Lagrangian multiform theory and pluri-Lagrangian systems

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Quote

Paul Dirac, in his seminal paper[1] of 1933, stated:

"The two formulations [namely that of Hamilton and of Lagrange] *are, of course, closely related but there are reasons for believing that the Lagrangian one is more fundamental."*

Dirac's paper contained already the key ideas underlying the path integral, later introduced by Feynman.

[1] P.A.M. Dirac, *The Lagrangian in Quantum Mechanics*, Physikalische Zeitschrift der Sowjetunion, Bd. 3, Heft 1, (1933)

The talk gives a brief overview of *Lagrangian multi-form theory*[2], which is a variational approach to integrability in the sense of *multidimensional consistency* (MDC).
It differs from the conventional variational approach in a number of respects:

- Lagrangians are no longer scalar objects (or volume forms) but genuine differential- or difference forms (with co-dimension nonzero) in the space of independent variables;
- the action is a functional not only of the dependent variables (the "fields") but also of (hyper)surfaces in the space of independent variables;
- at the critical point of the action, i.e., subject to solutions of a system of *generalized Euler-Lagrange (EL) equations*, the action is independent on local variations of the (hyper)surface in the space of multi-variables;
- the Lagrangians are no longer input (from tertiary considerations) but are to be viewed as solutions of the system of generalized EL equations.

Pluri-Lagrangian systems: The term was introduced by A. Bobenko & Yu. Suris, and it relaxes some of the assumptions of multiform theory, but there is also a subtly different perspective.

• First steps to a quantum multiform theory were undertaken in terms of Feynman propagators[3],[4].

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[2] S. Lobb & FWN: *Lagrangian multiforms and multidimensional consistency*, J. Phys. A:Math Theor. **42** (2009) 454013

[3] S.D. King and FWN, *Quantum variational principle and quantum multiform structure: The case of quadratic Lagrangians*, Nucl Phys. **B947** (2019) 114686.

[4] T. Kongkoom and S. Yoo-Kong, *Quantum integrability: Lagrangian 1-form case*, Nucl. Phys. **B987** (2023) 116101.

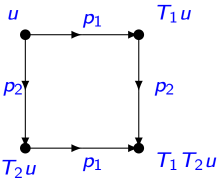
Multidimensional consistency on the lattice

quadrilateral P∆Es on the 2D lattice:

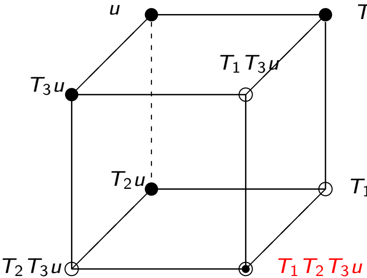
$$\boldsymbol{Q}(u, T_1 u, T_2 u, T_1 T_2 u; {\color{red} p_1, p_2}) = 0$$

notation of shifts on the elementary quadrilateral on a rectangular lattice:

$$u := u(n_1, n_2),\ \ T_1 u = u(n_1+1, n_2)$$
$$T_2 u := u(n_1, n_2+1),\ \ T_1 T_2 u = u(n_1+1, n_2+1)$$



Consistency-around-the cube:



Verifying consistency: Values at the black disks are initial values, values at open circles are uniquely determined from them, but there are three different ways to compute $T_1 T_2 T_3 u$.

Conventional variational formalism: discrete Euler-Lagrange equations

Define an action functional:

$$S[u(n_1, n_2)] = \sum_{n_1, n_2 \in \mathbb{Z}} \mathscr{L}(u, T_1 u, T_2 u; p_1, p_2) \ .$$

Following the usual *least-action principle*, the lattice equations for u are determined by the demand that S attains a minimum under local variations $u(n_1, n_2) \to u(n_1, n_2) + \delta u(n_1, n_2)$. Thus,

$$\begin{aligned} \delta S &= \sum_{n_1, n_2 \in \mathbb{Z}} \Big\{ \frac{\partial}{\partial u} \mathscr{L}(u, T_1 u, T_2 u; p_1, p_2) \delta u + \frac{\partial}{\partial T_1 u} \mathscr{L}(u, T_1 u, T_2 u; p_1, p_2) \delta(T_1 u) \\ &\qquad + \frac{\partial}{\partial T_2 u} \mathscr{L}(u, T_1 u, T_2 u; p_1, p_2) \delta(T_2 u) \Big\} = 0 \end{aligned}$$

Setting $\delta(T_i u) = T_i \delta u$, and resumming each of the terms we get:

$$\begin{aligned} 0 &= \sum_{n_1, n_2 \in \mathbb{Z}} \Big\{ \frac{\partial}{\partial u} \mathscr{L}(u, T_1 u, T_2 u; p_1, p_2) + \frac{\partial}{\partial u} \mathscr{L}(T_1^{-1} u, u, T_1^{-1} T_2 u; p_1, p_2) \\ &\qquad + \frac{\partial}{\partial u} \mathscr{L}(T_2^{-1} u, T_1 T_2^{-1} u, u; p_1, p_2) \Big\} \delta u \end{aligned}$$

(ignoring boundary terms) and since δu is arbitrary the discrete Euler-Lagrange (EL) equation follow:

$$\begin{aligned} \frac{\partial}{\partial u} \Big(&\mathscr{L}(u, T_1 u, T_2 u; p_1, p_2) + \mathscr{L}(T_1^{-1} u, u, T_1^{-1} T_2 u; p_1, p_2) \\ &+ \mathscr{L}(T_2^{-1} u, T_1 T_2^{-1} u, u; p_1, p_2) \Big) = 0 \end{aligned}$$

The problem

Multidimensional consistency: We know that many "integrable" equations, discrete and continuous possess the property of multidimenional consistency.

- continuous: commuting flows, higher symmetries & master symmetries, hierarchies;
- discrete: consistency-around-the-cube, Bäcklund transforms, higher continuous symmetries, commuting discrete flows

In all these cases we can think of the *dependent* variable a (possibly vector-valued) function of many (discrete and continuous) variables

$$u = u(n_1, n_2, \dots; x, t_1, t_2, \dots)$$

on which we can impose many equations simultaneously, and it is the *compatibility* of those equations that makes the integrability manifest.

Key question: How to capture the property of multidimensional consistency within a Lagrange formalism?

Main problem: The EL equations, only produces one equation per component of the dependent variables, but not an entire system of compatible equations on one and the same dependent variable!
Answer: Lagrangians of an integrable theory (in the sense of MDC) must be differential- or difference forms in space of multi-variables!
A key discovery was that Lagrangians for MDC quad-lattice equations obey a *closure relation*:

$$\Delta_i \mathscr{L}_{jk} + \Delta_j \mathscr{L}_{ki} + \Delta_k \mathscr{L}_{ij} = 0 \ , \quad (\text{where } \Delta_i = T_i - \text{id}) \ .$$

Simple example of a MDC system: linear case

here we consider multi-variable functions

$$u = u(n_1, n_2, \ldots; p_1, p_2, \ldots)$$

of discrete ($\{n_i\}$) and continuous ($\{p_i\}$) variables.

$\boxed{\text{P}\Delta\text{E}}$ Fully discrete Lagrangian:

$$\mathscr{L}_{ij} = u(T_i - T_j)u - \frac{1}{2}\left(\frac{p_i + p_j}{p_i - p_j}\right)\left((T_i - T_j)u\right)^2 \ ,$$

Linear quadrilateral lattice equation:

$$(p_i + p_j)(T_i - T_j)u - (p_i - p_j)(\mathrm{id} - T_i T_j)u = 0 \ .$$

The lattice Lagrangian $\mathscr{L}_{ij}$ obeys the closure relation:

$$\Delta_i \mathscr{L}_{jk} + \Delta_j \mathscr{L}_{ki} + \Delta_k \mathscr{L}_{ij} = 0$$

(where $\Delta_i = T_i - \mathrm{id}$) on solutions of the lattice equation.

$\boxed{\text{D}\Delta\text{E}}$ Linear differential-difference equation:

$$2p_i \frac{\partial}{\partial p_i} u = n_i (T_i^{-1} - T_i)u \ ,$$

Semi-discrete Lagrangian:

$$\mathscr{L}_i = n_i u \frac{\partial}{\partial p_i} T_i u - p_i \left(\frac{\partial}{\partial p_i} u\right)^2 \ .$$

$\boxed{\text{PDE}}$ Fully continuous equation;

$$\partial_{p_i}\partial_{p_j}(p_i^2-p_j^2)\partial_{p_i}\partial_{p_j}w=4(n_j\partial_{p_i}-n_i\partial_{p_j})\frac{1}{p_i^2-p_j^2}(n_jp_i^2\partial_{p_i}-n_ip_j^2\partial_{p_j})w\ ,$$

arises as (conventional) EL equation from the Lagrangian:

$$\mathscr{L}_{ij}=\frac{1}{n_jn_i}\left\{\frac{1}{2}(p_i^2-p_j^2)(\partial_{p_i}\partial_{p_j}w)^2+(n_j^2(\partial_{p_i}w)^2-n_i^2(\partial_{p_j}w)^2)+\frac{p_i^2+p_j^2}{p_i^2-p_j^2}(n_j\partial_{p_i}w-n_i\partial_{p_j}w)^2\right\}$$

The continuous Lagrangian $\mathscr{L}_{ij}$ obeys the closure relation:

$$\partial_{p_i}\mathscr{L}_{jk}+\partial_{p_j}\mathscr{L}_{ki}+\partial_{p_k}\mathscr{L}_{ij}=0$$

on solutions of the continuous equation.
In fact, *off-shell* we have

$$\begin{aligned}&\partial_{p_i}\mathscr{L}_{jk}+\partial_{p_j}\mathscr{L}_{ki}+\partial_{p_k}\mathscr{L}_{ij}=\\&\frac{1}{n_in_jn_k}\left[n_k(p_i^2-p_j^2)\partial_{p_i}\partial_{p_j}w+n_i(p_j^2-p_k^2)\partial_{p_j}\partial_{p_k}w+n_j(p_k^2-p_i^2)\partial_{p_i}\partial_{p_k}w\right]\\&\times\left[\partial_{p_i}\partial_{p_j}\partial_{p_k}w+\frac{4n_jn_kp_i^2\partial_{p_i}w}{(p_k^2-p_i^2)(p_i^2-p_j^2)}+\frac{4n_kn_ip_j^2\partial_{p_j}w}{(p_i^2-p_j^2)(p_j^2-p_k^2)}+\frac{4n_in_jp_k^2\partial_{p_k}w}{(p_j^2-p_k^2)(p_k^2-p_i^2)}\right]\ ,\end{aligned}$$

which implies that $\mathrm{d}\mathsf{L}$ has a 'double zero' and vanishes on the solutions of either factor.
These factors are the multiform EL equations, and imply the 2-variable EL equation.

Continuous 2-form action

The Lagrangian 2-formaction functional, defined on an arbitrary surface σ (embedded in a space of independent variables $\{\boldsymbol{p}\}$ of arbitrary dimension), e.g. parametrised as:

$$\sigma : \quad \boldsymbol{p} = \boldsymbol{p}(s,t) = (p_i(s,t)) \ , \quad (s,t) \in \Omega \subset \mathbb{R}^2 \ ,$$

takes the form

$$S[u(\boldsymbol{p});\sigma] = \int_\sigma \sum_{i<j} \mathscr{L}_{i,j} dp_i \wedge dp_j = \iint_\Omega \sum_{i<j} \left\{ \mathscr{L}_{i,j} \frac{\partial(p_i,p_j)}{\partial(s,t)} \right\} \mathrm{d}s\,\mathrm{d}t \ ,$$

where typically (2-jet case)

$$\mathscr{L}_{ij} = \mathscr{L}(u, \partial_{p_i} u, \partial_{p_j} u, \partial_{p_i}\partial_{p_j} u; p_i, p_j; n_i, n_j) \ .$$

We have two types of variations:

• Variations of the surface: $\sigma \to \sigma + \delta\sigma$, (i.e., $\boldsymbol{p} \mapsto \boldsymbol{p} + \delta\boldsymbol{p}$, in the parametrisation). This gives the closure relation as variational equations of the Lagrangian as a function of the independent variables

$$\mathsf{L}(\boldsymbol{p}(s,t)) := \sum_{i<j} \left\{ \mathscr{L}_{i,j} \frac{\partial(p_i,p_j)}{\partial(s,t)} \right\} \ ,$$

and apply the variational derivative:

$$\frac{\delta \mathsf{L}}{\delta \boldsymbol{p}(s,t)} = 0 \quad \Rightarrow \quad \partial_{p_i} \mathscr{L}_{j,k} + \partial_{p_j} \mathscr{L}_{k,i} + \partial_{p_k} \mathscr{L}_{i,j} = 0 \ .$$

• Infinitesimal variations of the dependent variable $u \mapsto u + \delta u$, on an arbitrary, but fixed, surface. This has two contributions:

◊ tangential contributions, i.e. from components $(\nabla \delta u)_{\|}$ along the surface;

◊ orthogonal contributions, i.e. from components $(\nabla \delta u)_{\perp}$ orthogonal to the surface.

Lagrange 2-form in 3D space

In the simple case of smooth 2D surfaces σ embedded in R^3, and $\mathscr{L}$ depending only on the first jet, we get the following set of equations:

- From the tangential contributions:

$$\sum_{i<j} \Bigg[\frac{\partial(p_i,p_j)}{\partial(s,t)} \frac{\partial \mathscr{L}_{ij}}{\partial u} - \frac{\partial}{\partial s} \left(\frac{\partial(p_i,p_j)}{\partial(s,t)} \frac{\boldsymbol{p}_t \times \boldsymbol{n}}{\|\boldsymbol{p}_s \times \boldsymbol{p}_t\|} \cdot \frac{\partial \mathscr{L}_{ij}}{\partial \nabla u} \right) \\ + \frac{\partial}{\partial t} \left(\frac{\partial(p_i,p_j)}{\partial(s,t)} \frac{\boldsymbol{p}_s \times \boldsymbol{n}}{\|\boldsymbol{p}_s \times \boldsymbol{p}_t\|} \cdot \frac{\partial \mathscr{L}_{ij}}{\partial \nabla u} \right) \Bigg] = 0$$

where $\boldsymbol{n}$ is the unit normal to the surface,

- From the transversal contributions:

$$\sum_{i<j} \frac{\partial(p_i,p_j)}{\partial(s,t)} \, \boldsymbol{n} \cdot \frac{\partial \mathscr{L}_{ij}}{\partial \nabla u} = 0 \ .$$

- From the variation of the surface we obtain the continuous closure relation

$$\partial_{p_i} \mathscr{L}_{jk} + \partial_{p_j} \mathscr{L}_{ki} + \partial_{p_k} \mathscr{L}_{ij} = 0$$

where $\mathscr{L}_{ji} = -\mathscr{L}_{ij}$, means that the *Lagrangian 2-form* with components $\mathscr{L}_{ij}$ is closed but only *on the solutions of the equations of the motion.*

The latter guarantees that at critical point the action is stationary under changes of the surface σ.

Question: Can one solve the Lagrangian components $\mathscr{L}_{ij}$ from this system of generalised EL equations?

A different approach to the variational system for the general continuous 2-form case was proposed[5], using "stepped surfaces", amounting to choosing piecewise flat surfaces for the action functional along the coordinate patches.
The resulting EL equations for a Lagrangian 2-form an actions defined on two-dimensional surfaces embedded in D-dimensional space, are:

$$\frac{\delta_{ij}\mathscr{L}_{ij}}{\delta\boldsymbol{\varphi}_I}=0\quad\forall I\not\ni i,j\ ,\qquad\frac{\delta_{ij}\mathscr{L}_{ij}}{\delta\boldsymbol{\varphi}_{Ij}}=\frac{\delta_{ik}\mathscr{L}_{ik}}{\delta\boldsymbol{\varphi}_{Ik}}\quad\forall I\not\ni i\ ,$$
$$\frac{\delta_{ij}\mathscr{L}_{ij}}{\delta\boldsymbol{\varphi}_{Iij}}+\frac{\delta_{jk}\mathscr{L}_{jk}}{\delta\boldsymbol{\varphi}_{Ijk}}+\frac{\delta_{ki}\mathscr{L}_{ki}}{\delta\boldsymbol{\varphi}_{Iki}}=0\quad\forall I$$

where $I=(i_1,\ldots,i_D)$ and

$$\boldsymbol{\varphi}_I=\frac{\partial^{|I|}\boldsymbol{u}}{(\partial p_1)^{i_1}\ldots(\partial p_D)^{i_D}}$$

with $|I|=i_1+\ldots+i_D$ and $I_{i_k}=(i_1,\ldots,i_{k+1},\ldots,i_D)$ and the variational derivative

$$\frac{\delta_{ij}\mathscr{L}_{ij}}{\delta\boldsymbol{\varphi}_I}=\sum_{\alpha,\beta\geq 0}(-1)^{\alpha+\beta}D_{p_i}^{\alpha}D_{p_j}^{\beta}\frac{\partial\mathscr{L}_{(ij)}}{\partial\boldsymbol{\varphi}_{Ii^\alpha j^\beta}}\ .$$

where D_{p_i}, D_{p_j} are the total derivative operators w.r.t. the variables p_i,p_j.
A yet alternative approach employs the variational bicomplex, and encodes the system of EL eqs. in the formula $\boxed{\delta\mathrm{d}\mathsf{L}=0}$.

[5] Yu. Suris & M. Vermeeren, *On the Lagrangian structure of integrable hierarchies, in: Ed A. Bobenko, "Advances in Discrete Differential Geometry",(Springer, 2016) pp 347–78.*

Application: KdV generating PDE system

The higher-order Lagrangian [6]

$$\mathscr{L}_{ij} = \frac{1}{4}(p_i^2 - p_j^2)\frac{(\partial_{p_i}\partial_{p_j}u)^2}{(\partial_{p_i}u)\partial_{p_j}u} + \frac{1}{p_i^2 - p_j^2}\left(n_i^2 p_i^2 \frac{\partial_{p_j}u}{\partial_{p_i}u} + n_j^2 p_j^2 \frac{\partial_{p_i}u}{\partial_{p_j}u}\right) ,$$

through the Euler-Lagrange equation:

$$\frac{\partial}{\partial p_i}\frac{\partial}{\partial p_j}\left(\frac{\partial \mathscr{L}_{ij}}{\partial(\partial_{p_i}\partial_{p_j}u)}\right) - \frac{\partial}{\partial p_i}\left(\frac{\partial \mathscr{L}_{ij}}{\partial(\partial_{p_i}u)}\right) - \frac{\partial}{\partial p_j}\left(\frac{\partial \mathscr{L}_{ij}}{\partial(\partial_{p_j}u)}\right) = 0 ,$$

yields a 2nd order PDE in both independent variables, which generates the KdV hierarchy by multi-time expansion i.t.o. the Miwa variables p_i, p_j.
The generalised EL system yields 3D equations, similar to the linear case, namely

$$(p_i^2 - p_j^2)(\partial_{p_k}u)\partial_{p_i}\partial_{p_j}u + (p_j^2 - p_k^2)(\partial_{p_i}u)\partial_{p_j}\partial_{p_k}u + (p_k^2 - p_i^2)(\partial_{p_j}u)\partial_{p_i}\partial_{p_k}u = 0 ,$$

and

$$\begin{aligned} 2\partial_{p_i}\partial_{p_j}\partial_{p_k}u &= \frac{\partial_{p_i}\partial_{p_j}u}{\partial_{p_k}u} + \frac{\partial_{p_j}\partial_{p_k}u}{\partial_{p_i}u} + \frac{\partial_{p_i}\partial_{p_k}u}{\partial_{p_j}u} \\ &+ \frac{n_i^2/(\partial_{p_i}u)^2}{(p_i^2 - p_j^2)(p_i^2 - p_k^2)} + \frac{n_j^2/(\partial_{p_j}u)^2}{(p_j^2 - p_i^2)(p_j^2 - p_k^2)} + \frac{n_k^2/(\partial_{p_k}u)^2}{(p_k^2 - p_i^2)(p_k^2 - p_j^2)} , \end{aligned}$$

which imply the 2-variable EL equation, as well as the *closure relation* on the Lagrangian:

$$\partial_{p_k}\mathscr{L}_{ij} + \partial_{p_i}\mathscr{L}_{jk} + \partial_{p_j}\mathscr{L}_{ki} = 0 .$$

[6] FWN, A. Hone, N. Joshi, Phys. Lett. **267** (2000).

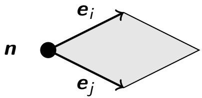
Surface-dependent actions

The closure relation suggests the introduction of surface-dependent action functionals.

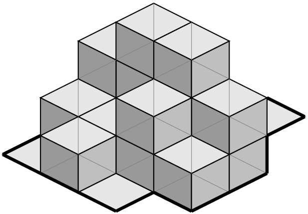
$$S[u(\boldsymbol{n});\sigma] = \sum_{\sigma} \mathsf{L} = \sum_{\sigma_{ij}(\boldsymbol{n})\in\sigma} \mathscr{L}_{ij}(\boldsymbol{n})$$

where $\mathscr{L}_{ij}(\boldsymbol{n})$ is a discrete Lagrangian 2-form [7]: $\mathsf{L} = \sum_{i<j} \mathscr{L}_{ij}\Delta_i \wedge \Delta_j$
These are oriented expressions of the form:

$$\mathscr{L}_{ij}(\boldsymbol{n}) = \mathscr{L}(u(\boldsymbol{n}), u(\boldsymbol{n}+\boldsymbol{e}_i), u(\boldsymbol{n}+\boldsymbol{e}_j); p_i, p_j)$$



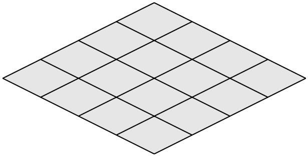
defined on elementary plaquettes, in a multidimensional lattice, characterized by triplets $\sigma_{ij}(\boldsymbol{n}) = (\boldsymbol{n}, \boldsymbol{n}+\boldsymbol{e}_i, \boldsymbol{n}+\boldsymbol{e}_j)$.
σ a quad surface consisting of (a connected configuration of) elementary plaquettes $\sigma_{ij}(\boldsymbol{n})$



[7] Notation borrowed from: E.L. Mansfield and P.E. Hydon, Difference forms, *Found. of Comp. Math.* **8** (2008) 427–467.

Surface independence on the latice

The closure relation implies the invariance of the action S under local deformations $S \to S'$ of the surface:

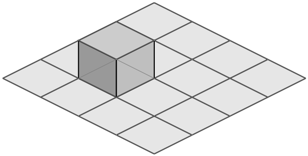


$$\begin{aligned} S' \quad = \quad & S - \mathscr{L}(u, u_i, u_j; p_i, p_j) + \mathscr{L}(u_k, u_{i,k}, u_{j,k}; p_i, p_j) + \mathscr{L}(u_i, u_{i,j}, u_{i,k}; p_j, p_k) \\ & + \mathscr{L}(u_j, u_{j,k}, u_{i,j}; p_k, p_i) - \mathscr{L}(u, u_j, u_k; p_j, p_k) - \mathscr{L}(u, u_k, u_i; p_k, p_i) \end{aligned}$$

taking into account the orientation of the plaquettes.

Surface independence on the lattice

The closure relation implies the invariance of the action S under local deformations $S \to S'$ of the surface:



$$\begin{aligned} S' \quad &= \quad S - \mathscr{L}(u, u_i, u_j; p_i, p_j) + \mathscr{L}(u_k, u_{i,k}, u_{j,k}; p_i, p_j) + \mathscr{L}(u_i, u_{i,j}, u_{i,k}; p_j, p_k) \\ &\quad + \mathscr{L}(u_j, u_{j,k}, u_{i,j}; p_k, p_i) - \mathscr{L}(u, u_j, u_k; p_j, p_k) - \mathscr{L}(u, u_k, u_i; p_k, p_i) \end{aligned}$$

taking into account the orientation of the plaquettes.

Fundamental EL system for quad-equations

We will now describe how to resolve the issue of "weak equations": that the closure requires stronger equations than the variational one[8].

Assuming the 3-point form of the Lagrangians:

$$\mathscr{L}_{i,j}(u, T_i u, T_j u) := \mathscr{L}(u, T_i u, T_j u; p_i, p_j) \ ,$$

the lattice EL can be written as:

$$(EL0) \quad \frac{\partial}{\partial u}\left(\mathscr{L}_{i,j}(T_i^{-1}u, u, T_i^{-1}T_j u) + \mathscr{L}_{i,j}(u, T_i u, T_j u) + \mathscr{L}_{i,j}(T_j^{-1}u, T_i T_j^{-1}u, u)\right) = 0 \ .$$

This represents the "planar" EL eqs, illustrated by the diagram (embedded in 3D lattice):

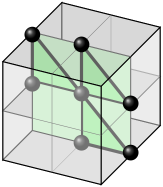


Figure: EL in flat 2D lattice.

This is the weak (non-quadrilateral) form of the equations. However, by the extended variational principle of the multiform structure, the quad-lattice equation is recovered.

[8] S.B.Lobb & F.W. Nijhoff. A variational principle for discrete integrable systems. ArXiv: 1312.1440.
R. Boll, M. Petrera and Yu. Suris, *What is integrability of discrete variational systems?*, arXiv:1307.0523.

Lattice action for the closed cube surface

To derive elementary configurations we need action over the (decorated) full oriented cube:

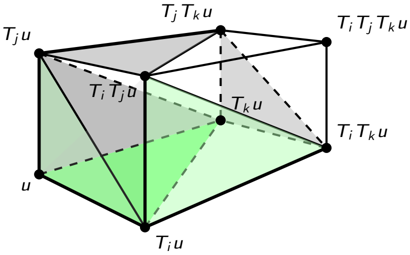


Figure: Decorated cube.

This gives rise to a lattice action functional:

$$
\begin{aligned}
S[u; cube] = {} & \mathscr{L}_{i,j}(u, T_i u, T_j u) + \mathscr{L}_{j,k}(u, T_j u, T_k u) + \mathscr{L}_{k,i}(u, T_k u, T_i u) \\
& - \mathscr{L}_{i,j}(T_k u, T_i T_k u, T_j T_k u) - \mathscr{L}_{j,k}(T_i u, T_i T_j u, T_i T_k u) - \mathscr{L}_{k,i}(T_j u, T_j T_k u, T_i T_j u).
\end{aligned}
$$

The faces joining each vertex involved in the action will give rise to the various elementary surface configurations: the elementary actions that will lead to the fundamental system of EL equations.

Elementary configurations for lattice action

Over curved quad-surfaces we need the following types of elementary configurations:

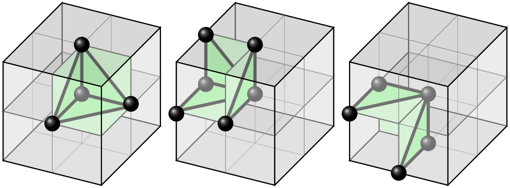


Figure: Elementary lattice configurations in 3D.

The action functionals corresponding to these configurations give rise to the fundamental system of EL equations:

$$(EL1) \qquad \frac{\partial}{\partial u}\Big(\mathscr{L}_{i,j}(u, T_i u, T_j u) + \mathscr{L}_{j,k}(u, T_j u, T_k u) + \mathscr{L}_{k,i}(u, T_k u, T_i u)\Big) = 0,$$

$$(EL2) \qquad \frac{\partial}{\partial u}\Big(\mathscr{L}_{i,j}(T_i^{-1} u, u, T_i^{-1} T_j u) - \mathscr{L}_{j,k}(u, T_j u, T_k u) + \mathscr{L}_{k,i}(T_i^{-1} u, T_i^{-1} T_k u, u)\Big) = 0,$$

$$(EL3) \qquad \frac{\partial}{\partial u}\Big(\mathscr{L}_{j,k}(T_j^{-1}(u), u, T_j^{-1} T_k u) + \mathscr{L}_{k,i}(T_i^{-1} u, T_i^{-1} T_k u, u)\Big) = 0.$$

(up to permutations of the lattice indices).
Furthermore, imposing that the action remains invariant under (discrete) deformations of the surface (allowing the above equations to hold simultaneously) the system is supplemented with the closure relation:

$$(EL4) \quad \Delta_i \mathscr{L}(u, T_j u, T_k u; p_j, p_k) + \Delta_j \mathscr{L}(u, T_k u, T_i u; p_k, p_j) + \Delta_k \mathscr{L}(u, T_i u, T_j u; p_i, p_j) = 0\ .$$

General quad-lattice Lagrangian

Main hypothesis: The solutions of above *linear system of equations for the Lagrangians* $\mathscr{L}$ correspond exactly to the Lagrangians for integrable (in the sense of multidimensional consistency) quadrilateral lattice systems.

For *affine-linear* D_4-symmetric quad-lattice equations

$$0 = Q_{\mathfrak{p}_i,\mathfrak{p}_j}(u,u_i,u_j,u_{ij}) = Q_{\mathfrak{p}_i,\mathfrak{p}_j}(u_i,u,u_{ij},u_j) = Q_{\mathfrak{p}_i,\mathfrak{p}_j}(u_j,u_{ij},u,u_i) = -Q_{\mathfrak{p}_j,\mathfrak{p}_i}(u,u_j,u_i,u_{ij})$$

for scalar dependent variable $u = u(\boldsymbol{n})$, with
$u_i := u(\boldsymbol{n}+\boldsymbol{e}_i)$, $\quad u_{ij} := u(\boldsymbol{n}+\boldsymbol{e}_i+\boldsymbol{e}_j)$, introducing the biquadratic functions:

$$\begin{aligned} Q_{u_j} Q_{u_{ij}} - Q\, Q_{u_j u_{ij}} &=: K_{\mathfrak{p}_i,\mathfrak{p}_j} h_{\mathfrak{p}_i}(u,u_i) \\ Q_{u_i} Q_{u_{ij}} - Q\, Q_{u_i u_{ij}} &=: K_{\mathfrak{p}_j,\mathfrak{p}_i} h_{\mathfrak{p}_j}(u,u_j) \\ Q_u Q_{u_{ij}} - Q\, Q_{u u_{ij}} &=: -K_{\mathfrak{p}_i,\mathfrak{p}_j} h_{\mathfrak{p}_{ij}}(u_i,u_j) \end{aligned}$$

where $K_{\mathfrak{p},\mathfrak{q}} = -K_{\mathfrak{q},\mathfrak{p}}$ is a function of the lattice parameters $\mathfrak{p}$, $\mathfrak{q}$ only, we have the following general solution *for the Lagrangian* of the multiform EL system[9]

$$\mathscr{L}(u,u_i,u_j;u^0,u_i^0,u_j^0) = \int_{u^0}^{u}\int_{u_i^0}^{u_i} \frac{dx\,dy}{h_{\mathfrak{p}_i}(x,y)} - \int_{u^0}^{u}\int_{u_j^0}^{u_j} \frac{dx\,dy}{h_{\mathfrak{p}_j}(x,y)} - \int_{u_i^0}^{u_i}\int_{u_j^0}^{u_j} \frac{dx\,dy}{h_{\mathfrak{p}_{ij}}(x,y)}$$
$$+ \int_{u_i^0}^{u_i} dx \int_{u_j^0}^{Y(u^0,x,u_{ij}^0)} \frac{dy}{h_{\mathfrak{p}_{ij}}(x,y)} + \int_{u_j^0}^{u_j} dy \int_{u_i^0}^{X(u^0,y,u_{ij}^0)} \frac{dx}{h_{\mathfrak{p}_{ij}}(x,y)}$$

where the functions X and Y are solutions of the equations

$$Q_{\mathfrak{p}_i,\mathfrak{p}_j}(u^0,x,Y,u_{ij}^0) = 0 \quad \text{respectively} \quad Q_{\mathfrak{p}_i,\mathfrak{p}_j}(u^0,X,y,u_{ij}^0) = 0 \ .$$

[9] P. Xenitidis, FWN & S. Lobb, *On the Lagrangian formulation of multidimensionally consistent systems*, Proc. Roy. Soc. **A467** # 2135 (2011) 3295-3317.

Quantisation of the Lattice Equation

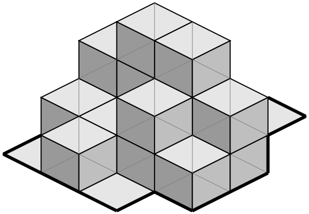
Defining a multiform path integral approach to quantising the linear lattice equation[10].

$$(p_i + p_j)(u_i - u_j) = (p_i - p_j)(u - u_{ij}) \ , \quad \text{where} \quad u_i = T_i u, u_{ij} = T_i T_j u \ ,$$

through a quadratic Lagrangian:

$$\mathscr{L}_{ij}(u, u_i, u_j; p_i, p_j) = u(u_i - u_j) - \frac{1}{2} s_{ij}(u_i - u_j)^2 \ ; \qquad s_{ij} = \frac{p_i + p_j}{p_i - p_j}$$

Thus, we have a discrete 2D quantum field theory with fields variables $u(\boldsymbol{n})$ with $\boldsymbol{n}$ coordinates of the lattice sites.Consider a quad surface σ with boundary $\partial\sigma$.



Action: $\mathscr{S}[u_{n,m}; \sigma] = \sum_\sigma \mathscr{L}(\boldsymbol{n})$.
Propagator (all interior field variables are integrated over):

$$\begin{aligned} K_\sigma(\partial\sigma) &= \int [\mathscr{D} u_{n,m}] \ e^{i\mathscr{S}[u_{n,m};\sigma]/\hbar} \\ &= \mathscr{N}_\sigma \prod_{\boldsymbol{n}\in\sigma} \int \mathrm{d}u(\boldsymbol{n}) \ e^{i\mathscr{S}[u(\boldsymbol{n});\sigma]/\hbar} \end{aligned}$$

with normalisation factor $\mathscr{N}_\sigma$.

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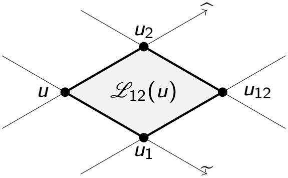
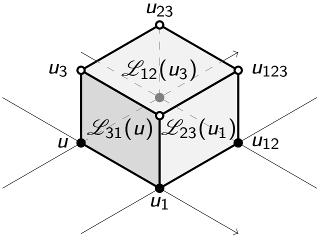
[10] S. King and F.W. Nijhoff, *Quantum variational principle and quantum multiform structure: the case of quadratic Lagrangians*, arXiv: 1702.08709.

Surface-independence of the propagator

Main question: What happens to propagator $K_\sigma(\partial\sigma)$ under variation of the surface σ?
Consider a pop-up cube, and performing Gaussian integrals:



(a)



(b)

$$\mathscr{S}_{pop}[u_{n,m}] = \mathscr{L}_{23}(u_1) + \mathscr{L}_{31}(u_2) + \mathscr{L}_{12}(u_3) - \mathscr{L}_{23}(u) - \mathscr{L}_{31}(u)$$

$$K_\sigma = \mathscr{N} \exp\left(\frac{i}{\hbar}\mathscr{L}_{12}(u, u_1, u_2)\right)$$

$$K_{pop} = \mathscr{N} \iiiint \mathrm{d}u_3 \mathrm{d}u_{31} \mathrm{d}u_{23} \mathrm{d}u_{123} \ \exp\left(\frac{i}{\hbar}\mathscr{S}_{pop}[u_{n,m}]\right)$$

$$= \mathscr{N} V^2 \frac{2\pi\hbar}{s_{23}} \exp\left(\frac{i}{\hbar}\mathscr{L}_{12}(u, u_1, u_2)\right)$$

The contributions to the propagator from each surface are (up to normalisation) the same. Thus, *the propagator invariant under this surface-move!* In fact, this also holds for all elemntary surface moves (as in the classical case).

Quantum variational principle for surface actions

This result suggests a quantum variational principle in analogy to the classical case, containing the following ingredients:

• Propagator for general quadratic Lagrangian 2-form over discrete surface σ, with action $\mathscr{S}[u(\boldsymbol{n});\sigma]$ as defined before;

• Path integral over interior field variables:

$$K_\sigma(\partial\sigma) = \int [\mathscr{D}u_{n,m}]\ e^{i\mathscr{S}[u_{n,m};\sigma]/\hbar} \ := \mathscr{N} \prod_{\boldsymbol{n}\in\sigma} \int \mathrm{d}u(\boldsymbol{n})\ e^{i\mathscr{S}[u(\boldsymbol{n});\sigma]/\hbar}\ .$$

In general $K_\sigma(\partial\sigma)$ is a function of the field variables on the boundary $\partial\sigma$ and also depends on the surface σ itself;

• For a special choice of discrete Lagrangian 2-form the propagator $K_\sigma(\partial\sigma)$ is *independent of the surface* σ. This Lagrangian exists at a *critical point* of the variation of the surface, such that some of the integrations over field variables reduce to volume factors;

• The condition of stationarity of propagator under surface moves determines (up to equivalence) the Lagrangian form (this has been demonstrated for the case of quadratic 3-point Lagrangians), leading to Lagrangian of the form:

$$\mathscr{L}_{ij}(u,u_i,u_j;p_i,p_j) = u(u_i-u_j) - \frac{1}{2}s_{ij}(u_i-u_j)^2\ ; \qquad s_{ij} = \frac{p_i+p_j}{p_i-p_j}\ .$$

• The invariance under surface deformation suggests that one could consider a novel quantum object obtained by a *sum over all surfaces*, leading to a functional of the Lagrangian 2-form, and which attains a critical point for Lagrangians for which the usual surface-dependent propagator $K_\sigma(\partial\sigma)$ becomes invariant.

Possible quantum 1-form theory

In the Lagrangian 1-form case[11] we have an action functional

$$S[\boldsymbol{x}(\boldsymbol{n});\Gamma] \sum_{\gamma(\boldsymbol{n})\in\Gamma} \mathscr{L}_i(\boldsymbol{x}(\boldsymbol{n}),\boldsymbol{x}(\boldsymbol{n}+\boldsymbol{e_i}))\ ,$$

over a discrete curve Γ or in the continuous-time case, an action functional:

$$S[\boldsymbol{x}(\boldsymbol{t});\Gamma\,] = \int_\Gamma \mathsf{L}(\boldsymbol{x}(\boldsymbol{t}),\boldsymbol{x}_{\boldsymbol{t}}) = \int_{s_a}^{s_b}\sum_k \left(\mathscr{L}_k(\boldsymbol{x}(\boldsymbol{t}(s)),\boldsymbol{x}_{t_1}(\boldsymbol{t}(s)),\boldsymbol{x}_{t_2}(\boldsymbol{t}(s)),\dots)\frac{dt_k}{ds}\right)\,ds$$

for a system with commuting flows in higher-time variables $\boldsymbol{t}=(t_1,t_2,\dots)$.
This structure, and the corresponding multi-time EL equations, applies to the CM and Ruijsenaars system in both discrete as well as continuous time.
A tentative proposal for a quantum Lagrangian 1-form structure is the Feynman type propagator[12]:

$$K(\boldsymbol{x}_b,\boldsymbol{t}_b,s_b;\boldsymbol{x}_a,\boldsymbol{t}_a,s_a) = \int_{\boldsymbol{t}(s_a)=\boldsymbol{t}_a}^{\boldsymbol{t}(s_b)=\boldsymbol{t}_b}[\mathscr{D}\boldsymbol{t}(s)]\int_{\boldsymbol{x}(\boldsymbol{t}_a)=\boldsymbol{x}_a}^{\boldsymbol{x}(\boldsymbol{t_b})=\boldsymbol{x}_b}[\mathscr{D}_\Gamma\boldsymbol{x}(\boldsymbol{t})]\,\exp\left(\frac{i}{\hbar}S[\boldsymbol{x}(\boldsymbol{t});\Gamma]\right)\ .$$

Here:

- $[\mathscr{D}_\Gamma\boldsymbol{x}(\boldsymbol{t})]$ is some path integral measure along a curve Γ in the space of dependent variables $\boldsymbol{x}(\boldsymbol{t})$;
- Γ is a curve in the space of independent variables, parametrised by the parameter $s\in[s_a,s_b]$, bounded by the points $\boldsymbol{t}(s_a)=\boldsymbol{t}_a$ and $\boldsymbol{t}(s_b)=\boldsymbol{t}_b$;
- $[\mathscr{D}\boldsymbol{t}(s)]$ is some path integral measure in the space of independent variables.

[11] S. Yoo-Kong, S. Lobb and FWN , Discrete Caloger-Moser system and Lagrangian 1-form structure, J. Phys A: Math Theor. **44** (2011) 365203.

[12] FWN, talk given at the 2013 Newton Institute meeting on Discrete Integrable Systems

Discussion

Some of the main points resulting from Lagrangian multiform theory are the following:

- The many explicit examples studied so far seem to indicate that the multiform structure is a universal aspect of integrability;
- The main motivation was to formulate a least-action principle that produces the whole system of multidimensionally consistent equations, rather than a single equation of the motion;
- This new variational principle brings in an essential way *the geometry of the independent variables* into play: a kind of "democracy between independent and dependent variables";
- The variational principle determines not only the equations for the classical trajectories of the system, but more prominently it selects the *admissable Lagrangians* as solutions of the system of generalized EL equations;
- The close interplay between compatible *continuous* and *discrete* structures (exhibiting a role reversal of parameters and independent variables) indicates that both are intechangeable aspects of one and the same structure;
- Because of the determinacy of the Lagrangian components, in a way the multiform structure provides a partial answer to the question of the *inverse problem of Lagrangian dynamics*.
- The main motivation comes from quantum theory: the multiform structure seems to point to novel quantum objects which are a kind of "sum over geometries" (here there may be parallels with LQG).