Cluster structures on braid varieties

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Bases for Cluster Algebras, CMO
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Goal and plan

Goal: For any positive braid β, construct a cluster algebra structure on $\mathbb{C}[X(\beta)]$.

Plan:

1. Braid varieties
    - Definition.
    - Double Bott-Samelson cells as braid varieties
    - Richardson varieties as braid varieties.
2. Cluster structures on braid varieties
    - Algebraic weaves.
    - Lusztig cycles.
    - Constructing the cluster variables.
3. Properties
    - Local acyclicity.
    - Existence of reddening sequences.
    - Polinomiality.

Braid varieties: notation

Before defining braid varieties, let us fix some notation:

- G is a simple algebraic group with Dynkin diagram D (e.g. $G = \mathrm{SL}_n$).
- $B \subseteq G$ is a Borel subgroup (e.g. B = upper triangular matrices)
- $T \subseteq B$ is a maximal torus (e.g. T = diagonal matrices)
- $W = \langle s_i, i \in D \mid s_i^2 = 1, \dots \rangle$ is the Weyl group (e.g. $W = \langle s_1, \dots, s_{n-1} \rangle = S_n$), $w_0 \in W$ is its longest element (e.g. $w_0 = [n, n-1, \dots, 2, 1]$)
- $\mathrm{Br} = \langle \sigma_i, i \in D \mid \dots \rangle$ is the positive braid monoid.
- $\mathcal{B} := G/B$ is the flag variety, that admits the Bruhat decomposition:

$$\mathcal{B} := \bigsqcup_{w \in W} BwB/B$$

Demazure products

Let $\mathbf{i} = (i_1, \ldots, i_\ell) \in D^\ell$. We define the *Demazure product* of $\mathbf{i}$, $\delta(\mathbf{i}) \in W$ inductively on ℓ as follows:

- $\delta(\emptyset) = e \in W$.
- $\delta(\mathbf{i}, i_{\ell+1}) = \begin{cases} \delta(\mathbf{i}) & \text{if } \delta(\mathbf{i})s_{i_{\ell+1}} < \delta(\mathbf{i}) \\ \delta(\mathbf{i})s_{i_{\ell+1}} & \text{else.} \end{cases}$

If $\beta_{\mathbf{i}} := \sigma_{i_1} \cdots \sigma_{i_\ell} \in \mathrm{Br}$, then one can check that:

$$\beta_{\mathbf{i}} = \beta_{\mathbf{j}} \Rightarrow \delta(\mathbf{i}) = \delta(\mathbf{j})$$

so that we have a well-defined notion of $\delta(\beta) \in W$ for $\beta \in \mathrm{Br}$.

Example

For $W = S_3$, $\delta(\sigma_1^2\sigma_2^3) = s_1s_2$.

Braid varieties: Definition

Let us recall that two flags $xB, yB \in G/B$ are said to be in position $w \in W$ if $x^{-1}y \in BwB$ (e.g. for $G = \mathrm{SL}_n$, two flags are in position s_i if they differ in precisely the i-th subspace). We denote this by $xB \xrightarrow{w} yB$.

Definition

Let $\mathbf{i} := (i_1, \ldots, i_\ell) \in D^\ell$. The *braid variety* $X(\mathbf{i})$ is the space of $\ell + 1$-tuples of flags $(x_1B, x_2B, \ldots, x_{\ell+1}B) \in \mathcal{B}^{\ell+1}$ such that:

1. $x_1B = B$.
2. $x_{\ell+1}B = \delta(\mathbf{i})B$.
3. $x_jB \xrightarrow{s_{i_j}} x_{j+1}B$.

Braid varieties

Theorem (Escobar, Casals-Gorsky-Gorsky-S., Mellit, Shen-Weng)

- *The braid variety* $X(\mathbf{i})$ *is a smooth, affine variety of dimension* $\ell - \ell(\delta(\mathbf{i}))$.
- *If* $\beta_{\mathbf{i}} = \beta_{\mathbf{j}}$ *then the braid varieties* $X(\mathbf{i})$ *and* $X(\mathbf{j})$ *are canonically isomorphic.*
- *If* $\delta(\mathbf{i}, j) = \delta(\mathbf{i})s_j$ *then* $X(\mathbf{i}) \cong X(\mathbf{i}, j)$.

The second bullet point justifies the name braid variety, and we have a well-defined notion of $X(\beta)$ *for* $\beta \in \mathrm{Br}$.
The third bullet point allows us to assume wlog that $\delta(\beta) = w_0$.

Example

For $G = \mathrm{SL}_2$, let $\beta = \sigma^2$. Then, $X(\beta) = \mathbb{C}^{\times}$.

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Example

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Braid varieties: coordinates

To give coordinates to braid varieties, we use a *pinning*. These are a family of compatible maps,

$$\varphi_i : \mathrm{SL}_2 \to G, \quad i \in D.$$

and we define

$$B_i(z) := \varphi_i \begin{pmatrix} z & -1 \\ 1 & 0 \end{pmatrix} \in G, \quad z \in \mathbb{C}$$

And for $\beta = \sigma_{i_1} \cdots \sigma_{i_\ell} \in \mathrm{Br}$ define

$$B_\beta(z) := B_{i_1}(z_1) \cdots B_{i_\ell}(z_\ell), \quad z = (z_1, \ldots, z_\ell) \in \mathbb{C}^\ell.$$

So that

$$X(\beta) = \{z \in \mathbb{C}^\ell \mid \delta(\beta)^{-1} B_\beta(z) \in B\}.$$

It is known (*Lusztig 1994*) that a pinning always exists, and any two pinnings are conjugate.

Double Bott-Samelson cells

Definition (Shen-Weng)

Let $\beta \in \mathrm{Br}$. The *(half-decorated) double Bott-Samelson cell* is the locus:

$$\mathrm{Conf}(\beta) := \{z \in \mathbb{C}^\ell \mid B_\beta(z) \in B_- B\}$$

This is a Zariski (principal) open set in $\mathbb{C}^\ell$, given by the non-vanishing of several generalized minors of $B_\beta(z)$.

It is not hard to show that, if $\Delta \in \mathrm{Br}$ denotes a minimal lift of w_0 then:

$$X(\Delta\beta) \cong \mathrm{Conf}(\beta).$$

Theorem (Shen-Weng)

The variety $\mathrm{Conf}(\beta)$ *admits a cluster structure.*

Richardson varieties

Let $v, w \in W$. The *open Richardson variety* is the intersection:

$$R(v,w) := (BwB)/B \cap (B_-vB)/B \subseteq \mathcal{B}$$

of a Schubert cell and an opposite Schubert cell. It is known that this is an affine variety that is nonempty if and only if $v \leq w$. In this case, $\dim(R(v,w)) = \ell(w) - \ell(v)$.

Theorem

Let $\beta(w)$ be a reduced lift to Br of w, and similarly for $\beta(v^{-1}w_0)$. Then:

$$X(\beta(w)\beta(v^{-1}w_0)) \cong R(v,w)$$

$$B \xrightarrow{s_{i_1}} x_1B \xrightarrow{s_{i_2}} \cdots \xrightarrow{s_{i_k}} x_kB \xrightarrow{s_{j_1}} x_{k+1}B \xrightarrow{s_{j_2}} \cdots \xrightarrow{s_{j_t}} x_{k+t}B = B_-$$

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That open Richardson varieties admit cluster structures was conjectured by Leclerc in 2016. The case of *positroids* is known thanks to work of (*Galashin–Lam 2019*, *Serhiyenko–Sherman-Bennett–Williams 2019*)

Cluster structures on braid varieties

Theorem (Casals–Gorsky–Gorsky–Le–Shen–S.)

For any simple algebraic group G and any $\beta \in \mathrm{Br}$, the braid variety $X(\beta)$ admits a cluster structure.

Remark

As we have seen this morning, independent work of Galashin–Lam–Sherman-Bennett–Speyer constructs a cluster structure on $X(\beta)$. It would be interesting to compare these cluster structures.

To prove the theorem, one needs to:

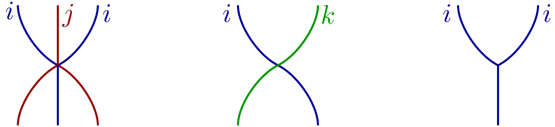
1. Find candidates for cluster tori in $X(\beta)$.
2. Find a system of coordinates for each cluster tori, that are *regular functions* on $X(\beta)$.
3. Find a mutation rule, and show that the coordinates from (2) remain regular upon mutation.

Algebraic weaves

For simplicity, we will assume that G is simply laced.

An *algebraic weave* $\mathfrak{w} : \beta \to \delta(\beta)$ is a graph on a rectangle R, whose edges are colored by the vertices of the Dynkin diagram D and whose vertices are of the following type:

- *Univalent vertices*, which are located only on the top and bottom sides of R. On the top, the colors of the edges adjacent to these vertices spell β from left-to-right. On the bottom, they spell $\delta(\beta)$.
- *Trivalent vertices*, located in the interior of R.
- *Tetravalent vertives*, located in the interior of R.
- *Hexavalent vertices*, located in the interior of R.



Algebraic weaves

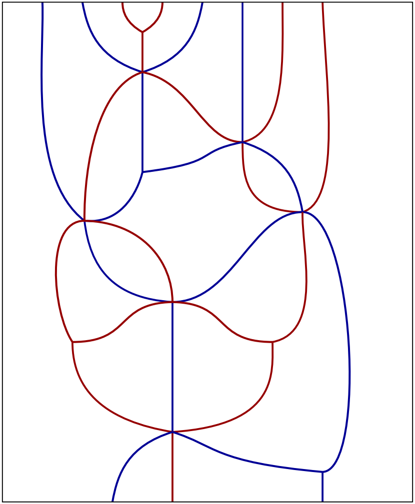


Figure: A weave $\mathfrak{w} : \sigma_1^2\sigma_2^2\sigma_1^2\sigma_2^2 \to \sigma_1\sigma_2\sigma_1$

Weaves as flag moduli

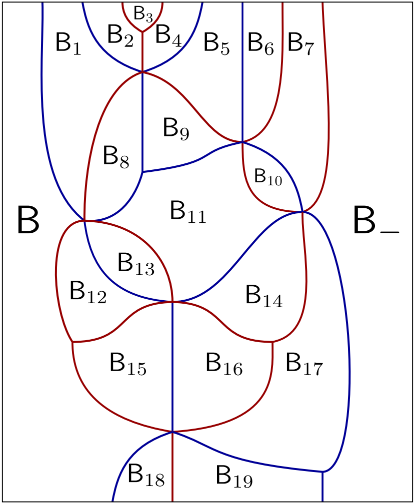
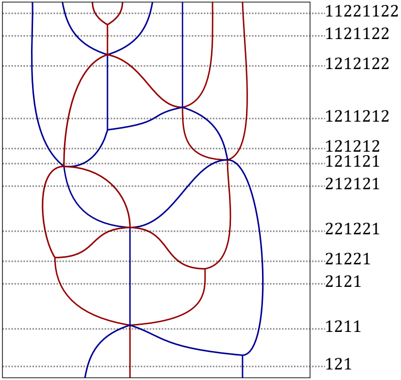


Figure: Moduli of flags determined by a weave. All flags are completely determined by those on top, and this determines an open torus $T_{\mathfrak{w}}$ in $X(\beta)$

Weaves as paths on words

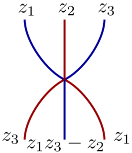
From another point of view, a weave is a sequence of braid words starting at (a word for) β and finishing at (a word for) $\delta(\beta)$, using the following types of local steps:

- *Trivalent vertices*: $\sigma_i\sigma_i \mapsto \sigma_i$.
- *Tetravalent vertices*: $\sigma_i\sigma_j \mapsto \sigma_j\sigma_i$.
- *Hexavalent vertices*: $\sigma_i\sigma_j\sigma_i \mapsto \sigma_j\sigma_i\sigma_j$.

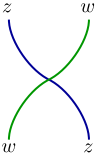


Weaves as equations of braid elements

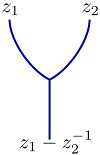
One last viewpoint on weaves is that they encode certain equations among products of elements of the form $B_i(z)$. To do this, we label every edge by (several) variables which are rational functions on $z_1, \dots, z_\ell$, the top labels being $z_1, \dots, z_\ell$.



$B_i(z_1)B_j(z_2)B_i(z_3) =$
$B_j(z_3)B_i(z_1z_3 - z_2)B_j(z_1)$



$B_i(z)B_k(w) = B_k(w)B_i(z)$



$B_i(z_1)B_i(z_2) = B_i(z_1 - z_2^{-1})U$
$z_2 \neq 0, U \in B$

Warning: A trivalent vertex is going to affect all labels to its right!

s-variables

Definition

For a trivalent vertex v, we define its *s-variable* s_v to be the label on its right incoming edge.

The s-variables are coordinates for the torus $T_{\mathfrak{w}}$ defined by the weave $\mathfrak{w}$, but they are only *rational* functions on $X(\beta)$.

Simultaneously, we will define a quiver $Q_{\mathfrak{w}}$ and create an upper unitriangular change of variables that gives a system of coordinates in $T_{\mathfrak{w}}$ consisting of *regular* functions on $X(\beta)$.

Lusztig cycles

For any trivalent vertex v, we define a function $\gamma_v : \text{edges}(\mathfrak{w}) \to \mathbb{Z}_{\geq 0}$ as follows

- For any edge above v, $\gamma_v(e) = 0$.
- For the outgoing edge of v, $\gamma_v(e) = 1$.
- Below v, γ_v satisfies a tropical version of Lusztig's coordinates:
  - If e_1, e_2 are the incoming edges of a trivalent vertex v' and e_3 is the outgoing edge then $\gamma_v(e_3) = \min(\gamma_v(e_1), \gamma_v(e_2))$.
  - If e_1, e_2 are the incoming edges of a tetravalent vertex, and e_1', e_2' the outgoing edges, then $\gamma_v(e_1') = \gamma_v(e_2)$, $\gamma_v(e_2') = \gamma_v(e_1)$.
  - If e_1, e_2, e_3 are the incoming edges of a hexavalent vertex and e_1', e_2', e_3' the outgoing edges, then

$$\gamma_v(e_1') = \gamma_v(e_2) + \gamma_v(e_3) - \min(\gamma_v(e_1), \gamma_v(e_3)),$$

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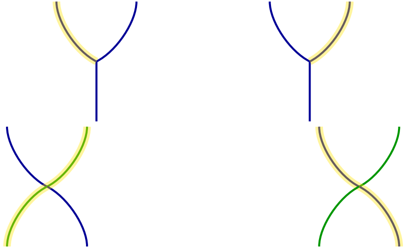

  - If e_1, e_2, e_3 are the incoming edges of a hexavalent vertex and e'_1, e'_2, e'_3 the outgoing edges, then

$$\gamma_v(e'_1) = \gamma_v(e_2) + \gamma_v(e_3) - \min(\gamma_v(e_1), \gamma_v(e_3)),$$
$$\gamma_v(e'_2) = \min(\gamma_v(e_1), \gamma_v(e_3)),$$
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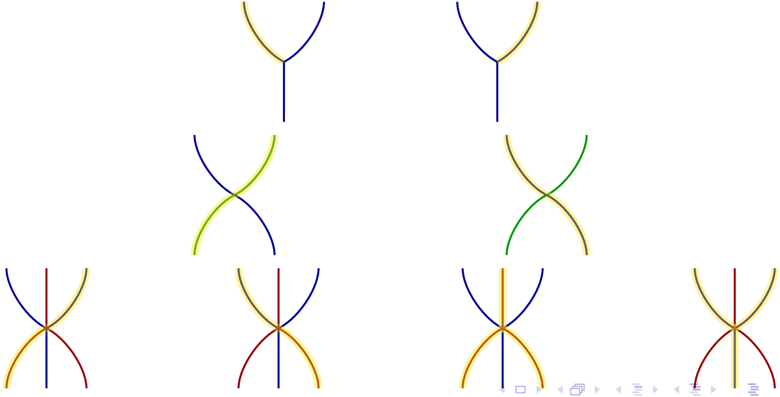

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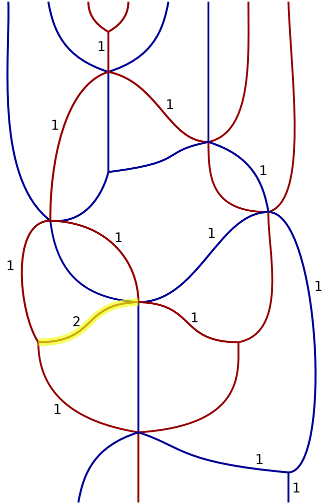
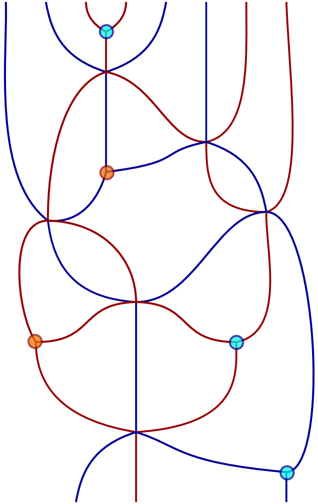


Figure: The cycle γ_v for the topmost trivalent vertex of $\mathfrak{w}$.

Frozen and mutable

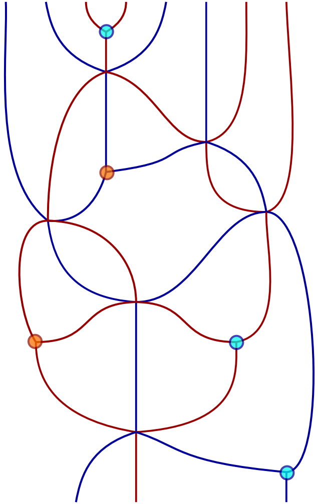
We say that a trivalent vertex v of $\mathfrak{w}$ is *frozen* if there exists an edge e on the bottom of $\mathfrak{w}$ such that $\gamma_v(e) \neq 0$. Else, we say that v is *mutable*.



Frozen and mutable

Equivalently, a trivalent vertex $\beta_1\sigma_i\sigma_i\beta_2 \to \beta_1\sigma_i\beta_2$ is frozen if

$$\delta(\beta_1\beta_2) < \delta(\beta_1\sigma_i\beta_2)$$

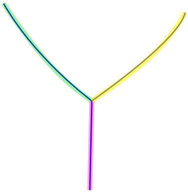


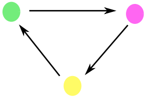
Intersections

Now we define a skew-symmetric matrix ε using *intersections* of cycles at tri- and hexa-valent vertices.

- If t is a trivalent vertex of $\mathfrak{w}$ with incoming edges e_1, e_2 and outgoing edge e_3 then:

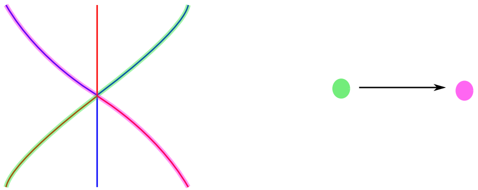
$$\#_t(\gamma_v, \gamma_{v'}) = \begin{vmatrix} 1 & 1 & 1 \\ \gamma_v(e_1) & \gamma_v(e_3) & \gamma_v(e_2) \\ \gamma_{v'}(e_1) & \gamma_{v'}(e_3) & \gamma_{v'}(e_2) \end{vmatrix}$$





- If t is a hexavalent vertex of $\mathfrak{w}$ with incoming edges e_1, e_2, e_3 and outgoing edges e_1', e_2', e_3' then $\#_t(\gamma_v, \gamma_{v'})$ is:

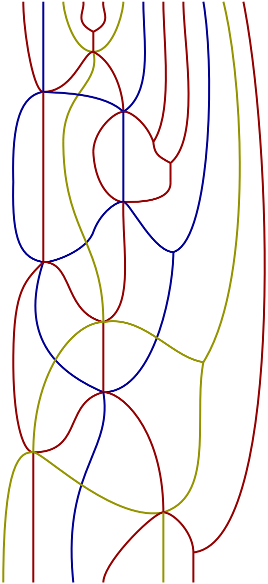
$$\frac{1}{2}\left(\begin{vmatrix} 1 & 1 & 1 \\ \gamma_v(e_1) & \gamma_v(e_2) & \gamma_v(e_3) \\ \gamma_{v'}(e_1) & \gamma_{v'}(e_2) & \gamma_{v'}(e_3) \end{vmatrix} - \begin{vmatrix} 1 & 1 & 1 \\ \gamma_v(e_1') & \gamma_v(e_2') & \gamma_v(e_3') \\ \gamma_{v'}(e_1') & \gamma_{v'}(e_2') & \gamma_{v'}(e_3') \end{vmatrix}\right)$$



And we define

$$\varepsilon_{v,v'} := \sum_t \#_t(\gamma_v, \gamma_{v'}).$$

Example



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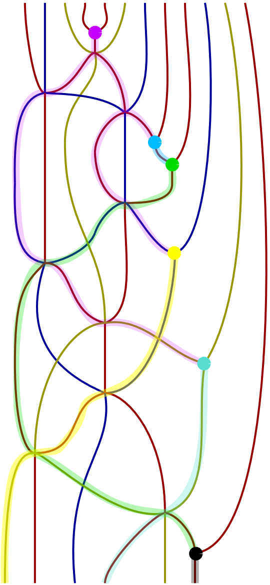

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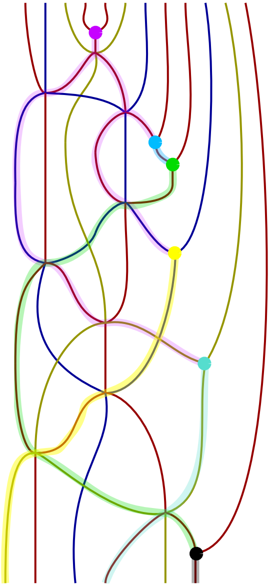
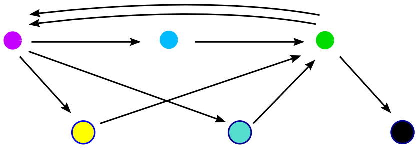




Cluster variables

Recall that the s-vairable s_v of a trivalent vertex v is the rational function labeling its right incoming edge.

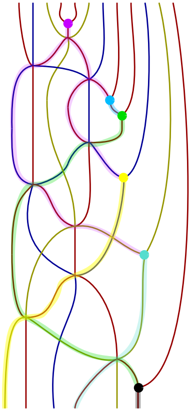
Theorem (Casals-Gorsky-Gorsky-Le-Shen-S.)

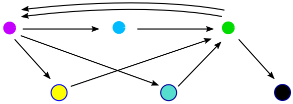
Let $\mathfrak{w}$ be a weave such that, for each trivalent vertex v, either its right arm e_r^v or its left arm e_l^v goes all the way to the top. Then,

$$A_v := s_v \times \prod_{v'} A_{v'}^{\gamma_{v'}(e_r^v)+\gamma_{v'}(e_l^v)}$$

is a regular function on $X(\beta)$, and together with the intersection form give $X(\beta)$ a cluster structure.

Example





$A_1 = z_5$, $A_2 = -z_6z_7 + z_5z_8$, $A_3 = -z_6z_7z_9 + z_5z_8z_9 - z_5$,
$A_4 = -z_6z_9 + z_5z_{10}$, $A_5 = -z_7z_9 + z_5z_{11}$, $A_6 = z_6z_7z_{10}z_{11} - z_5z_8z_{10}z_{11} - z_6z_7z_9z_{12} + z_5z_8z_9z_{12} - z_8z_9 + z_7z_{10} + z_6z_{11} - z_5z_{12} + 1$.

Example

Mutating:

$$
\begin{aligned}
A_1' &= \frac{A_2A_4A_5+A_3^2}{A_1} \\
&= z_6z_7z_8z_9^2 + z_5z_8^2z_9^2 + z_6z_7^2z_9z_{10} - z_5z_7z_8z_9z_{10} + z_6^2z_7z_9z_{11} + \\
&\quad -z_5z_6z_8z_9z_{11} - z_5z_6z_7z_{10}z_{11} + z_5^2z_8z_{10}z_{11} + 2z_6z_7z_9 - 2z_5z_8z_9 + z_5
\end{aligned}
$$

$$A_2' = \frac{A_1+A_3}{A_2} = z_9.$$

$$A_3' = \frac{A_2A_4A_5+A_1^2A_6}{A_3} = z_6z_7z_9 - z_5z_7z_{10} - z_5z_6z_{11} + z_5^2z_{12} - z_5.$$

These are all regular, and in fact *polynomials!*

Weave mutation

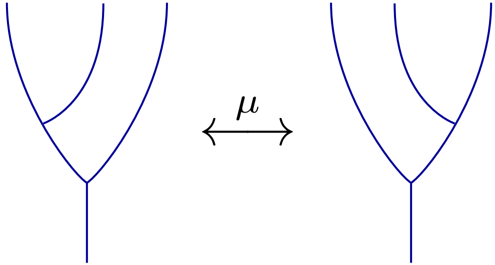


Figure: Weave mutation corresponds to cluster mutation.

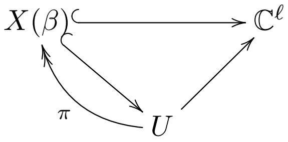
Theorem (Elias, CGGLSS)

For a fixed expression for $\delta(\beta)$, *any two weaves* $\mathfrak{w}, \mathfrak{w}' : \beta \to \delta(\beta)$ *are related by a sequence of equivalences and mutations.*

Polinomiality

Theorem (CGGLSS)

The way we have defined cluster variable starting from s-variables, the exchange relations are already valid in the polynomial algebra $\mathbb{C}[z_1, \ldots, z_\ell]$.



Polinomiality

Theorem (CGGLSS)

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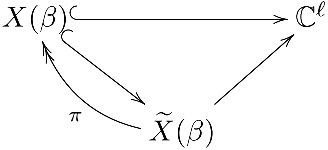


$$\widetilde{X}(\beta) := \{B \xrightarrow{s_{i_1}} x_1 B \xrightarrow{s_{i_2}} \cdots \xrightarrow{s_{i_\ell}} x_\ell B \mid x_\ell B \in B w_0 B/B\}$$

Fibers of π are affine spaces of dimension $\ell(w_0)$. When $\beta = \Delta\beta'$ in fact $\widetilde{X}(\beta) = \mathbb{C}^{\ell(w_0)} \times X(\beta)$.

Properties

The cluster structure on $X(\beta)$ satisfies the following properties:

- *Cyclic rotation.* If $s_{i^*} = w_0 s_i w_0$ then we have an isomorphism $\mathbb{C}[X(\beta\sigma_i)] \to \mathbb{C}[X(\sigma_{i^*}\beta)]$. This is a quasi-cluster isomorphism. (see also *(Casals-Weng '22)*)
- $\mathcal{A} = \mathcal{U}$. We have $\mathbb{C}[X(\beta)] = \mathcal{A}(Q_{\mathfrak{w}}) = \mathcal{U}(Q_{\mathfrak{w}})$ for any weave $\mathfrak{w}$. Moreover, the elements $z_i \in \mathbb{C}[X(\beta)]$ are cluster monomials (for probably different clusters).
- *Full rank.* The exchange matrix $\varepsilon_{\mathfrak{w}}$ has full rank.
- *Local acyclicity.* The cluster algebra $\mathcal{A}(Q_{\mathfrak{w}})$ is locally acyclic. In fact, $X(\beta)$ can be covered with cluster open sets of the form $X(\beta')$ for smaller braids β'.

Reddening sequences

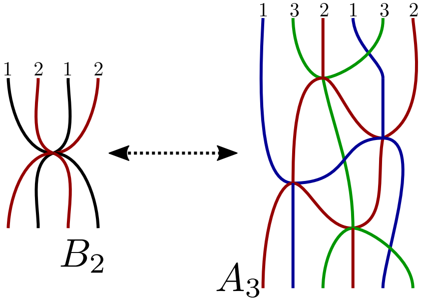
- Upon the identification $X(\Delta\beta) \cong \mathrm{Conf}(\beta)$, we obtain the same cluster structure as Shen-Weng. Moreover, if $\mathfrak{w}$ is a weave on $\Delta\beta$ such that, for every trivalent vertex v, its *right* arm goes all the way to the top, then we obtain the quiver associated to the wiring diagram of β.
- If $\delta(\beta\sigma_i) = \delta(\beta)$, then the quiver for $X(\beta)$ is obtained from that of $X(\beta\sigma_i)$ by deleting a frozen sink and freezing all variables adjacent to this frozen variable.
- If $\delta(\sigma_i\beta) = \delta(\beta)$, then the quiver for $X(\beta)$ is obtained from that of $X(\sigma_i\beta)$ by deleting a frozen source and freezing all variables adjacent to this frozen variable.
- It follows that this cluster structure admits a reddening sequence.
- It also follows that $\mathbb{C}[X(\beta)]$ admits a basis of ϑ-functions.

Thanks for your attention!

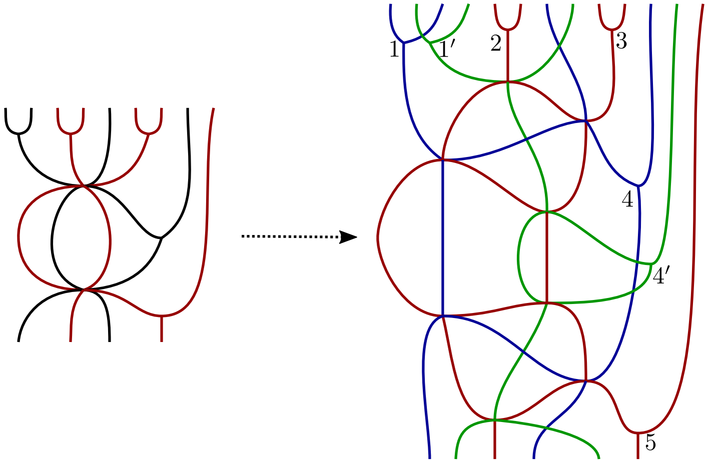
Happy Birthday Professor Leclerc!

The non-simply laced case

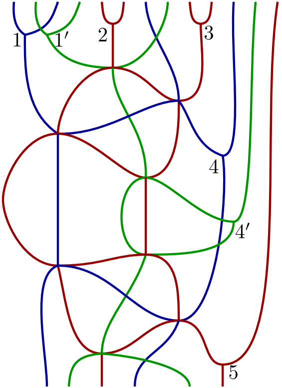
If G is non-simply laced, we still have the notion of a weave, where now we have $(2d)$-valent vertices as well. Any weave in non-simply laced type unfolds to one in simply-laced type, and we obtain the cluster structure by identifying cluster variables in the simply-laced type.

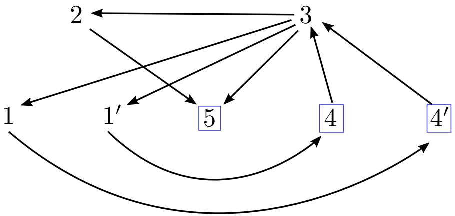


Non-simply laced example



Non-simply laced example





Non-simply laced example