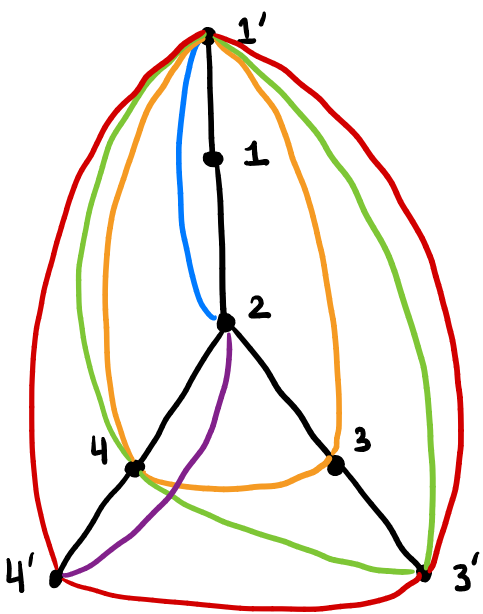
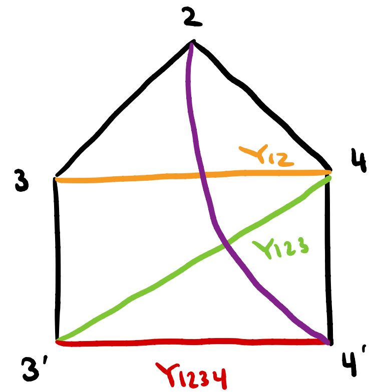
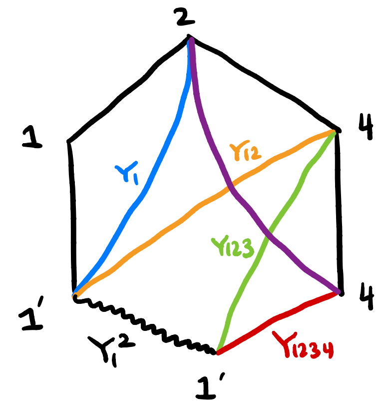
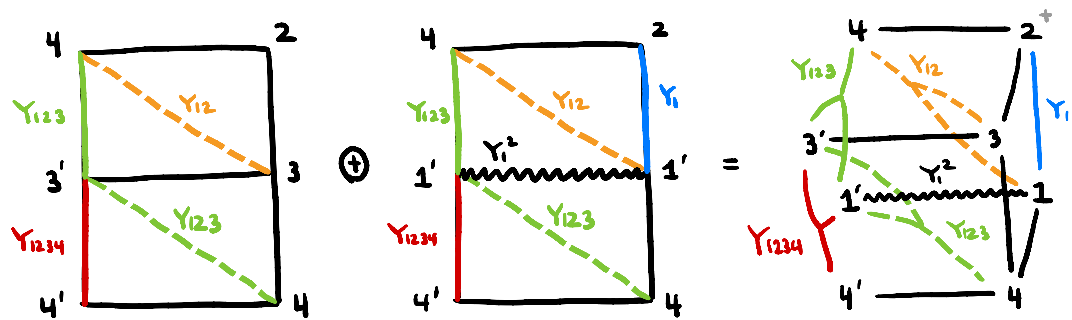




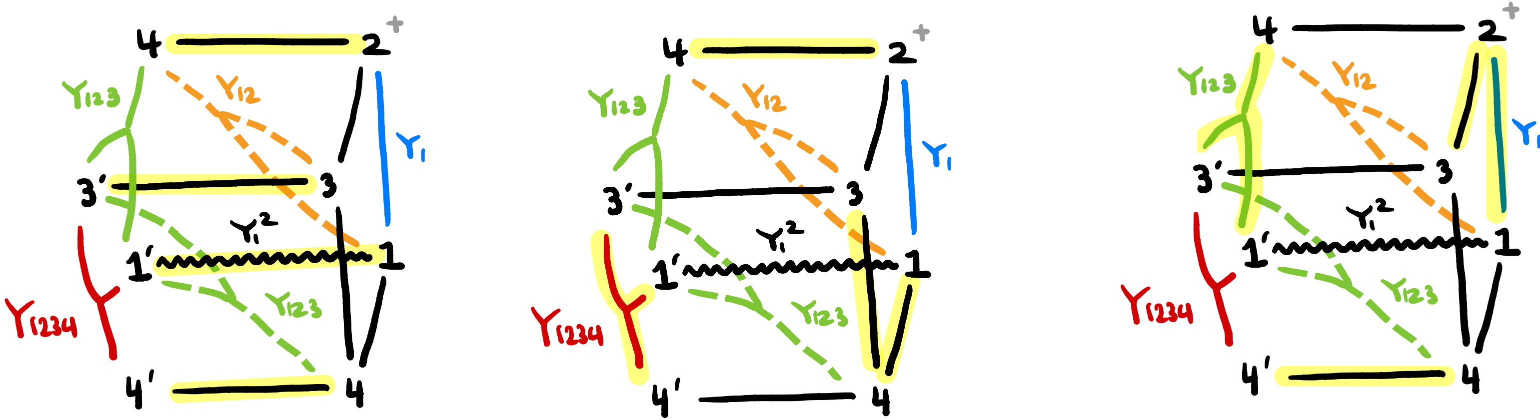


We then construct component snake graphs as usual and glue them to obtain the composite snake graph, G_4.



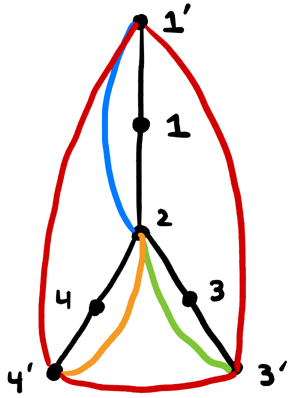
When we identify two vertices with the same label that have different edges in the same direction, that vertex picks up a $+$.

A vertex with k "$+$"s is allowed to have valency anywhere from 1 to $k-1$.



and we get the expansion $Y_4 = \dfrac{1}{Y_{12}\,Y_{123}}\left[\,Y_1^{\,2} + Y_1\,Y_{123} + Y_{1234}\,\right]$

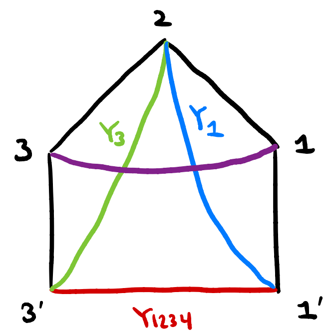
To see the other possible type of valency requirement in the composite snake graph, we'll look at another cluster.

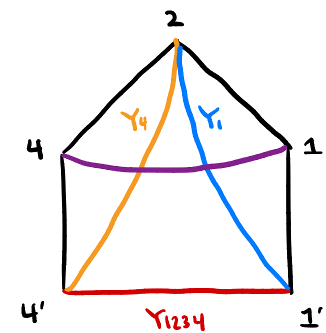


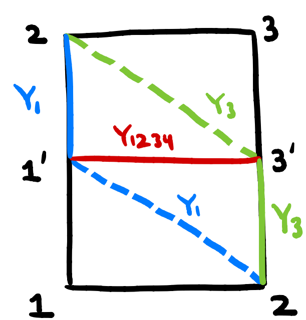
$e = \{\gamma_4, \gamma_3, \gamma_1, \gamma_{1234}\}$

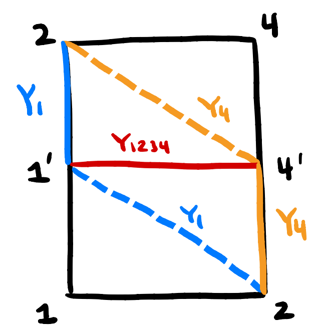
Let's construct G_2.

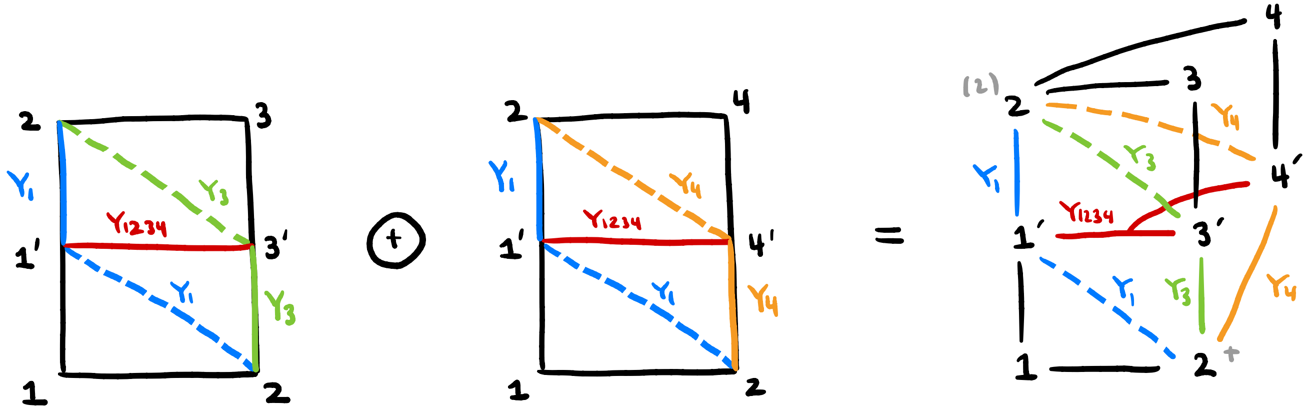
Choose incompatible neighbor 1, so $B = \{1'\}$, $E = \{3', 4'\}$.



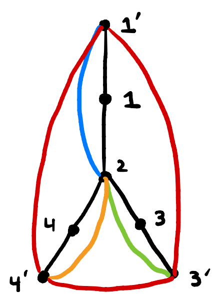








When we identify two vertices with the same label i that are incident to different diagonals in this way, we decorate that vertex with $\ell-1$, where ℓ is the number of neighbors of i in I_i.



Here, $I_2 = \{1,2,3,4\}$

$\ell = 3$

Lemma: (Banaian - Chepuri - K. - Zhang, 2022[+])

Let $\mathcal{I}$ be any maximal nested collection on a tree Γ.

Then for any $i \in V(\Gamma)$,

$$Y_i = \frac{1}{\ell(G_i)} \sum_P wt(P)$$

where $\ell(G_i)$ is the product of the diagonal labels of G_i and the summation is over all allowed matchings.

Next, we would like to construct G_S, where $S = \{s_1, \ldots, s_k\}$ is a connected vertex subset.

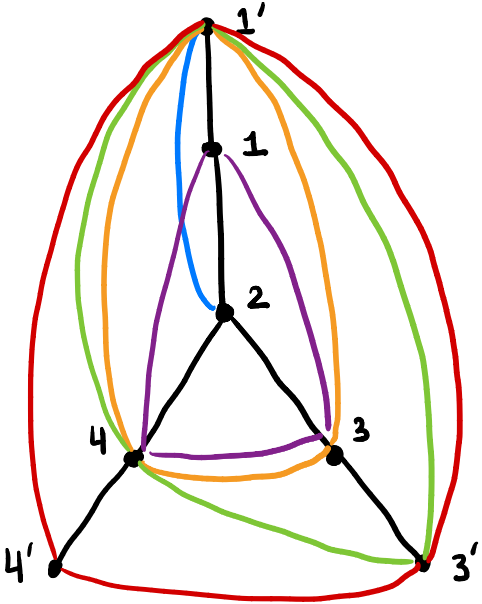
One strategy for constructing γ_S is to construct and glue together the singleton snake graphs $G_{s_1}, \ldots, G_{s_k}$.

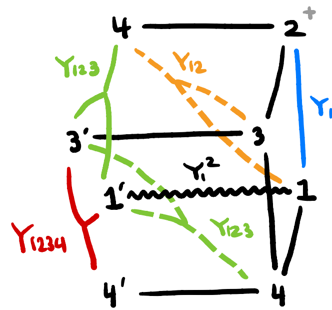
One helpful fact –

<u>Proposition</u>: (Banaian-Chepuri-K.-Zhang, 2022+)

If j is a neighbor of i, then G_i contains a unique edge $i - j$.

In our first example,





G_4

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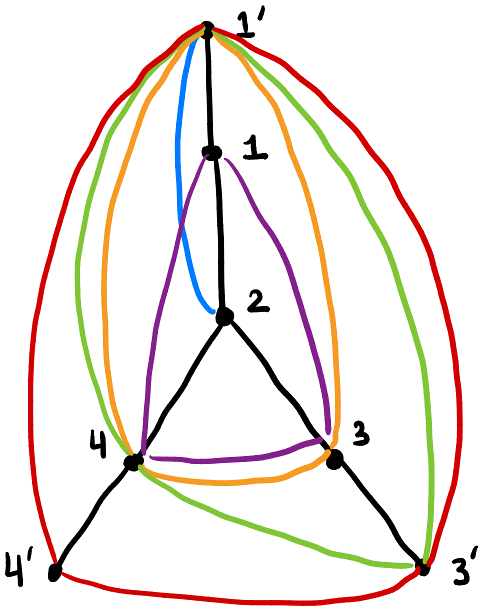
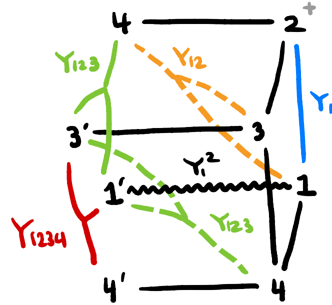
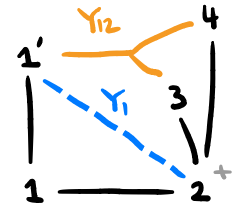




G_4



G_2

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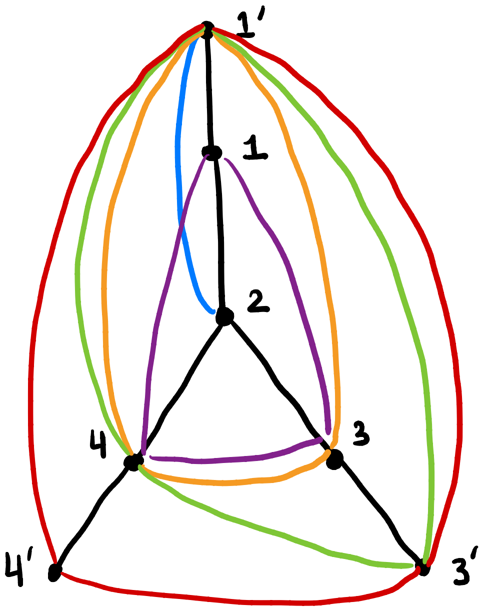
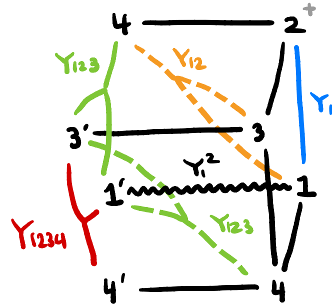
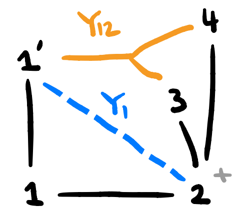


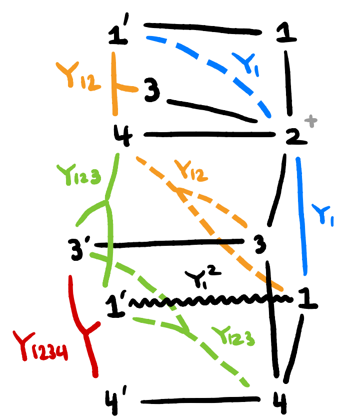


G_4

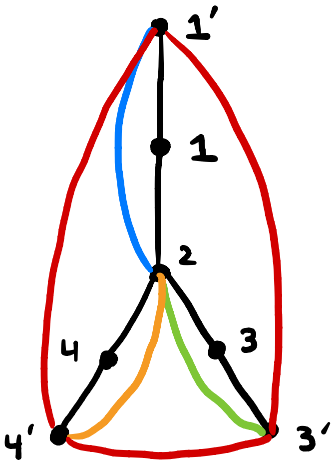


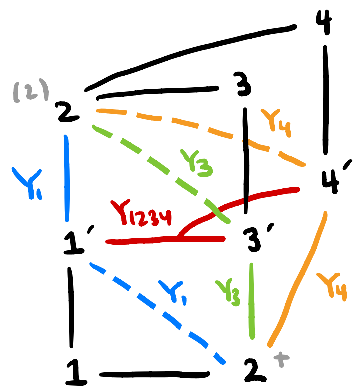
G_2

which we glue together along the 2-4 edge:



In our second example, what if we wanted to construct $G_{\{1,2\}}$?





G_2

Now, Y_1 is already in our cluster so we can't really construct "G_1".

Instead, we:

- remove the diagonal Y_1 and everything on the "opposite" side of it from the internal edge $Y_{I_2} = Y_{1234}$.
- reduce the valence of the vertex 2 that was incident to the diagonal Y_1 by one.

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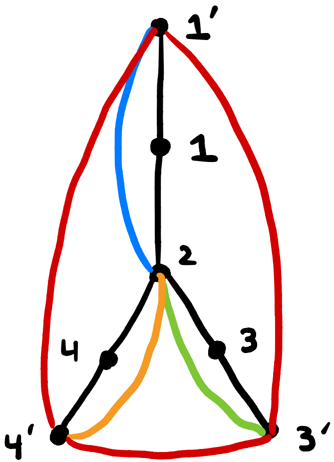

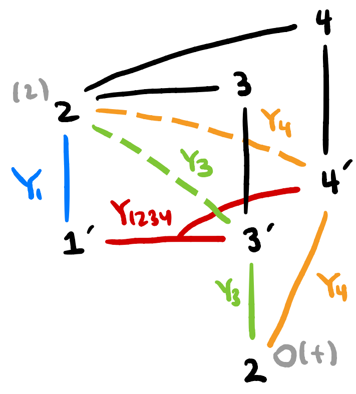

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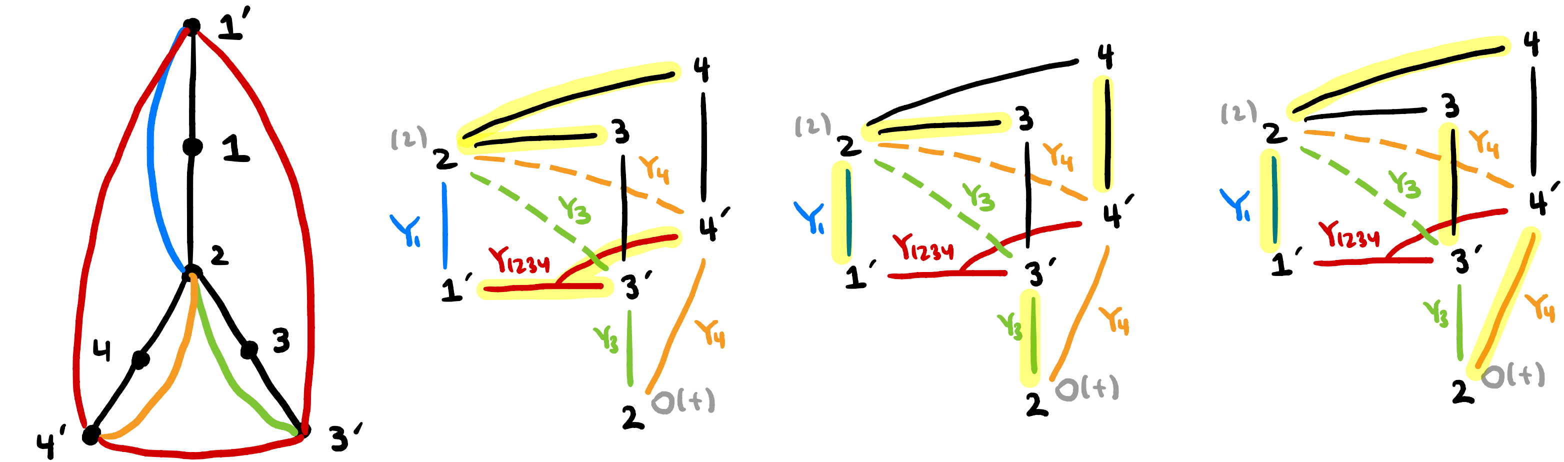

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Theorem: (Banaian - Chepuri - K. - Zhang, 2022+)

For any maximal nested collection $\mathcal{I}$ and rooted set S,

$$Y_S = \frac{1}{\ell(G_S)} \sum_P wt(P)$$

where G_S is the snake graph obtained by gluing.

Corollary: (Banaian - Chepuri - K. - Zhang, 2022+)

For any maximal nested collection $\mathcal{I}$ and rooted set S, Y_S can be expressed as a Laurent polynomial in $\mathcal{I}$ with positive coefficients.

Wrapping up loose ends...

In general, we know how to:

- Glue G_i and G_j when j is the only vertex that covers i in $P_{\mathcal{I}}$.
- "Adjoin" sets from $\mathcal{I}$.

In progress:

- Gluing for weakly rooted sets (and beyond?)

  weakly rooted: vertices i, j meet the condition for being in a rooted set OR are both in some $I \in \mathcal{I}$.

- Another method of constructing G_S ("growing snakes")

  This has some advantages & some disadvantages compared to the gluing method.

Thanks!

(and HAPPY BELATED 60th BIRTHDAY

to Professor Bernard Leclerc!)

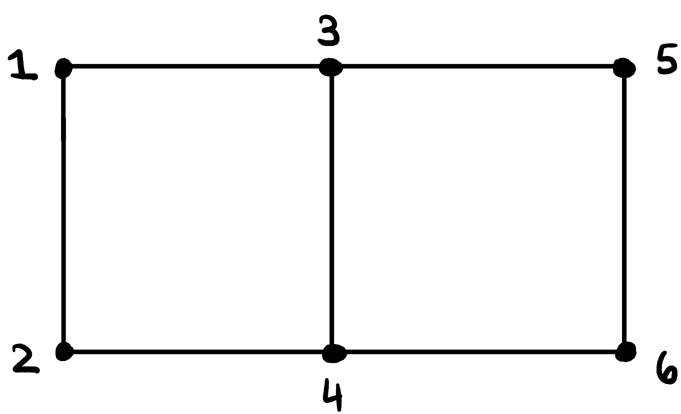
Appendix (exchange relation notation)

The actual exchange relations require a little bit more notation to state.

Let $S_i := \{S \cup i\}$

$S \oplus i$ be the connected component of S_i containing i

$S \ominus i := S_i \setminus (S \oplus i)$ i.e., the connected components of S that do not contain i.



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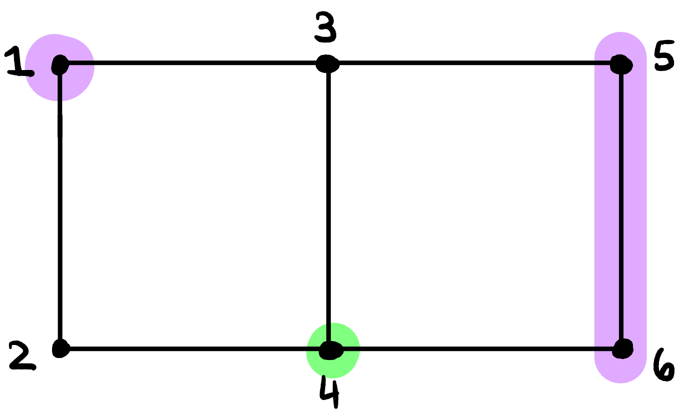

$i = 4$

$S = \{1, 5, 6\}$

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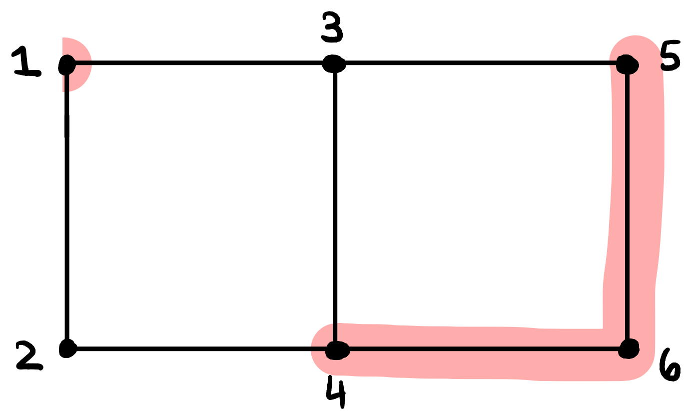

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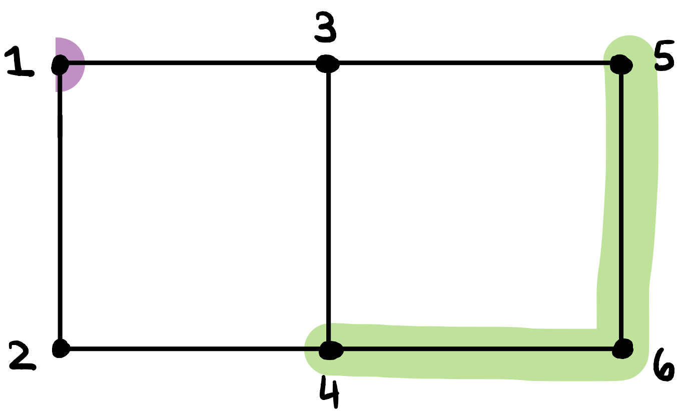


$i = 4$

$S = \{1,5,6\}$

$S_i = \{1,4,5,6\}$

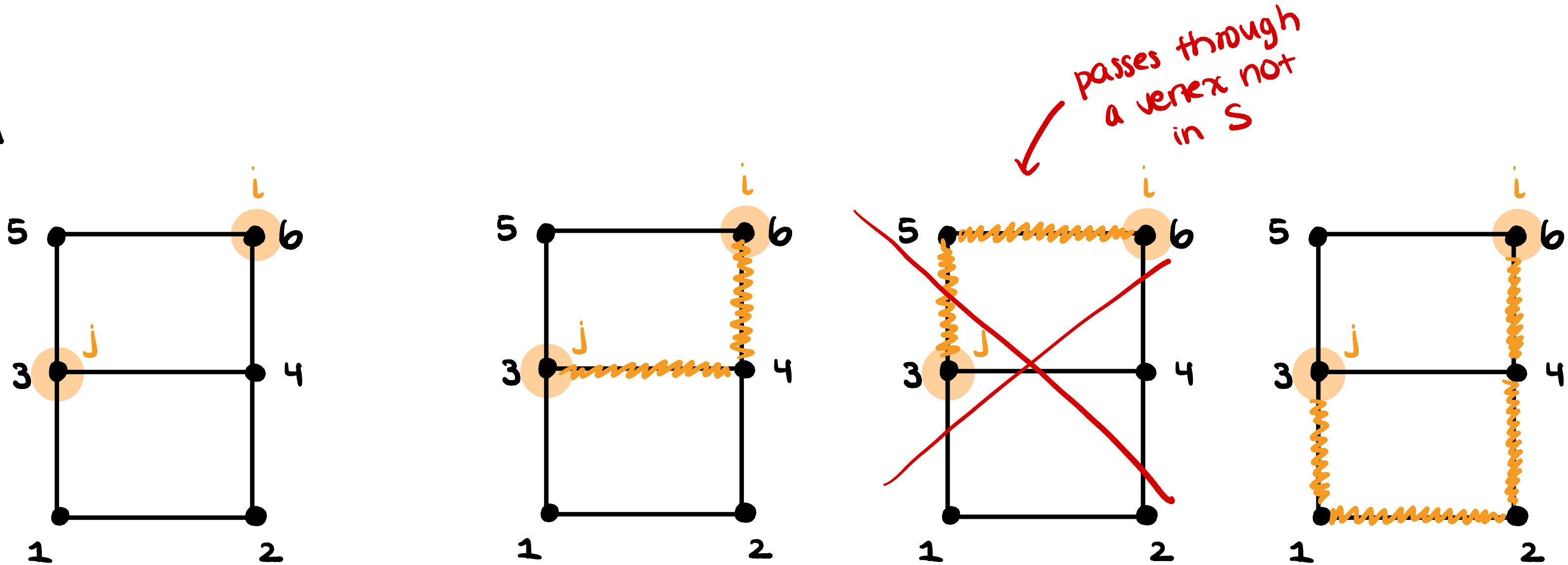
$S \oplus i = \{4,5,6\}$

$S \ominus i = \{1\}$

Let $p_S^{ij} := \{$ vertex non-repeating paths from i to j whose intermediate vertices are in S $\}$

Then define $P_S^{ij} := \sum_{p \in p_S^{ij}} Y_{S \setminus \{k \in p\}}$.

<u>EX II</u>



$S = \{1, 2, 3, 4\}$

$$P^{3,6}_{\{1,2,3,4\}} = Y_{12} + 1$$