Homotopic Distance and Generalized Motion Planning

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Homotopic Distance

Let X and Y be path-connected topological spaces and $f, g\colon X \to Y$ two continuous maps.

The ***homotopic distance*** $\mathrm{D}(f,g)$ between f and g is the least integer $k \geq 1$ such that there exists an open covering $U_1 \cup \cdots \cup U_k = X$ with the property that $f_{|U_j}$ and $g_{|U_j}$ are homotopic for all j.

Let $A \subset X$ be a subspace. The ***subspace distance*** between the two maps f, g on A, is defined as

$$\mathrm{D}_X(A; f, g) := \mathrm{D}(f_{|A}, g_{|A}) = \mathrm{D}(f \circ i_A, g \circ i_A),$$

where $i_A\colon A \hookrightarrow X$ is the inclusion.

- The ***L-S-category*** of X is the distance between the identity id_X and any constant map, $\operatorname{cat} X = \mathrm{D}(\mathrm{id}_X, x_0)$.
- Let $p_1, p_2\colon X \times X \to X$ be the projections. The ***topological complexity*** of X is $\mathrm{TC}(X) = \mathrm{D}(p_1, p_2)$.

Properties of the homotopic distance

Let $\mathcal{P}(f,g)$ be the space of pairs (x, γ) where $x \in X$ and γ is a continuous path on Y, such that $\gamma(0) = f(x)$ and $\gamma(1) = g(x)$. Notice that $\pi^* = (f,g)^*\pi\colon \mathcal{P}(f,g) \to X$ is the pullback fibration of the path fibration $\pi\colon Y^{[0,1]} \to Y \times Y$, where $\pi(\gamma) = (\gamma(0), \gamma(1))$, by the map $(f,g)\colon X \to Y \times Y$:

$$\begin{array}{ccc} \mathcal{P}(f,g) & \longrightarrow & Y^{[0,1]} \\ {\scriptstyle \pi^*}\downarrow & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{(f,g)} & Y \times Y. \end{array}$$

Theorem 1 $\mathrm{D}(f,g)$ equals the *Švarc genus* of π^*, that is, the minimum number $k \geq 1$ such that there exists an open covering $U_1 \cup \cdots \cup U_k = X$, where for each U_j there is a continuous section $s_j\colon U_j \to \mathcal{P}(f,g)$ of the pullback fibration π^*.

Note that if X is not connected and $\{A_i\}_{i=1}^n$ are the connected components of X, then

$$\mathrm{D}(f,g) = \mathrm{D}_X(X; f, g) = \max_i \mathrm{D}_X(A_i; f, g).$$

Let $\{V_i\}_{i=1}^k$ be a finite open covering of X. Then:

$$\mathrm{D}(f,g) \leq \sum_{i=1}^{k} \mathrm{D}_X(V_i; f, g).$$

Let M be a compact differentiable manifold. A smooth function $\Phi\colon M \to \mathbb{R}$ is called a *Morse-Bott function* if the critical set $\operatorname{Crit}\Phi$ is a disjoint union of connected submanifolds S_i and for each critical point $p \in S_i \subset \operatorname{Crit}\Phi$ the Hessian is non-degenerate in the directions transverse to S_i.

Topological Complexity and Motion Planning
BIRS (México), May 29 to June 3, 2022

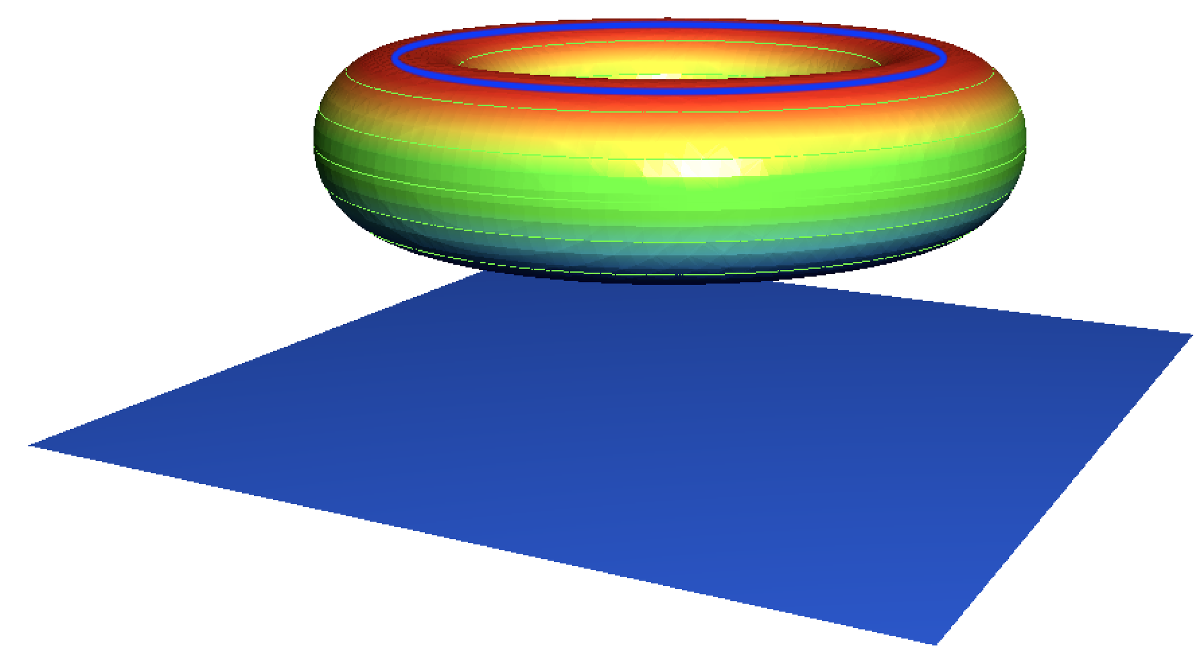

Fig. 1: Morse-Bott function on the torus.

Theorem 2 Let $\Phi\colon M \to \mathbb{R}$ be a Morse-Bott function in a compact smooth manifold M. Let $c_1 < \cdots < c_p$ be its critical values, and let $\Sigma_i = \Phi^{-1}(c_i) \cap \operatorname{Crit}\Phi$ be the set of critical points in the level $\Phi = c_i$. If $f, g\colon M \to Y$ are two continuous maps, then

$$\mathrm{D}(f,g) \leq \sum_{i=1}^{p} \mathrm{D}_M(\Sigma_i; f, g).$$

This extends analogous results for the topological complexity ([1]) and the L-S-category ([6]).

Navigation functions and generalized motion planning problem

We can interpret that the homotopic distance between f and g solves the following:

Generalized planning problem: Let $f, g\colon X \to Y$ be two continuous maps between topological spaces. Given an arbitrary point $x \in X$ find a continuous path $s(x)$, joining $f(x)$ and $g(x)$ in Y, in such a way that the path $s(x)$ depends continuously on x.

Assume that we have two continuous maps $f, g\colon M \to Y$, defined on the manifold M, and a Morse-Bott function $\Phi\colon M \to \mathbb{R}$, with critical values $c_1, \ldots, c_p$. The generalized motion planning problem can be reduced to the critical set, by using the gradient flow of Φ as in Fig. 2.

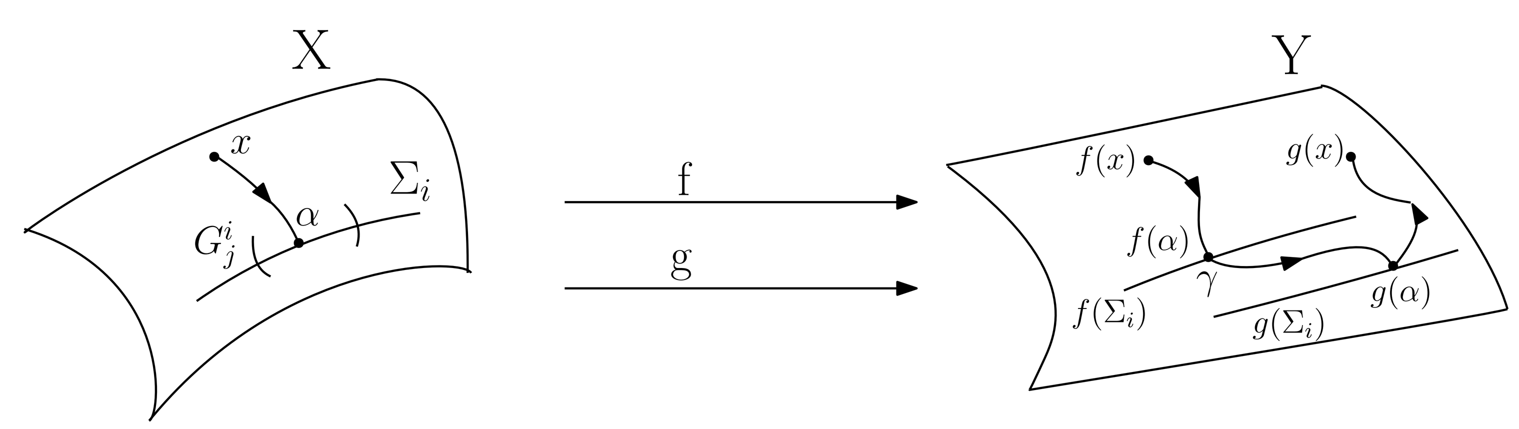

Fig. 2: Navigation function.

Note that G_j^i is a covering by ENRs for each critical submanifold Σ_i.

Examples

1. If $A \subset X$ is a subspace, the ***LS category of*** A ***in*** X, $\operatorname{cat}_X A$, is the minimum integer $k \geq 1$ such that there is a covering $U_1 \cup \cdots \cup U_k = A$, verifying that each subset U_j is open in A and the inclusion map $U_j \subset X$ is homotopic to a constant map.
$$\operatorname{cat}_X A = \mathrm{D}_X(A; \mathrm{id}_X, x_0) = \mathrm{D}(i_A, x_0).$$
For $x_0 \in X$, we define the axis inclusion maps $i_1, i_2\colon X \to X \times X$ as $i_1(x) = (x, x_0)$ and $i_2(x) = (x_0, x)$. Then, $\mathrm{D}(i_1, i_2) = \operatorname{cat} X$.
2. Let $A \subset X \times X$ be a subspace. The ***subspace topological complexity*** of A, $\mathrm{TC}_X(A)$, is the smallest integer $k \geq 1$ such that there is a cover $U_1 \cup \cdots \cup U_k = A$ with the property that each $U_j \subset A$ is open in A, and the projections $U_j \rightrightarrows X$ on the first and the second factors are homotopic to each other.
If $p_1, p_2\colon X \times X \to X$ are the projections, then
$$\mathrm{TC}_X(A) = \mathrm{D}_{X \times X}(A; p_1, p_2).$$
3. Let us suppose that X is the configuration space of a multi-arm robot and Y is the workspace (the spacial region that can be effectively attained by the end device of the arm). A ***work map*** $f\colon X \to Y$ assigns to each state of X the corresponding position of the end effector.
The ***topological complexity of*** f, $\mathrm{tc}(f)$, is the least integer n such that $X \times X$ can be covered by n open subsets $\{U_i\}_{i=1}^n$ such that for each U_i there exists a continuous map $f_i\colon U_i \to Y^I$ satisfying $f_i(x_0, x_1)(0) = f(x_0)$ and $f_i(x_0, x_1)(1) = f(x_1)$ ([4]).
The topological complexity of f equals the distance of the projections composed with the map f. That is,
$$\mathrm{tc}(f) = \mathrm{D}(f \circ p_1, f \circ p_2).$$
4. The ***topological complexity of a fibration*** $f\colon X \to Y$, $\mathrm{cx}(f)$, is the number of partial solutions to the motion planning problem $\pi_f\colon X^I \to X \times Y$, $\pi(\gamma) = (\gamma(0), f(\gamma(1)))$, which assigns to each path in the configuration space the initial state x and the end effector position $f(y)$ at the final state ([5]).
If $\pi_X\colon X \times Y \to X$ and $\pi_Y\colon X \times Y \to Y$ are the projections, then
$$\mathrm{cx}(f) = D(f \circ \pi_X, \pi_Y).$$

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