Simple integral fusion categories

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Fusion rings

A *fusion ring* is a based $\mathbb{Z}$-module $\mathcal{F} = \mathbb{Z}\mathcal{B}$ with $\mathcal{B} = \{b_1, \dots, b_r\}$ finite, together with *fusion rules* (generalizing the multiplication on a finite group, or the tensor product on its representations):

$$b_i \cdot b_j = \sum_{k=1}^{r} N_{ij}^k b_k$$

with $N_{ij}^k \in \mathbb{Z}_{\geq 0}$ such that:

- **Associativity.** $b_i \cdot (b_j \cdot b_k) = (b_i \cdot b_j) \cdot b_k$,
- **Neutral.** $b_1 \cdot b_i = b_i \cdot b_1 = b_i$,
- **Inverse/Adjoint/Dual.** $\forall i \ \exists! i^*$ with $N_{i,k}^1 = N_{k,i}^1 = \delta_{i^*,k}$,
- **Frobenius reciprocity.** $N_{ij}^k = N_{i^*k}^j = N_{kj^*}^i$.

It may be understood as a representation ring of a 'virtual' group.

Frobenius-Perron dimension

The adjoint $*$ induces a structure of finite dim. $*$-algebra on $\mathbb{C}\mathcal{B}$,

Frobenius-Perron theorem

$\exists!$ $*$-homomorphism $d : \mathbb{C}\mathcal{B} \to \mathbb{C}$ such that $d(\mathcal{B}) \subset (0, \infty)$.

- the **Frobenius-Perron dim** (FPdim) of b_i is $d_i := d(b_i)$,
- the FPdim of $\mathcal{F}$ is $\sum_i d_i^2$,
- the *type* of $\mathcal{F}$ is $[d_1, d_2, \dots, d_r]$,

The fusion ring $\mathcal{F}$ is called:

- of *Frobenius type* if for all i, $\frac{\mathrm{FPdim}(\mathcal{F})}{d_i}$ is an algebraic integer,
- *integral* if for all i the number d_i is an integer.

The "golden" fusion ring (Yang-Lee rules)

$\mathcal{B} = \{b_1, b_2\}$, with $b_2^2 = b_1 + b_2$, type $[1, \phi]$ with ϕ golden ratio.

Simple integral fusion rings

A fusion ring w/o proper non-trivial fusion subring is called **simple**.
The fusion ring of $\mathrm{Rep}(G)$ is simple iff the finite goup G is simple.

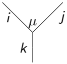
Theorem (Liu-P.-Wu, *Adv. Math.* 2021)

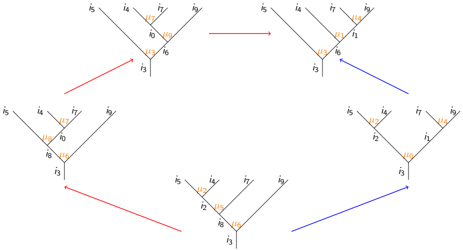
rank	≤ 5	6	7	8	9	10	all
$\mathrm{FPdim} <$	1000000	150000	15000	4080	504	240	132

With the above bounds, there are exactly 34 (perfect) simple integral fusion rings of Frobenius type (4 of which $\mathrm{Rep}(G)$).

#	rank	FPdim	type	group Rep
1	5	60	$[1,3,3,4,5]$	$\mathrm{PSL}(2,5)$
1	6	168	$[1,3,3,6,7,8]$	$\mathrm{PSL}(2,7)$
2	7	210	$[1,5,5,5,6,7,7]$	
2	7	360	$[1,5,5,8,8,9,10]$	$\mathrm{PSL}(2,9)$
4	7	7980	$[1,19,20,21,42,42,57]$	
15	8	660	$[1,5,5,10,10,11,12,12]$	$\mathrm{PSL}(2,11)$
5	8	990	$[1,9,10,11,11,11,11,18]$	
2	8	1260	$[1,6,7,7,10,15,20,20]$	
2	8	1320	$[1,6,6,10,11,15,15,24]$	

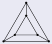
Fusion category (up to equivalence)

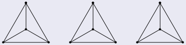
A *fusion category* $\mathcal{C}$ is a fusion ring and a solution of its *pentagon equations*, where $\mu \in \hom_{\mathcal{C}}(X_i \otimes X_j, X_k)$ is represented as 



Such a fusion ring is called a *Grothendieck ring* (i.e. *categorifiable*).

In the pseudo-unitary case ($\mathrm{FPdim} = \dim_{\mathcal{C}}$), it is equiv. to that every (labeled oriented) trivalent graph admits a unique evaluation by (l.o.) tetrahedrons  (complex numbers called F-symbols).

Two evaluations of the triangular prism  recovers the PE:

(TPE) $\sum \cdots$  $= \sum \cdots$ 

Unitary case: mirror image (of tetrahedron) $=$ complex conjugate.

Categorification criterion from *Quantum Fourier Analysis*

Here is the Commutative Schur Product Criterion:

Theorem (Liu-P.-Wu, *Adv. Math.* 2021)

Let $\mathcal{F}$ be a commutative fusion ring, let (M_i) be its fusion matrices, and let $(\lambda_{i,j})$ be the table given by their simultaneous diagonalization, with $\lambda_{i,1} = \|M_i\|$. If $\exists (j_1, j_2, j_3)$ such that

$$\sum_i \frac{\lambda_{i,j_1}\lambda_{i,j_2}\lambda_{i,j_3}}{\lambda_{i,1}} < 0$$

then $\mathcal{F}$ admits no unitary categorification.

This criterion rules out 28 among the 30 non group-like simple integral fusion rings of the previous classification (more than 93%).

The remaining 2 are denoted $\mathcal{F}_{210}$ and $\mathcal{F}_{660}$ (according to FPdim)

$\mathcal{F}_{660}$ is excluded (over any field) by the *zero-spectrum criterion*. Note that $\mathcal{F}_{210}$ cannot be excluded by known criteria (see why later), this requires the use a *localization strategy* involving TPE.

Zero-Spectrum Criterion

It is about the existence of a PE of the form $xy = 0$ with $x, y \neq 0$:

Zero-spectrum criterion (Liu, P., Ren, *in preparation*)

For a fusion ring $\mathcal{F}$, if there are indices i_j, $1 \leq j \leq 9$, such that $N_{i_4,i_1}^{i_6}$, $N_{i_5,i_4}^{i_2}$, $N_{i_5,i_6}^{i_3}$, $N_{i_7,i_9}^{i_1}$, $N_{i_2,i_7}^{i_8}$, $N_{i_8,i_9}^{i_3}$ are non-zero, and

$$\sum_k N_{i_4,i_7}^{k} N_{i_5^*,i_8}^{k} N_{i_6,i_9^*}^{k} = 0,$$

$$N_{i_2,i_1}^{i_3} = 1,$$

$$\sum_k N_{i_5,i_4}^{k} N_{i_3,i_1^*}^{k} = 1 \text{ or } \sum_k N_{i_2,i_4^*}^{k} N_{i_3,i_6^*}^{k} = 1 \text{ or } \sum_k N_{i_5^*,i_2}^{k} N_{i_6,i_1^*}^{k} = 1,$$

$$\sum_k N_{i_2,i_7}^{k} N_{i_3,i_9^*}^{k} = 1 \text{ or } \sum_k N_{i_8,i_7^*}^{k} N_{i_3,i_1^*}^{k} = 1 \text{ or } \sum_k N_{i_2^*,i_8}^{k} N_{i_1,i_9^*}^{k} = 1,$$

then $\mathcal{F}$ cannot be categorified (at all) over any field.

It excludes $\mathcal{F}_{660}$. Idem for "$0 = xyz$" (one-spectrum criterion).

Localization strategy

In general, the system of pentagon equations is too big to be attacked head-on, but the TPE framework reveals some symmetries allowing us to get local subsystems.

Theorem (Liu-P.-Ren, *in preparation*)

Let $\mathcal{C}$ be pseudo-unitary fusion category over $\mathbb{C}$ (so spherical). Let x be a self-adjoint simple object such that for all simple object $a \le x^2$, then $a^\star = a$ and $\langle x^2, a\rangle \le 1$. Let S_x be the set of simple components of x^2 and S'_x be a subset of S_x such that for all $a, b, c \in S'_x$ then $\langle bc, a\rangle \le 1$. Then we can consider the subsystem E_x of PE, with variables $X(i,j)$ and $Y(i,j)$ with $(i,j) \in S_x \times S'_x$ such that for all $a, b \in S'_x$

$$\delta_{a,b} = d_b \sum_{i \in S_x} d_i Y(i,a)Y(i,b),$$

$$X(a,b) = \sum_{i \in S_x} d_i Y(i,a)Y(i,b)^2,$$

$$Y(a,b)^2 = \sum_{i \in S_x} d_i Y(i,a)X(i,b)$$

with $X(a,x) = Y(a,x)^2$; $X(a,b) = 0$ if $\langle b^2, a\rangle = 0$; $Y(a,b) = Y(b,a)$; $Y(1,b) = d_x^{-1}$; $X(1,b) = (d_b d_x)^{-1}$.

Application to $\mathcal{F}_{210}$

Let call $1, 5_1, 5_2, 5_3, 6_1, 7_1, 7_2$ the simple objects of $\mathcal{F}_{210}$. Consider E_x where $x = 5_1$, $S_x = \{1, 5_1, 5_3, 7_1, 7_2\}$ and $S'_x = \{1, 5_1, 5_3\}$. It has 10 variables and 12 equations:

$$
\begin{aligned}
5u_0 + 7u_1 + 7u_2 - 4/25 &= 0, \\
5v_0 + 5v_1 + 7v_3 + 7v_5 + 1/5 &= 0, \\
25v_0^2 + 25v_1^2 + 35v_3^2 + 35v_5^2 - 4/5 &= 0, \\
5v_0^3 + 5v_1^3 + 7v_3^3 + 7v_5^3 - v_0^2 + 1/125 &= 0, \\
5v_0v_1^2 + 5v_1v_2^2 + 7v_3v_4^2 + 7v_5v_6^2 + 1/125 &= 0, \\
5u_0v_1 - v_1^2 + 7u_1v_3 + 7u_2v_5 + 1/125 &= 0, \\
5v_1 + 5v_2 + 7v_4 + 7v_6 + 1/5 &= 0, \\
25v_0v_1 + 25v_1v_2 + 35v_3v_4 + 35v_5v_6 + 1/5 &= 0, \\
5v_0^2v_1 + 5v_1^2v_2 + 7v_3^2v_4 + 7v_5^2v_6 - v_1^2 + 1/125 &= 0, \\
25v_1^2 + 25v_2^2 + 35v_4^2 + 35v_6^2 - 4/5 &= 0, \\
5v_1^3 + 5v_2^3 + 7v_4^3 + 7v_6^3 - u_0 + 1/125 &= 0, \\
5u_0v_2 - v_2^2 + 7u_1v_4 + 7u_2v_6 + 1/125 &= 0
\end{aligned}
$$

It admits 14 solutions in char. 0, which can be written as a Gröbner basis.

Theorem (Liu-P.-Ren, *in preparation*)

(assumption of previous theorem) Let x, S_x, S'_x and E_x as above, and let $z \in S'_x$ with S_z, S'_z and E_z as above. Then there is an extra equation linking the two independent subsystems E_x and E_z:

$$X_x(z,z) = \sum_{i \in S_x \cap S_z} d_i Y_z(i,z) X_x(i,z)$$

Let us apply above theorem to $\mathcal{F}_{210}$ with E_x as above, $z = 5_3$, $S_z = \{1, 5_2, 5_3, 7_1, 7_2\}$ and $S'_z = \{1, 5_2, 5_3\}$. By putting together the Gröbner bases of E_x, E_z and the extra, we quickly show the absence of solution in char. 0; and so $p > 0$ (in the pivotal case) by lifting theorem (below) and a quick check on $p|210$.

Theorem (ENO, 2005)

Let $\mathcal{C}$ be a fusion category over $\overline{\mathbb{F}}_p$. If $\dim(\mathcal{C}) \neq 0$ then it lifts into a Grothendieck-equivalent fusion category in char. 0.

Note that $\dim(\mathcal{C}) = 0$ iff p divides $\mathrm{FPdim}(\mathcal{C})$, by pseudo-unitarity.

Classification of unitary simple integral fusion categories

Previous classification + criteria + localization leads to:

Corollary (Liu-P.-Ren, *in preparation*)

A unitary simple (perfect) integral fusion category of Frobenius type, rank ≤ 8 and $\mathrm{FPdim} < 4080$ is Grothendieck equivalent to $\mathrm{Rep}(\mathrm{PSL}(2, q))$ with $4 \leq q \leq 11$ prime power.

The existence of a non group-like (unitary) simple integral fusion category is related to a famous open problem of the theory:
A fusion category is *weakly group-theoretical* if its Drinfeld center is equivalent to the one coming from a sequence of group extensions.

Theorem (ENO, 2011)

A weakly group-theoretical simple fusion category is Grothendieck equivalent to $\mathrm{Rep}(G)$, with G a finite simple group.

Question (ENO, 2011)

Is there an integral fusion category not weakly group-theoretical?

Formal table characterization of commutative fusion ring

Let $\mathcal{F}$ be a commutative fusion ring. Let (M_i) be its fusion matrices, and let $D_i = diag(\lambda_{i,j})$, be their simultaneous diagonalization. The *eigentable* of $\mathcal{F}$ is the table given by $(\lambda_{i,j})$.

Theorem (Folklore?; Liu-P.-Ren, *under review*)

Let $(\lambda_{i,j})$ be a formal $r \times r$ table. Consider the space of functions from $\{1, \dots, r\}$ to $\mathbb{C}$ with some inner product $\langle f, g \rangle$. Consider the functions (λ_i) defined by $\lambda_i(j) = \lambda_{i,j}$, and assume that $\langle \lambda_i, \lambda_j \rangle = \delta_{i,j}$. Consider the pointwise multiplication $(fg)(i) = f(i)g(i)$, and the multiplication operator $M_f : g \mapsto fg$. Consider $M_i := M_{\lambda_i}$, and assume that for all i there is j (automatically unique, denoted i^*) such that $M_i^* = M_j$. Assume that M_1 is the identity. Assume that for all i, j, k, $N_{i,j}^k := \langle \lambda_i \lambda_j, \lambda_k \rangle$ is a nonnegative integer. Then $(N_{i,j}^k)$ are the structure constants of a commutative fusion ring and $(\lambda_{i,j})$ is its eigentable. Moreover, every eigentable of a commutative fusion ring satisfies all the assumptions above.

In previous Theorem, the inner product can be taken of the form

$$\langle f, g \rangle := \sum_j \frac{1}{\mathfrak{c}_j} f(j)\overline{g(j)}$$

with $\mathfrak{c}_j = \sum_i |\lambda_{i,j}|^2$ (*formal codegrees*). So (Verlinde-like formula):

$$N_{i,j}^k = \sum_s \frac{\lambda_{i,s}\lambda_{j,s}\overline{\lambda_{k,s}}}{\mathfrak{c}_s} = \sum_s \frac{\lambda_{i,s}\lambda_{j,s}\overline{\lambda_{k,s}}}{\sum_l |\lambda_{l,s}|^2}$$

Generic character table of $\mathrm{Rep}(\mathrm{PSL}(2,q))$, q even

charparam c \ classparam k	$\{1\}$	$\{1\}$	$\{1,\ldots,\frac{q-2}{2}\}$	$\{1,\ldots,\frac{q}{2}\}$
$\{1\}$	1	1	1	1
$\{1,\ldots,\frac{q}{2}\}$	$q-1$	-1	0	$-2\cos(\frac{2\pi kc}{q+1})$
$\{1\}$	q	0	1	-1
$\{1,\ldots,\frac{q-2}{2}\}$	$q+1$	1	$2\cos(\frac{2\pi kc}{q-1})$	0
class size	1	q^2-1	$q(q+1)$	$q(q-1)$

There are also tables for $q \equiv 1$ or $3 \mod 4$. Above Theorem applies on these tables (even when q is not a prime-power).

Interpolated simple integral fusion rings of Lie type

Theorem (Liu-P.-Ren, *under review*)

The ring of $\mathrm{Rep}(\mathrm{PSL}(2,q))$ **interpolate** to q non prime-power as a non group-like simple integral fusion ring (∞ family). If q even:

$$\begin{aligned}
x_{q-1,c_1}x_{q-1,c_2} &= \delta_{c_1,c_2}x_{1,1} + \sum_{\substack{c_3 \text{ such that} \\ c_1+c_2+c_3\neq q+1 \text{ and } 2\max(c_1,c_2,c_3)}} x_{q-1,c_3} + (1-\delta_{c_1,c_2})x_{q,1} + \sum_{c_3} x_{q+1,c_3}, \\
x_{q-1,c_1}x_{q,1} &= \sum_{c_2}(1-\delta_{c_1,c_2})x_{q-1,c_2} + x_{q,1} + \sum_{c_2} x_{q+1,c_2}, \\
x_{q-1,c_1}x_{q+1,c_2} &= \sum_{c_3} x_{q-1,c_3} + x_{q,1} + \sum_{c_3} x_{q+1,c_3}, \\
x_{q,1}x_{q,1} &= x_{1,1} + \sum_{c} x_{q-1,c} + x_{q,1} + \sum_{c} x_{q+1,c}, \\
x_{q,1}x_{q+1,c_1} &= \sum_{c_2} x_{q-1,c_2} + x_{q,1} + \sum_{c_2}(1+\delta_{c_1,c_2})x_{q+1,c_2}, \\
x_{q+1,c_1}x_{q+1,c_2} &= \delta_{c_1,c_2}x_{1,1} + \sum_{c_3} x_{q-1,c_3} + (1+\delta_{c_1,c_2})x_{q,1} + \sum_{\substack{c_3 \text{ such that} \\ c_1+c_2+c_3\neq q-1 \\ \text{and } 2\max(c_1,c_2,c_3)}} x_{q+1,c_3} + \sum_{\substack{c_3 \text{ such that} \\ c_1+c_2+c_3=q-1 \\ \text{or } 2\max(c_1,c_2,c_3)}} 2x_{q+1,c_3},
\end{aligned}$$

They automatically check all the known categorification criteria, and $\mathcal{F}_{210}$ corresponds to $q=6$. Idem q odd (and all Lie families?).

Project: application of the localization strategy to others q (all?).