Intersections in Projective Spaces (23frg006)

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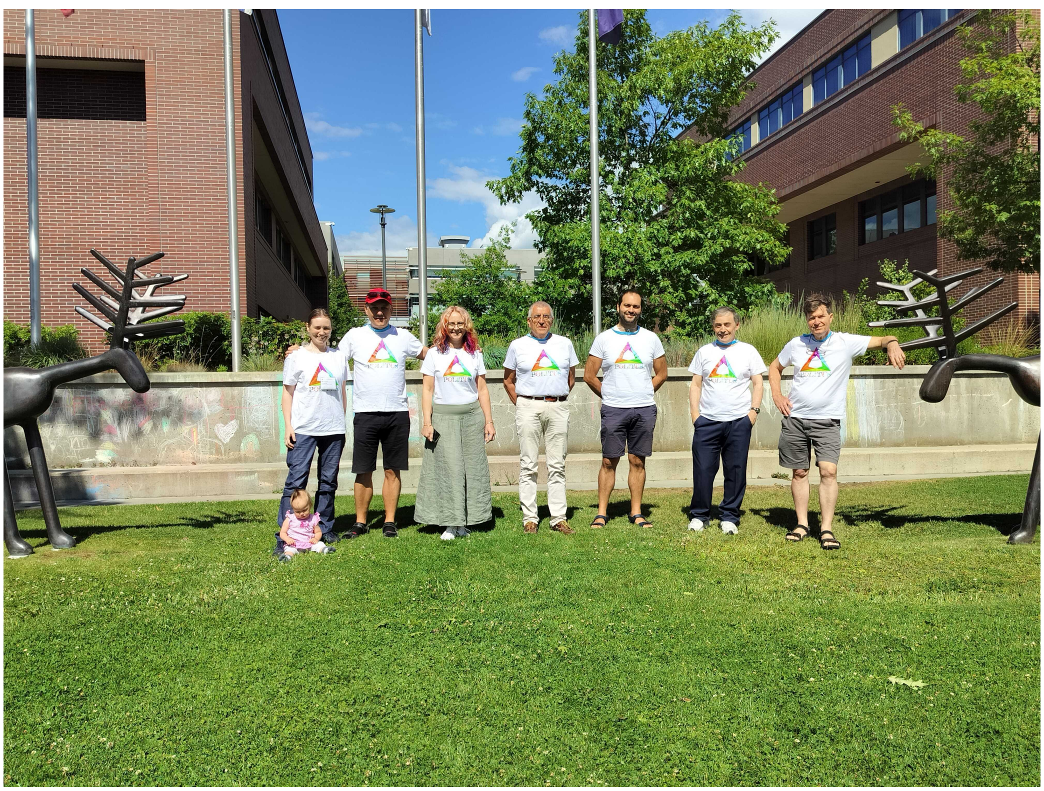
1 POLITUS Research Group



From the left: Farnik(s), Szemberg, Szpond, Chiantini, Favacchio, Migliore and Harbourne

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During our residence in Kelowna BIRS location we achieved multiple goals. We made substantial progress on collaborations begun before we arrived in Kelowna. The opportunity to work face to face for an extended period greatly accelerated the progress with respect to our previous zoom meetings. We also had an opportunity to discuss future paths for collaborative research in depth and we initiated two new projects.

Our stay in Kelowna would not have been as productive as it was without the generous support BIRS provided in terms of housing, lodging and on-site support by Chad Davis, to whom we express particular thanks.

2 Overview of the Field

Understanding 3-dimensional structure from 2-dimensional photos is the essential issue behind CT scans and is a significant hurdle in computer vision. A version of this problem has arisen in recent research in algebraic geometry: if the photographic image of a finite set of points has a nice structure (known technically as being a complete intersection) no matter what angle the photo is taken from, what can we say about the arrangement of the original set of points?

Although there were significant obstructions to extending what was known in dimension 3 to higher dimensions, the research undertaken under this proposal successfully explored how to relax the condition of being a complete intersection in interesting ways.

Indeed, it is well worth mentioning that topics studied in our Focussed Research Group form a new, still evolving area of research at the crossroads of algebraic geometry geometry, commutative algebra and combinatorics. Its roots go back only to a 2018 Workshop on Lefschetz Properties, from which the initial publication [6] originated. More recently several links have been established to the area of unexpected hypersurfaces [7, 1, 8]. A provisional version of the foundations of the theory has been established in work of our group [5] in the last year.

3 Proposed Goals and Open Problems

We proposed to study the following problems.

Problem A: Characterize/classify subschemes of $\mathbb{P}^N$ with the geproci property.

Problem B: Characterize sets of points in $\mathbb{P}^N$ with the geprofi property.

Problem C: Generalize upper bounds on the number of intersection points along the lines of theorems of Diaz and Giuffrida on subvarieties U, V in $\mathbb{P}^N$ with $\dim(U) + \dim(V) < N$.

Problem D: Generalize the notion of grids to higher dimensional projective spaces and apply generalized grids to answer Problems A and B.

4 Scientific Progress Made and Outcome of the Meeting

Problem A: We finished classifying geproci half grids on four lines, posted the resulting paper on the arXiv [2] and submitted it for publication during our residence in Kelowna.

Problem B: We made substantial progress on understanding geprofi sets in $\mathbb{P}^4$ under a natural geometric assumption (i.e., the points being in linear general position), and started writing up our results in a new paper [3].

Problem C: We initiated discussion as a foundation for future work and established an outline for moving forward. More specifically we explored homological invariants of intersections of two curves, [4].

Problem D: We started a collaboration extending the notion of grids to nonequidimensional intersections by introducing a definition for hypergrids.

References

[1] Th. Bauer, G. Malara, T. Szemberg, J. Szpond, Quartic unexpected curves and surfaces, *Manuscr. Math.* **161** (2020), 283–292.

[2] L. Chiantini, Ł. Farnik, G. Favacchio, B. Harbourne, J. Migliore, T. Szemberg, and J. Szpond. *Geproci sets and the combinatorics of skew lines in* $\mathbb{P}^3$, arXiv:2308.00761.

[3] L. Chiantini, Ł. Farnik, G. Favacchio, B. Harbourne, J. Migliore, T. Szemberg, and J. Szpond. *Geprofi sets of points in* $\mathbb{P}^4$, in progress.

[4] L. Chiantini, Ł. Farnik, G. Favacchio, B. Harbourne, J. Migliore, T. Szemberg, and J. Szpond. *Intersection of curves in* $\mathbb{P}^4$, in progress.

[5] L. Chiantini, Ł. Farnik, G. Favacchio, B. Harbourne, J. Migliore, T. Szemberg, and J. Szpond. *Configurations of points in projective space and their projections*, arXiv:2209.04820

[6] L. Chiantini and J. Migliore. Sets of points which project to complete intersections. *Trans. AMS* **374** (2021), 2581–2607. With an appendix by the authors and A. Bernardi, G. Denham, G. Favacchio, B. Harbourne, T. Szemberg and J. Szpond.

[7] D. Cook II, B. Harbourne, J. Migliore, and U. Nagel, Line arrangements and configurations of points with an unexpected geometric property, *Compos. Math.*, **154** (2018), 2150–2194

[8] B. Harbourne, J. Migliore, U. Nagel, Z. Teitler, Unexpected hypersurfaces and where to find them, *Mich. Math. J.* **70** (2021), 301–339.