# Cohomology of stacks of shtukas II

§0.

~~Recall~~ Reminder and remark:

Yesterday, for any $I=\{1,2,\cdots,n\}$ finite set, we defined the stack of shtukas with $I$ paws: $Sht_I$. In fact, usually the notation is $Sht_I^{(1,2,\cdots,n)}$. Recall that for any affine scheme $S$ over $\mathbb{F}_q$,

$$Sht_I^{(1,2,\cdots,n)}(S) = \left\{ (x_1,\cdots,x_n),\ \mathcal{G}_0 \overset{\phi_1}{\dashrightarrow} \mathcal{G}_1 \overset{\phi_2}{\dashrightarrow} \cdots \dashrightarrow \mathcal{G}_{n-1} \overset{\phi_n}{\dashrightarrow} {}^{\tau}\mathcal{G}_0 \right\}$$

where $x_i \in X(S)$

$\mathcal{G}_0, \mathcal{G}_1, \cdots, \mathcal{G}_{n-1}$ are $G$-bundles over $X\times S$

$\mathcal{G}_{i-1} \overset{\phi_i}{\dashrightarrow} \mathcal{G}_i$ ~~are~~ is isomorphism ~~outside~~ over $X\times S - \Gamma_{x_i}$

${}^{\tau}\mathcal{G}_0 := (Id_X \times Frob_S)^* \mathcal{G}_0$

$\Gamma_{x_i}$: graph of $x_i$ in $X\times S$

---

We can also define ~~the~~ a stack of shtukas without intermediate modifications:

$$Sht_I^{(I)}(S) = \left\{ (x_1,\cdots,x_n),\ \mathcal{G}_0 \overset{\phi}{\dashrightarrow} {}^{\tau}\mathcal{G}_0 \right\}$$

where $\phi$ is isomorphism over $X\times S - \bigcup_{i\in I} \Gamma_{x_i}$

$$\begin{tikzcd}
\mathrm{Sht}_I^{(1,2,\cdots,n)} \arrow[d, "\pi"] \arrow[dd, bend right=60, "p^{(1,2,\cdots,n)}"'] & \left( (x_1,\cdots,x_n),\ \mathcal{E}_0 \overset{\phi_1}{\dashrightarrow} \mathcal{E}_1 \overset{\phi_2}{\dashrightarrow} \cdots \dashrightarrow \mathcal{E}_{n-1} \overset{\phi_n}{\dashrightarrow} {}^{\tau}\mathcal{E}_0 \right) \arrow[d, mapsto] \\
\mathrm{Sht}_I^{(I)} \arrow[d, "p^{(I)}"] & \left( (x_1,\cdots,x_n),\ \mathcal{E}_0 \overset{\phi_n\circ\cdots\circ\phi_1}{\dashrightarrow} {}^{\tau}\mathcal{E}_0 \right) \arrow[d, mapsto] \\
X^I & (x_1,\cdots,x_n)
\end{tikzcd}$$

Recall that we have for any $W \in \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\widehat{G}^I)$

$$\begin{tikzcd}
\mathcal{F}_{I,W}^{(1,2,\cdots,n)} := \epsilon^* \mathcal{S}_{I,W}^{(1,2,\cdots,n)} & & \mathcal{S}_{I,W}^{(1,2,\cdots,n)} \leftarrow \text{Satake sheaf} \\
\mathrm{Sht}_I^{(1,2,\cdots,n)} \arrow[r, "\epsilon"] \arrow[d, "\pi"] \arrow[dr, phantom, "\square"] & \left[ (L^+G)_I \backslash \mathrm{Gr}_{G,I}^{(1,2,\cdots,n)} \right] \arrow[d, "\pi"] \\
\mathrm{Sht}_I^{(I)} \arrow[r, "\epsilon"] & \left[ (L^+G)_I \backslash \mathrm{Gr}_{G,I}^{(I)} \right] \\
\mathcal{F}_{I,W}^{(I)} := \epsilon^* \mathcal{S}_{I,W}^{(I)} & & \mathcal{S}_{I,W}^{(I)} \leftarrow \text{Satake sheaf}
\end{tikzcd}$$

Fact: $\pi$ is a small morphism.

Geometric Satake: $\mathcal{S}_{I,W}^{(I)} = \pi_! \mathcal{S}_{I,W}^{(1,2,\cdots,n)}$

$\Big\}$ non trivial

By proper base change, $\mathcal{F}_{I,W}^{(I)} = \pi_! \mathcal{F}_{I,W}^{(1,2,\cdots,n)}$

So $\mathcal{H}_{I,W} = Rp^{(1,2,\cdots,n)}_! \mathcal{F}_{I,W}^{(1,2,\cdots,n)} = Rp^{(I)}_! \mathcal{F}_{I,W}^{(I)}$

## §1. Partial Frobenius morphisms

We have a commutative diagram:

$$\Big((x_1,\cdots,x_n),\ \boxed{\mathcal{E}_0 \overset{\phi_1}{\dashrightarrow} \mathcal{E}_1} \overset{\phi_2}{\dashrightarrow} \cdots \dashrightarrow \mathcal{E}_{n-1} \overset{\phi_n}{\dashrightarrow} {}^\tau\mathcal{E}_0\Big) \mapsto \Big((\mathrm{Frob}(x_1), x_2,\cdots,x_n),\ \mathcal{E}_1 \overset{\phi_2}{\dashrightarrow} \mathcal{E}_2 \dashrightarrow \cdots \overset{\phi_n}{\dashrightarrow} \boxed{{}^\tau\mathcal{E}_0 \overset{{}^\tau\phi_1}{\dashrightarrow} {}^\tau\mathcal{E}_1}\Big)$$

$$\begin{tikzcd}
\mathrm{Sht}_I^{(1,2,\cdots,n)} \arrow[r, "\mathrm{Frob}_{\{1\}}"] \arrow[d, "p^{(1,\cdots,n)}"] & \mathrm{Sht}_I^{(2,3,\cdots,n,1)} \arrow[d, "p^{(2,\cdots,n,1)}"] \\
X^I \arrow[r, "\mathrm{Frob}_{\{1\}}"] & X^I
\end{tikzcd}$$

$$(x_1,\cdots,x_n) \longmapsto (\mathrm{Frob}(x_1), x_2,\cdots,x_n)$$

Similarly, we can define $\mathrm{Frob}_{\{2\}}, \cdots, \mathrm{Frob}_{\{n\}}$. By definition,

$\mathrm{Frob}_{\{n\}} \circ \cdots \circ \mathrm{Frob}_{\{2\}} \circ \mathrm{Frob}_{\{1\}} = \mathrm{Frob} \leftarrow$ total Frobenius morphism on $\mathrm{Sht}_I^{(1,2,\cdots,n)}$, i.e.

sending $\Big((x_1,\cdots,x_n), \mathcal{E}_0 \overset{\phi_1}{\dashrightarrow} \mathcal{E}_1 \dashrightarrow \cdots \overset{\phi_n}{\dashrightarrow} {}^\tau\mathcal{E}_0\Big)$ to $\Big((\mathrm{Frob}(x_1),\cdots,\mathrm{Frob}(x_n)), {}^\tau\mathcal{E}_0 \overset{{}^\tau\phi_1}{\dashrightarrow} {}^\tau\mathcal{E}_1 \dashrightarrow \cdots \overset{{}^\tau\phi_n}{\dashrightarrow} {}^{\tau\tau}\mathcal{E}_0\Big)$

Fact: we can construct a canonical morphism

$$\mathrm{Frob}_{\{1\}}^* \mathcal{F}_{I,W}^{(2,\cdots,n,1)} \simeq \mathcal{F}_{I,W}^{(1,\cdots,n)}$$

It induces a morphism

$$\mathrm{Frob}_{\{1\}} : \mathrm{Frob}_{\{1\}}^* \mathcal{H}_{I,W} \xrightarrow{\sim} \mathcal{H}_{I,W} \quad \text{over } X^I.$$

$$\mathrm{Frob}_{\{n\}} \circ \cdots \circ \mathrm{Frob}_{\{2\}} \circ \mathrm{Frob}_{\{1\}} = \mathrm{Frob} : \quad \mathrm{Frob}^* \mathcal{H}_{I,W} \xrightarrow{\sim} \mathcal{H}_{I,W}$$

$\uparrow$ total Frobenius

~~Let~~ Moreover, ~~for any $\kappa$~~ there exists a dominant coweight $\kappa$ of $G$, s.t. for any dominant coweight $\mu$,

$$\mathrm{Frob}_{\{i\}} : \mathrm{Frob}_{\{i\}}^* \mathcal{H}_{I,W}^{\leq \mu} \longrightarrow \mathcal{H}_{I,W}^{\leq \mu+\kappa} \quad \text{over } X^I .$$

## § 2. Hecke operators

Let $v$ be a place of $X$.

Let $V \in \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\hat{G})$ be an irreducible representation (of highest weight $\lambda$).

We have

$$\begin{tikzcd} & \Gamma(\lambda) \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & \\ \mathrm{Sht}_I\big|_{(X-v)^I} & & \mathrm{Sht}_I\big|_{(X-v)^I} \end{tikzcd}$$

$$\Gamma(\lambda)(S) = \left\{ \begin{array}{l} (x_1, \cdots, x_n), \ x_i \in (X-v)(S) \\ \begin{tikzcd} \mathcal{E}_0 \arrow[r, dashed, "\phi"] \arrow[d, dashed, "\alpha_0"'] \arrow[dr, phantom, "\circlearrowright"] & {}^\tau\mathcal{E}_0 \arrow[d, dashed, "{}^\tau\alpha_0"] \\ \mathcal{E}_0' \arrow[r, dashed, "\phi'"] & {}^\tau\mathcal{E}_0' \end{tikzcd} \end{array} \right\}$$

where
- $\phi$, $\phi'$ ~~is~~ are isomorphisms over $X \times S - \bigcup_{i \in I} \Gamma_{x_i}$

- $\alpha_0$ is isomorphism over $(X-v) \times S$

- relative position of $\xi_0$ and $\xi_0'$ on $v$ is given by $\lambda$

( i.e. $\forall\ \bar{s}$ geo. point of $S$, $\alpha_0 : \xi_0|_{X \times \bar{s}} \dashrightarrow \xi_0'|_{X \times \bar{s}}$

~~note~~ let $D_v$ be the formal disc on $v$

$D_v^\circ$ be the punctured formal disc on $v$

then $\alpha_0 : \xi_0|_{D_v^\circ} \xrightarrow{\sim} \xi_0'|_{D_v^\circ}$ ~~is~~ determinates an element in

$G(\mathcal{O}_v) \backslash G(F_v) / G(\mathcal{O}_v)$, which is ~~$G(\mathcal{O}_v) t^\lambda G(\mathcal{O}_v)$~~ $G(\mathcal{O}_v)\, \varpi_v^\lambda\, G(\mathcal{O}_v)$. )

$\mathcal{O}_v$: completed local ring of $X$ ~~on~~ at $v$

$F_v$: fraction field

$\varpi_v$: uniformazor

$$\left( (x_1,\dots,x_n),\ \alpha_0 : \begin{tikzcd} \mathcal{G}_0 \arrow[r, dashed, "\phi"] \arrow[d, dashed] & {}^{\tau}\mathcal{G}_0 \arrow[d, dashed, "{}^{\tau}\alpha_0"] \\ \mathcal{G}_0' \arrow[r, dashed, "\phi'"] & {}^{\tau}\mathcal{G}_0' \end{tikzcd} \right) \qquad x_i \in (X-v)(S)$$

$$\begin{tikzcd}[column sep=small] & \Gamma(\lambda) \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & \\ \mathrm{Sht}_I|_{(X-v)^I} \arrow[dr, "\epsilon"'] & \circlearrowright & \mathrm{Sht}_I|_{(X-v)^I} \arrow[dl, "\epsilon"] \\ & \left[ (L^+G)_I \backslash \mathrm{Gr}_{G,I} \right]|_{(X-v)^I} & \end{tikzcd}$$

$\left( (x_1,\dots,x_n),\ \mathcal{G}_0 \overset{\phi}{\dashrightarrow} {}^{\tau}\mathcal{G}_0 \right)$ $\qquad$ $\left( (x_1,\dots,x_n),\ \mathcal{G}_0' \overset{\phi'}{\dashrightarrow} {}^{\tau}\mathcal{G}_0' \right)$

$$\left( (x_1,\dots,x_n),\ \begin{tikzcd} \mathcal{G}_0|_{\cup \Gamma_\infty x_i} \arrow[r, dashed, "\phi"] \arrow[d, phantom, "\wr"] & {}^{\tau}\mathcal{G}_0|_{\cup \Gamma_\infty x_i} \arrow[d, phantom, "\wr"] \\ \mathcal{G}_0'|_{\cup \Gamma_\infty x_i} \arrow[r, dashed, "\phi'"] & {}^{\tau}\mathcal{G}_0'|_{\cup \Gamma_\infty x_i} \end{tikzcd} \right)$$

$$pr_1^* \mathcal{F}_{I,W} = pr_1^* \epsilon^* S_{I,W} \simeq pr_2^* \epsilon^* S_{I,W} = pr_2^* \mathcal{F}_{I,W} \qquad \circledast$$

Then consider

$$\begin{tikzcd}[column sep=small] & \Gamma(\lambda) \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & \\ \mathrm{Sht}_I|_{(X-v)^I} \arrow[dr, "p"'] & & \mathrm{Sht}_I|_{(X-v)^I} \arrow[dl, "p"] \\ & (X-v)^I & \end{tikzcd}$$

$pr_1, pr_2$ are finite étale

$$\mathcal{H}_{I,W}\big|_{(X-v)^I} = p_! \mathcal{F}_{I,W} \longrightarrow p_!(pr_1)_*(pr_1)^* \mathcal{F}_{I,W} = p_!(pr_1)_!(pr_1)^* \mathcal{F}_{I,W} \overset{\circledast}{\xrightarrow{\sim}} p_!(pr_1)_!(pr_2)^* \mathcal{F}_{I,W}$$

$$\wr$$

$$p_!(pr_2)_!(pr_2)^* \mathcal{F}_{I,W}$$

$$\downarrow$$

$$\mathcal{H}_{I,W}\big|_{(X-v)^I} = p_! \mathcal{F}_{I,W}$$

This gives the Hecke operator associated to the ~~cara~~ characteristic

function $\mathbb{1}_{G(\mathcal{O}_v)\varpi_v^\lambda G(\mathcal{O}_v)} \in \mathrm{Funct}_c\left( G(\mathcal{O}_v)\backslash G(F_v)/G(\mathcal{O}_v), \overline{\mathbb{Q}_\ell} \right)$

$$\| \qquad \mathcal{H}_{G,v} \text{ local Hecke algebra}$$

In general, for any $f \in \mathcal{H}_{G,v}$,

we have a Hecke operator

$$T(f): \mathcal{H}_{I,W}^{\leq \mu}\Big|_{(X-v)^I} \longrightarrow \mathcal{H}_{I,W}^{\leq \mu+K}\Big|_{(X-v)^I}$$

In particular, for any $V \in \mathrm{Rep}_{\overline{\mathbb{Q}_\ell}}(\widehat{G})$, by the Satake isomorphism, we have $h_{V,v} \in \mathcal{H}_{G,v}$. We are interested in $T(h_{V,v})$.

We want to extend this operator to $X^I$. For this, we need a special case of excursion operator.

## §3. A special case of excursion operator.

Let $v$ be a place of $X$. Let $V \in \mathrm{Rep}_{\overline{\mathbb{Q}_\ell}}(\widehat{G})$.

Let $\delta_V : \mathbb{1} \longrightarrow V \otimes V^*$

$$1 \longmapsto \sum e_i \otimes e_i^*$$

Creation operator $\mathcal{C}^{\#}_{\delta_V}$ :

$$\Delta: X \xhookrightarrow{\text{diag}} X \times X$$

$$(\overline{\mathbb{Q}_\ell})_v \boxtimes \mathcal{H}_{I,W} \simeq \mathcal{H}_{\{0\}\cup I, \mathbb{1}\boxtimes W}\Big|_{v \times X^I} \xrightarrow{\delta_V \boxtimes Id_W} \mathcal{H}_{\{0\}\cup I, (V\otimes V^*)\boxtimes W}\Big|_{v\times X^I} \xleftarrow{\sim} \mathcal{H}_{\{1,2\}\cup I, V\boxtimes V^* \boxtimes W}\Big|_{\Delta(v)\times X^I}$$

functoriality

$\mathbb{1} \longrightarrow V\otimes V^*$

fusion

(factoriation structure)

Let $ev_V : V \otimes V^* \longrightarrow \mathbb{1}$ be the evaluation map.

$x \otimes \xi \mapsto \xi(x)$

functoriality

Annihilation operator $\mathcal{C}^b_{ev_V}$ :

$$\mathcal{H}_{\{1,2\}\cup I,\, V\boxtimes V^*\boxtimes W}\Big|_{\Delta(v)\times X^I} \underset{\text{fusion}}{\simeq} \mathcal{H}_{\{0\}\cup I,\, (V\otimes V^*)\boxtimes W}\Big|_{v\times X^I} \xrightarrow{ev_V \boxtimes Id_W} \mathcal{H}_{\{0\}\cup I,\, \mathbb{1}\boxtimes W}\Big|_{v\times X^I} \simeq (\overline{\mathbb{Q}}_\ell)_v \boxtimes \mathcal{H}_{I,W}$$

Let $S_{V,v}$ be the composition :

$$(\overline{\mathbb{Q}}_\ell)_v \boxtimes \mathcal{H}_{I,W} \xrightarrow{\mathcal{C}^\#_{S_V}} \mathcal{H}_{\{1,2\}\cup I,\, V\boxtimes V^*\boxtimes W}\Big|_{\Delta(v)\times X^I} \cancel{\xrightarrow{\mathcal{C}^b_{ev_V}} (\overline{\mathbb{Q}}_\ell)_v \boxtimes \mathcal{H}_{I,W}}$$

$$\Big\downarrow \mathrm{Frob}_{\{1\}}^{\deg(v)} \quad \text{partial Frobenius morphism}$$

( note that ~~deg~~ $\mathrm{Frob}^{\deg(v)}(v) = v$ )

$$\mathcal{H}_{\{1,2\}\cup I,\, V\boxtimes V^*\boxtimes W}\Big|_{\Delta(v)\times X^I} \xrightarrow[\mathcal{C}^b_{ev_V}]{} (\overline{\mathbb{Q}}_\ell)_v \boxtimes \mathcal{H}_{I,W}$$

$S_{V,v}$ descents to a morphism of sheaves over $X^I$ :

$$S_{V,v} : \mathcal{H}_{I,W}^{\leq \mu} \longrightarrow \mathcal{H}_{I,W}^{\leq \mu+\kappa}$$

Proposition (V. Lafforgue) : The operator $S_{V,v}$, which is a morphism of sheaves over $X^I$, extends the Hecke operator $T(h_{V,v})$, which is a morphism of sheaves over $(X-v)^I$.

**Proposition** (V. Lafforgue) Let $I = \{i\} \sqcup I^{\circ}$. Let ~~W∈P~~ $W = W_i \boxtimes W^{\circ} \in \mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}(\hat{G}^{I})$ with $W_i \in \mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}(\hat{G})$ and $W^{\circ} \in \mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}(\hat{G}^{I^{\circ}})$.

Then there exists dominant coweight $\kappa$, s.t. $\forall$ dominant coweight $\mu$, we have

$$\sum_{\alpha=0}^{\dim W_i} (-1)^{\alpha} S_{\Lambda^{\dim W_i - \alpha} W_i,\, v} \circ \left(\mathrm{Frob}_{\{i\}}^{\deg(v)}\right)^{\alpha} = 0 \quad \text{in}$$

$$\mathrm{Hom}\left( \mathcal{H}_{I,W}^{\leq \mu}\Big|_{v \times X^{I^{\circ}}},\ \mathcal{H}_{I,W}^{\leq \mu+\kappa}\Big|_{v \times X^{I^{\circ}}} \right)$$

**Remark:**
The combination of these two propositions is called the <u>Eichler - Shimura relations</u>.

## §4. A finiteness property of $\mathcal{H}_{I,W}\big|_{\bar{\eta}_I}$.

recall:

$$\bar{\eta}_I \to \eta_I \to X^I$$

generic point

**Proposition:**
$\mathcal{H}_{I,W}\big|_{\bar{\eta}_I}$ is an increasing union of $\bar{\mathbb{Q}}_\ell$-vector subspaces $\mathfrak{M}$ which are stable by the action of the partial Frobenius morphisms, and for which there exists a family $(v_i)_{i\in I}$ of closed points in $X$ s.t. $\mathfrak{M}$ is stable under the action of $\bigotimes_{i\in I} \mathcal{H}_{G,v_i}$ and is of finite type as module over $\bigotimes_{i\in I} \mathcal{H}_{G,v_i}$.

p.f. Since $\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\hat{G}^I)$ is semisimple, we can suppose $W = \boxtimes_{i \in I} W_i$.

$\forall \mu$, choose a dense open subscheme $\Omega$ of $X^I$ s.t.

$\mathcal{H}_{I,W}^{\leq \mu}\big|_{\Omega}$ is lisse. Choose $v \in \Omega$. Let $\begin{array}{ccc} X^I & \xrightarrow{pr_i} & X \\ v & \longmapsto & v_i \end{array}$.

$$\mathfrak{M}_\mu := \mathrm{Im}\left( \sum_{(n_i) \in \mathbb{N}^I} \left( \bigotimes_{i \in I} \mathcal{H}_{G,v_i} \right) \circ \left( \prod_{i \in I} \mathrm{Frob}_{\{i\}}^{n_i} \right) \mathcal{H}_{I,W}^{\leq \mu}\big|_{\overline{\eta_I}} \longrightarrow \mathcal{H}_{I,W}\big|_{\overline{\eta_I}} \right)$$

By the Eichler-Shimura relations, the sum is over finitely many $(n_i)$.

So finite type as $\bigotimes_{i \in I} \mathcal{H}_{G,v_i}$-module.

$$\mathcal{H}_{I,W}\big|_{\overline{\eta_I}} = \bigcup_{\mu} \mathfrak{M}_\mu. \qquad \square.$$

## §5. Drinfeld's lemma

Recall

$$\begin{tikzcd} \operatorname{Spec}\overline{F_I} \arrow[d, phantom, "\parallel"] & \operatorname{Spec} F_I \arrow[d, phantom, "\parallel"] & \\ \overline{\eta}_I \arrow[r] \arrow[d] & \eta_I \arrow[r] \arrow[d] & X^I \arrow[d, "pr_i"] \\ \overline{\eta} \arrow[r] & \eta \arrow[r] & X \end{tikzcd}$$

we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \pi_1^{geo}(\eta_I, \overline{\eta}_I) \arrow[r] \arrow[d] & \mathrm{Weil}(\eta_I, \overline{\eta}_I) \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \pi_1^{geo}(\eta, \overline{\eta})^I \arrow[r] & \mathrm{Weil}(\eta, \overline{\eta})^I \arrow[r] & \mathbb{Z}^I \arrow[r] & 0 \end{tikzcd}$$

Drinfeld: " action of $\mathrm{Weil}(\eta_I, \overline{\eta}_I)$ + partial Frobenius

$\Rightarrow$ action of $\mathrm{Weil}(\eta, \overline{\eta})^I$ "

Now we define

~~FWeil~~

$$\mathrm{FWeil}(\eta_I, \overline{\eta}_I) := \left\{ \varepsilon \in \mathrm{Aut}_{\overline{\mathbb{F}}_q}(\overline{F_I}) \;\middle|\; \exists (n_i)_{i \in I} \in \mathbb{Z}^I,\ \varepsilon|_{(F_I)^{perf}} = \prod_{i \in I} (\mathrm{Frob}_{\{i\}})^{n_i} \right\}$$

$\uparrow$ perfection of $F_I$

We have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \pi_1^{geo}(\eta_I, \overline{\eta}_I) \arrow[r] \arrow[d] & \mathrm{FWeil}(\eta_I, \overline{\eta}_I) \arrow[r] \arrow[d, "\psi"] & \mathbb{Z}^I \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \pi_1^{geo}(\eta, \overline{\eta})^I \arrow[r] & \mathrm{Weil}(\eta, \overline{\eta})^I \arrow[r] & \mathbb{Z}^I \arrow[r] & 0 \end{tikzcd}$$

$$\varepsilon \mapsto \left( \mathrm{Frob}_{\{i\}}^{-n_i} \circ \varepsilon_i \right)_{i \in I}$$

~~$\mathcal{H}_{I,W}$ is equiped~~

(Drinfeld's lemma) Lemma 1:

A continuous action of $FWeil(\eta_I, \bar{\eta}_I)$ on a finite dimensional $\bar{\mathbb{Q}}_\ell$-vector space factors through $Weil(\eta, \bar{\eta})^I$.

(Variant) Lemma 2:

Let $A$ be a finitely generated $\bar{\mathbb{Q}}_\ell$-algebra. Let $M$ be an $A$-module of finite type. Then a continuous $A$-linear action of $FWeil(\eta_I, \bar{\eta}_I)$ on $M$ factors through $Weil(\eta, \bar{\eta})^I$.

Application:

action of $FWeil(\eta_I, \bar{\eta}_I)$

$$\mathcal{H}_{I,W}\big|_{\bar{\eta}_I} = \bigcup_{\mu} \underbrace{\mathfrak{M}_\mu}$$

action of $\pi_1(\eta_I, \bar{\eta}_I)$

stable under the action of partial Frobenius morphisms

finite type module over $\bigotimes_{i \in I} \mathcal{H}_{G, v_i}$

apply lemma 2 to $A = \bigotimes_{i \in I} \mathcal{H}_{G, v_i}$, $M = \mathfrak{M}_\mu \Rightarrow$ ~~$\mathfrak{M}_\mu$ is equiped~~ ~~with an act~~ the action of $FWeil(\eta_I, \bar{\eta}_I)$ on $\mathfrak{M}_\mu$ factors through $Weil(\eta, \bar{\eta})^I$.

Proposition: The action of $FWeil(\eta_I, \bar{\eta}_I)$ on $\mathcal{H}_{I,W}\big|_{\bar{\eta}_I}$ factors through $Weil(\eta, \bar{\eta})^I$.