Boundary Lagrangian formalism for integrable quad-equations

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Outline

Integrable quad-graph systems

Quad-graph systems with an integrable boundary

Lagriangian formalism

Some perspectives

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Example: lattice KdV-type equations

- ♯ the potential KdV equation:

$$u_t = u_{xxx} + 3u_x^2\,, \quad u := u(x,t)$$

- ♯ potential dressing chain obtained by Darboux-Bäcklund transformation:

$$\widetilde{u}^2 + \widetilde{u}_x + u^2 + u_x = 2a_1 + \widetilde{u}u\,, \quad u := u(x,t,n)$$

or with $y = u^2 + u_x$

$$\widetilde{y} + y = 2a_1 + \widetilde{u}u$$

- ♯ the action of $\widetilde{\ }$ as a shift: $\widetilde{u}(n) = u(n+1)$ is accompanied by a parameter a_1, and n is the discrete variable induced by Darboux-Bäcklund transformation
- ♯ using additive and multiplicative compatibilities ($\widehat{\ }$ is associated with another discrete variable m accompanied by a_2)

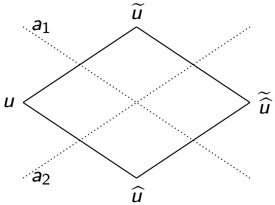
$$(\widetilde{y} + y) + (\widehat{\widetilde{y}} + \widehat{y}) = (\widehat{y} + y) + (\widetilde{\widehat{y}} + \widetilde{y})$$
$$\widehat{\widetilde{u}}\widehat{u} \times \widetilde{u}u = \widetilde{\widehat{u}}\widetilde{u} \times \widehat{u}u$$

yields (according to Adler-Bobenko-Suris classification)

H1: $(\widehat{\widetilde{u}} - u)(\widehat{u} - \widetilde{u}) = a_2 - a_1$

H2: $(\widehat{\widetilde{y}} - y)(\widehat{y} - \widetilde{y}) + 2(a_1 - a_2)(\widehat{\widetilde{y}} + \widehat{y} + \widetilde{y} + y) = 4(a_1^2 - a_2^2)$

Compatibility as Bianchi permutability



♯ Darboux-Bäcklund discretization induces two discrete variables

$$u(x,t) \to u(x,t;n,m,a_1,a_2)$$

and yields lattice potential KdV equation defined on a quadrilateral

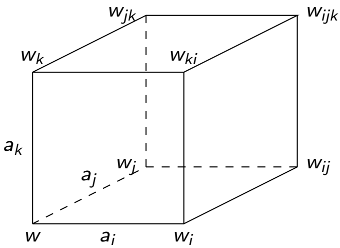
$$(\widehat{\widetilde{u}} - u)(\widehat{u} - \widetilde{u}) = a_2 - a_1$$

♯ Proper *double continuous limit* leads to the whole hierarchy of KdV equations

♯ Darboux-Bäcklund discretization could induce more than two discrete variables (discrete hierarchy)

3D-consistent equations

As an emerging notion: **three-dimensional consistency**, or **consistency around the cube**, as the **defining criterion** of discrete integrability (Nijhoff, Bobenko & Suris)



$$Q(w, w_i, w_j, w_{ij}, a_i, a_j) = 0, \quad Q(w_k, w_{ki}, w_{jk}, w_{kij}, a_i, a_j) = 0,$$
$$Q(w, w_j, w_k, w_{jk}, a_j, a_k) = 0, \quad Q(w_i, w_{ij}, w_{ki}, w_{ijk}, a_j, a_k) = 0,$$
$$Q(w, w_w, w_i, w_{wi}, a_w, a_i) = 0, \quad Q(w_j, w_{jk}, w_{ij}, w_{jki}, a_w, a_i) = 0.$$

Given w, w_i, w_j and w_k, the computations of w_{ijk} remains the same

$$w_{kij} = w_{ijk} = w_{jki}$$

3D-consistency implies multi-dimensional consistency (for instance on a hypercube), as a **discrete analogue of the infinite commuting flows**, and a natural derivation of the **discrete zero curvature condition**

$$M_{n+1,m}\, L_{n,m} = L_{n,m+1}\, M_{n,m}$$

Consistency around a hypercube

If a quadrilateral equation is 3D consistent, then it is also 4D consistent

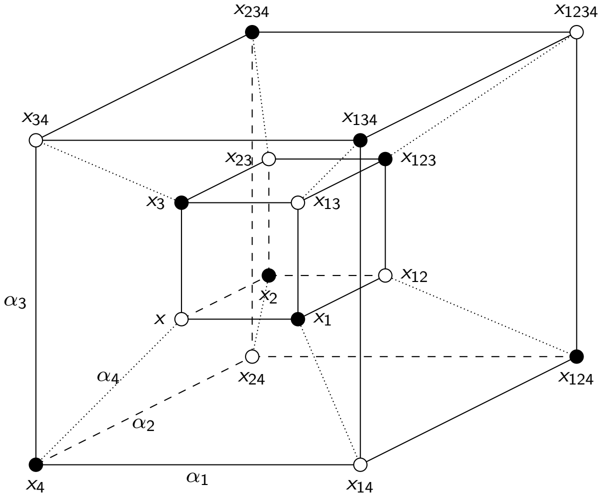
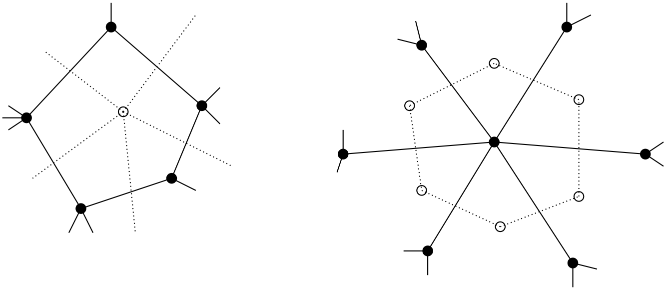


Figure: Consistency around a hypercube

Integrable quad-graph systems (Mercat, Bobenko & Suris)

- ♯ Quad-graph systems are discrete systems defined on a quad-graph (roughly a decomposition of a surface where elementary patterns are quadrilaterals)
- ♯ "Quad-graph discretization" of an arbitrary planar graph:



  the resulting quad-graph consists of quadrilaterals whose edges connecting adjacent black (original graph) and white (dual graph) dots
- ♯ this "simple" construction together with 3D-consistent equations lead to the notion of integrable quad-graph systems.
- ♯ some important results include **classification results** (Adler, Bobenko & Suris), **Lagrangian multiform or pluri-Lagrangian theory** (Lobbs & Nijhoff, Bobenko & Suris, etc.), connection to **discrete complex analytic functions and discrete Riemann surface theory** (Schramm, Mercat, Bobenko, Suris, Smirnov, etc), etc.

ABS classification for scalar quad-equations

- ♯ the classification [Adler Bobenko & Suris, '03] [Adler Bobenko & Suris, '09] is based on some basic techniques in projective multivariate polynomials and their invariants, in particular the field is living in $\mathbb{CP}^1$ (we lose the sense of real field in the continuous case)
- ♯ ABS list contains some equations, named as H1-H3, Q1-Q4, they are all KdV-type equations, Q4 is the top equation of the list where the parameters are sitting on elliptic curves

$$\text{H3:} s(v\widehat{v} - \widetilde{v}\widehat{\widetilde{v}}) - t(v\widetilde{v} - \widehat{v}\widehat{\widetilde{v}}) = (-1)^{n+m}\delta\left(\frac{s}{t} - \frac{t}{s}\right)$$

$$\text{Q1:} s^2(z - \widetilde{z})(\widehat{z} - \widehat{\widetilde{z}}) - t^2(z - \widehat{z})(\widetilde{z} - \widehat{\widetilde{z}}) = 4p^2\delta\left(\frac{1}{t^2} - \frac{1}{s^2}\right).$$

$$\text{Q2:} s^2(y - \widetilde{y})(\widehat{y} - \widehat{\widetilde{y}}) - t^2(y - \widehat{y})(\widetilde{y} + \widehat{\widetilde{y}}) + (\frac{1}{t^2} - \frac{1}{s^2})(y + \widetilde{y} + \widehat{y} + \widehat{\widetilde{y}}) = \frac{s^6 + 2s^2t^2(t^2 - s^2t^6)}{s^6t^6}$$

$$\text{Q3:} s\,\mu(x\widehat{x} + \widetilde{x}\widehat{\widetilde{x}}) - t\,\nu(x\widetilde{x} + \widehat{x}\widehat{\widetilde{x}}) = (\alpha^2 - \beta^2)\left(\widetilde{x}\widehat{x} + x\widehat{\widetilde{x}} + (p^2 - q^2)^2\frac{\delta}{s\mu t\nu}\right)$$

- ♯ Q4 (discrete Krichever-Novikov equation) (Hietarinta)

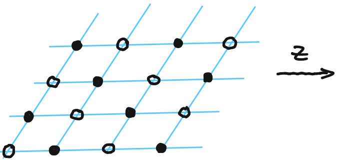
$$\text{sn}(s)(xu + vy) - \text{sn}(t)(xv + uy) - \text{sn}(s-t)(xy + uv) + \text{sn}(s-t)\text{sn}(s)\text{sn}(t)(1 + k^2xuvy) = 0$$

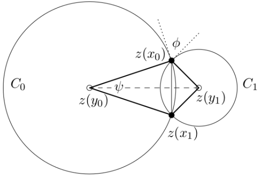
Cross-ratio as a *nonlinear* discrete analytic functions

♯ the cross-ratio, or lattice Schwarzian KdV, or Q0 equaiton:

$$\text{Q0}: \quad [u, \widetilde{u}, \widehat{u}, \widehat{\widetilde{u}}] = \frac{(u - \widehat{u})(\widetilde{u} - \widehat{\widetilde{u}})}{(u - \widetilde{u})(\widehat{u} - \widehat{\widetilde{u}})} = \frac{a}{b}\,.$$

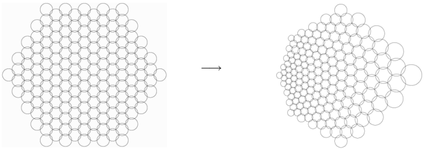
♯ z : quad-graph $\to \mathbb{C}$ such that (obeying the *Delauney decomposition*)





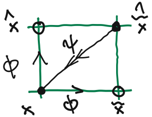
$$[z(y_0), z(x_1), z(x_0), z(y_1)] = \frac{(z(y_0) - z(x_0))(z(x_1) - z(y_1))}{(z(y_0) - z(x_1))(z(x_0) - z(y_1))} = e^{2i\phi}$$

♯ discrete analytic functions are cross-ratio preserving maps (idea due to Thurston to approximate Riemann mappings theorem using circle patterns, see for example [Bobenko, Mercat & Suris, 04])



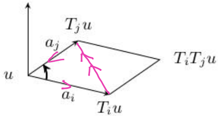
Lagrangian multi-form structures

Consider the ABS classification, all eqs admit a "three-leg" representation:



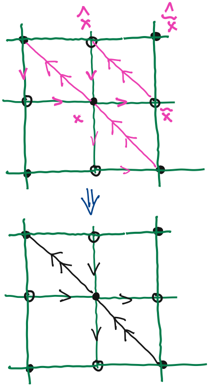
$$Q(x,\tilde{x},\hat{x},\hat{\tilde{x}};\alpha,\beta)=0$$

$$\phi(x,\tilde{x};\alpha)-\phi(x,\hat{x};\beta)=\psi(x,\hat{\tilde{x}};\alpha-\beta)$$



Lagrangian for a single quad-eq:

$$L(x,\tilde{x},\hat{x};\alpha,\beta)=C(x,\tilde{x};\alpha)-C(x,\hat{x};\beta)+V(\tilde{x},\hat{x};\alpha-\beta)$$



where
$$\frac{dC(x,\tilde{x};\alpha)}{dx}=\phi(x,\tilde{x};\alpha) \qquad \frac{d}{dx}\left(\overset{\alpha}{x\rightarrow\tilde{x}}\right)=\rightarrow \sim \phi(x,\tilde{x};\alpha)$$

$$\frac{dC(x,\hat{x};\beta)}{dx}=\phi(x,\hat{x};\beta) \qquad \frac{d}{d\tilde{x}}\left(\overset{\alpha}{x\rightarrow\tilde{x}}\right)=\rightarrow \sim \phi(x,\tilde{x};\alpha).$$

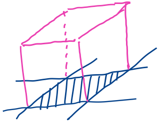
$$\frac{dV(\tilde{x},\hat{x};\alpha-\beta)}{d\tilde{x}}=\psi(\tilde{x},\hat{x};\alpha,\beta). \qquad \frac{d}{d\tilde{x}}\left(\overset{\alpha-\beta}{\tilde{x}\twoheadrightarrow\hat{x}}\right)=\twoheadrightarrow \sim \psi(\tilde{x},\hat{x};\alpha-\beta)$$

$S[x;\Sigma]=\sum L$, Euler-Lagrange eqs: $\delta S[x;\Sigma]=0$

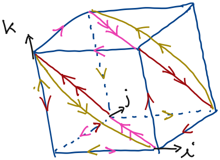
$$\frac{\partial}{\partial x}\left(L+T_i^{-1}(L)+T_j^{-1}(L)\right)=0$$ up to 2 copies of a quad-eq.

Closure relation, and star-triangle relations

closure relation: capture the property of 3D-consistency:



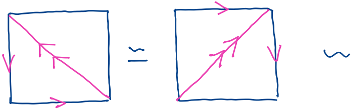
$$S(x,\Sigma) - S(x,\bar{\Sigma}) = 0$$

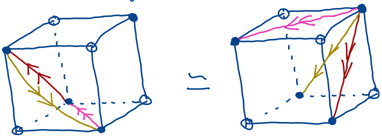


$$(T_j \mathcal{L} - \mathcal{L}) + (T_i \mathcal{L} - \mathcal{L}) + (T_k \mathcal{L} - \mathcal{L}) = 0$$

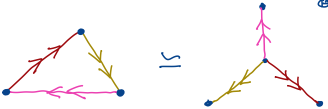
further relation for the Lagrangians of eq in the ABS list:

star-triangle relation:





or:



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Quad-graph discretization with a boundary

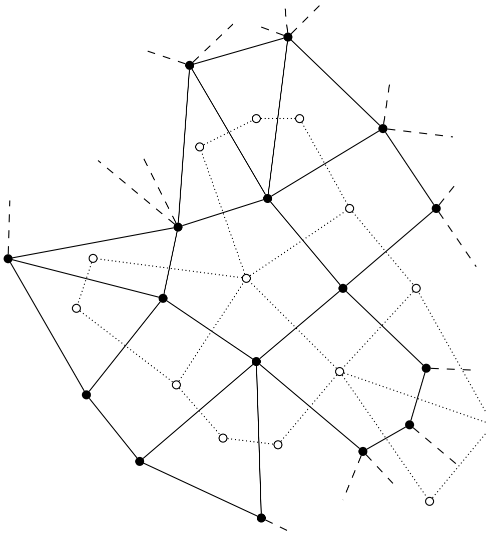


Figure: Planar graph with a boundary

Quad-graph discretization with a boundary

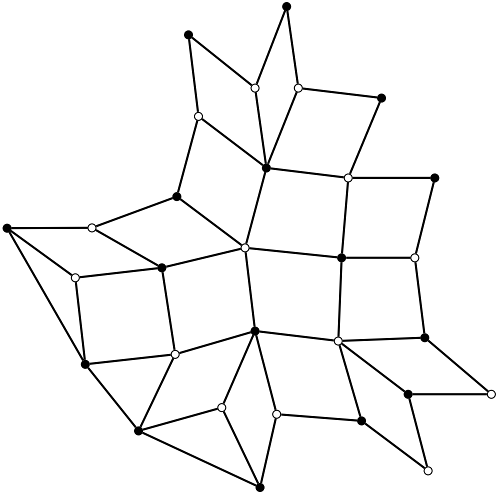
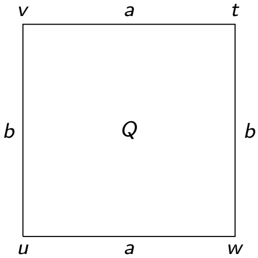
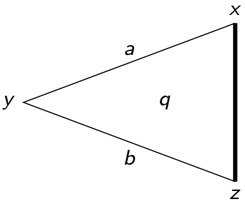


Figure: Quad-graph with a boundary

"Triangular equations" as discrete boundary conditions

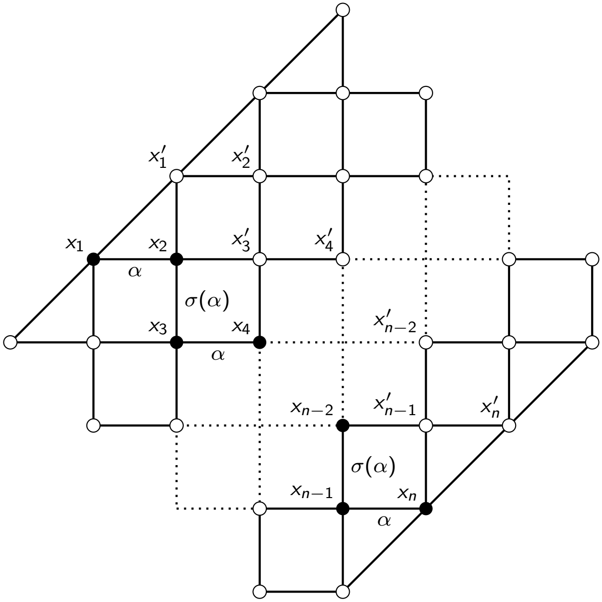




$$Q(u, w, v, t, a, b) = 0 \qquad q(x, y, z, a, b) = 0$$

We call $Q = 0$ *bulk* equation, $q = 0$ *boundary* equations. Given Q and q, a discrete initial-boundary value problem can be defined on a quad-graph, and q gives rise to the notion of **discrete boundary conditions**.

Example: a well-posed discrete interval problem ($n = 2k + 1$)



Non-degenerate ($\exists\ \sigma$) boundary consistency

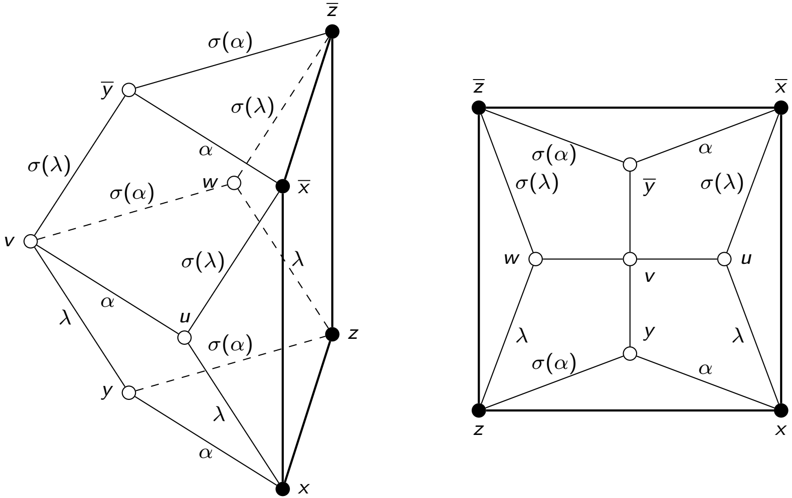
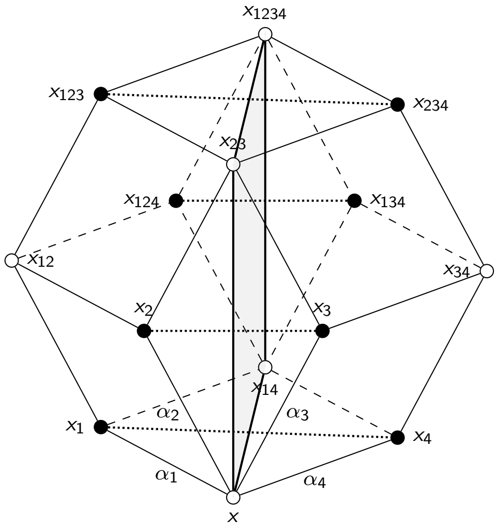


Figure: Boundary consistency around half of a rhombic dodecahedron (left) and its planar projection (right), here σ is an involution acting on the parameters.


x_{1234}
x_{123}
x_{234}
x_{23}
x_{124}
x_{134}
x_{12}
x_{34}
x_2
x_3
x_{14}
α_2
α_3
x_1
x_4
α_1
α_4
x

Some developments for quad-graph systems with a boundary

- ♯ we obtained an efficient method (classification) for obtaining integrable boundary equations for ABS classification
- ♯ Lax representation of integrable boundary equations, and an inverse scattering scheme for "interval problems"
- ♯ set up a Lagrangian formalism for the boundary equations
  - ⋄ set up the boundary Lagrangian
  - ⋄ the set of Euler-Lagrange equations (involving both bulk and boundary Lagrangians)
  - ⋄ systematic way to derive the boundary Lagrangians
  - ⋄ an integrable variational principle involving boundary

[Caudrelier, Crampé, **ZC**, 2014],
[Caudrelier, van der Kamp, **ZC**, 2022],
[Sun, **ZC**, 2023],
[Caudrelier, Nijhoff, **ZC**, in preparation]

Definition of boundary equations

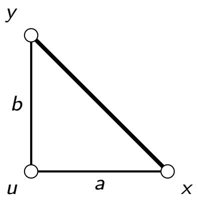


Figure: Elementary triangle supporting a boundary equation

Def 1: an equation defined on a triangle in the form $q(x, u, y; a, b) = 0$ is called a **boundary equation**, if 1) $q(x, u, y; a, b)$ is a multivariate polynomial in x, u, y, and affine-linear with respect to x and y, 2) q is Z^2-symmetric meaning there exists certain function γ uniquely depending on parameters such that

$$q(x, u, y; a, b) = \gamma(a, b) q(y, u, x; b, a) .$$

we call such q **boundary polynomial**.

Classification of factorized boundary equations

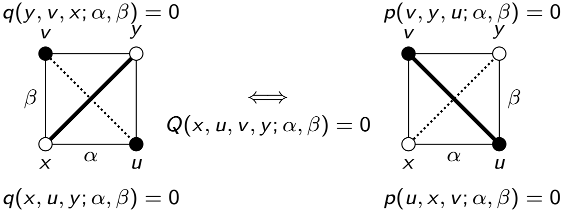
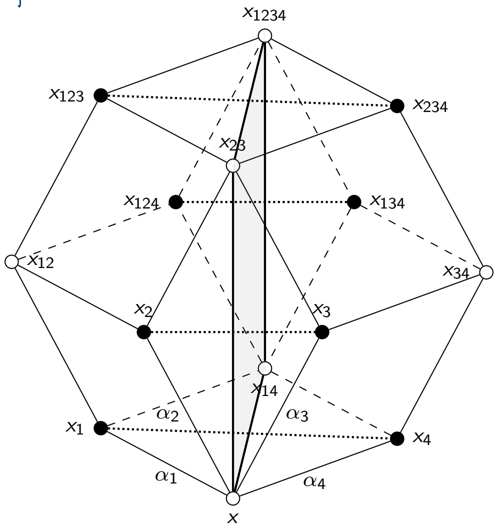


Figure: Factorization of Q along the two diagonals (thick lines).

- ♯ the boundary polynomials p and q are playing dual roles
- ♯ **Lemma**: degree of the middle fields in p (and q) is less or equal than 2 (by counting the degrees of polynomials)

$$\deg q_u + \deg p_x \leq 2\,.$$

Rough idea: cutting the 12-rhombic face object into 2 halves.



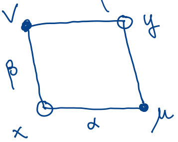
Full classification

One class (among three classes) of integrable boundary eqs reads like.

Q	$q = r_1 \boldsymbol{q}_1 + r_2 \boldsymbol{q}_2 + r_4 \boldsymbol{q}_4$	$\phi = r_1 xy + r_2(x+y) + r_4$
Q1(δ)	$\boldsymbol{q}_2$	$x+y$
Q3(0)	$\boldsymbol{q}_1 + \boldsymbol{q}_4\,,\,\boldsymbol{q}_2$	$xy+1\,,\,x+y$
Q3($\delta \neq 0$)	$\boldsymbol{q}_2$	$x+y$
Q4	$k\boldsymbol{q}_1 + 1\,,\,\boldsymbol{q}_2$	$kxy+1\,,\,x+y$
H1	$\boldsymbol{q}_1 + c\boldsymbol{q}_4\,,\,\boldsymbol{q}_2$	$xy+c\,,\,x+y$
H2	$\boldsymbol{q}_2 + c\boldsymbol{q}_4$	$x+y+c$
H3(0)	$\boldsymbol{q}_1 + \boldsymbol{q}_4\,,\,\boldsymbol{q}_2$	$xy+1\,,\,x+y$
H3($\delta \neq 0$)	$\boldsymbol{q}_1 + c\boldsymbol{q}_4\,,\,\boldsymbol{q}_2$	$xy+c\,,\,x+y$

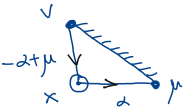
with $\sigma(\alpha) = -\alpha + \mu$

$\phi(x,y) = 0 \Rightarrow y = f(x)$.



$$q(\mu, x, v; \alpha) = Q(x, u, v, f(x); \alpha, \sigma(\alpha)) = 0$$

IMPORTANT CONSEQUENCE: "two-leg" form for $q=0$



$$\phi(x,\mu;\alpha) - \phi(x,v;\sigma(\alpha)) + \chi(x;\alpha) = \phi(x,u;\alpha) - \phi(x,v;\sigma(\alpha)) + \psi(x, f(y), \alpha - \sigma(\alpha))$$

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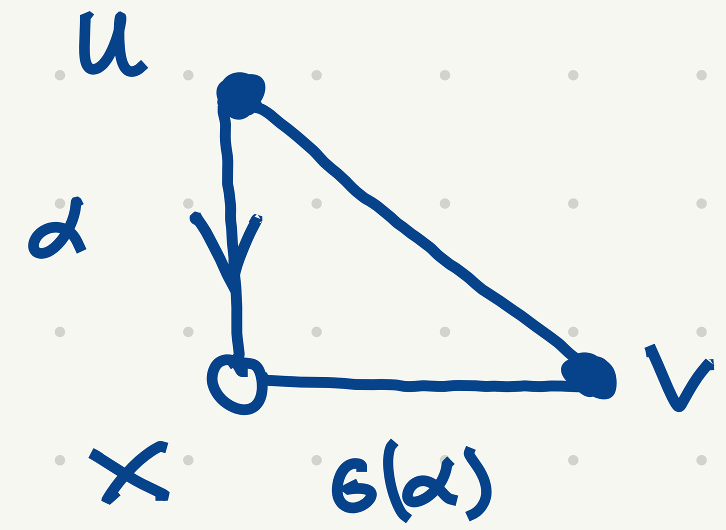
Boundary Lagrangian



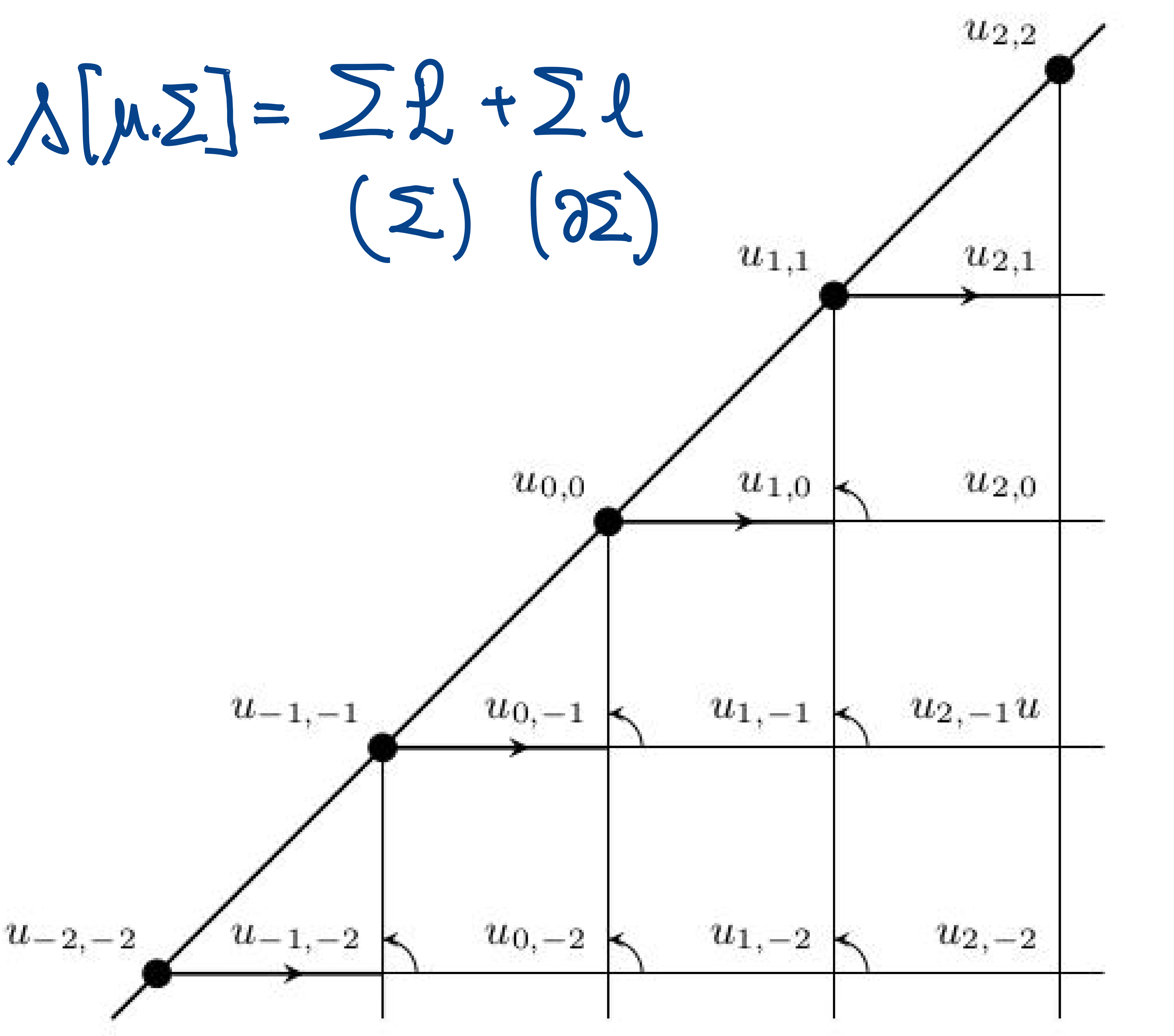
$$\ell := \ell(u,x;\alpha)$$
$$= C(u,x;\alpha) + H(x;\alpha)$$

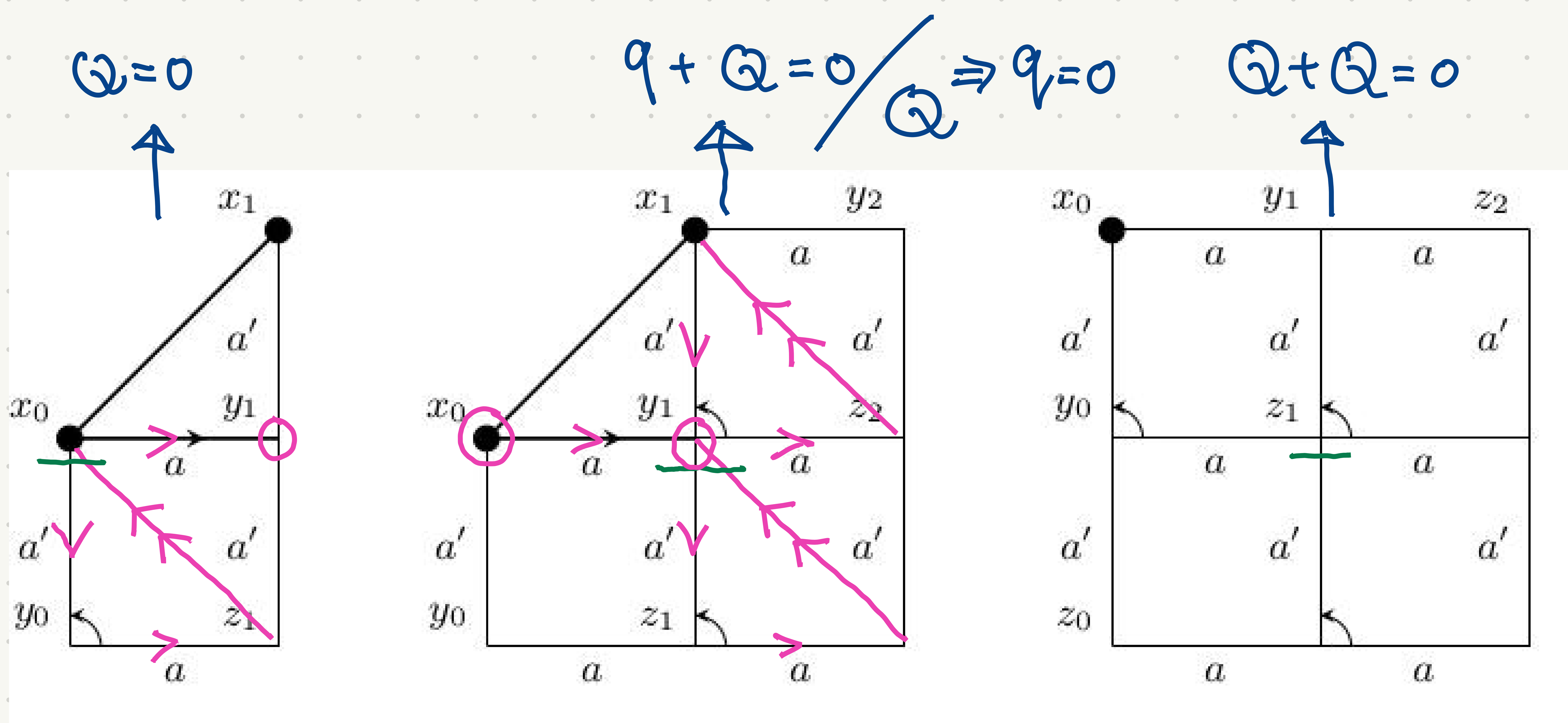
where $\frac{d}{d\mu} C(u,x;\alpha) = \Phi(u,x;\alpha)$

$\frac{d}{dx} H(x;\alpha) = \Lambda(x;\alpha)$.

$\delta S = 0 \Rightarrow$ A set of Euler-Lagrange equations:

$$S[u,\Sigma] = \sum_{(\Sigma)} \mathcal{L} + \sum_{(\partial\Sigma)} \ell$$



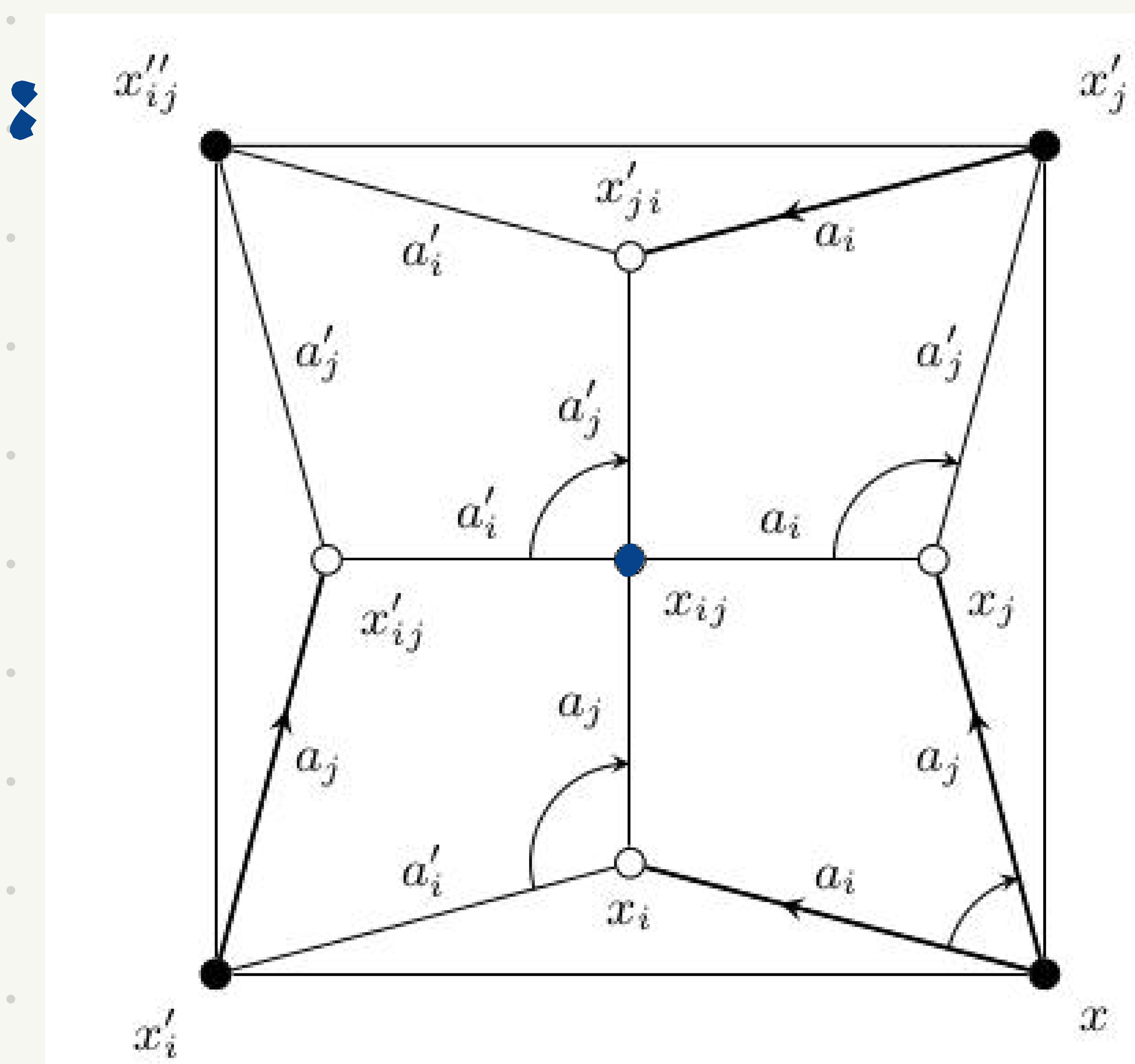


- When there is a "two-leg" representation of $q=0$ that is compatible with the "three-leg" representation of $Q=0$ (same "kinetic part")

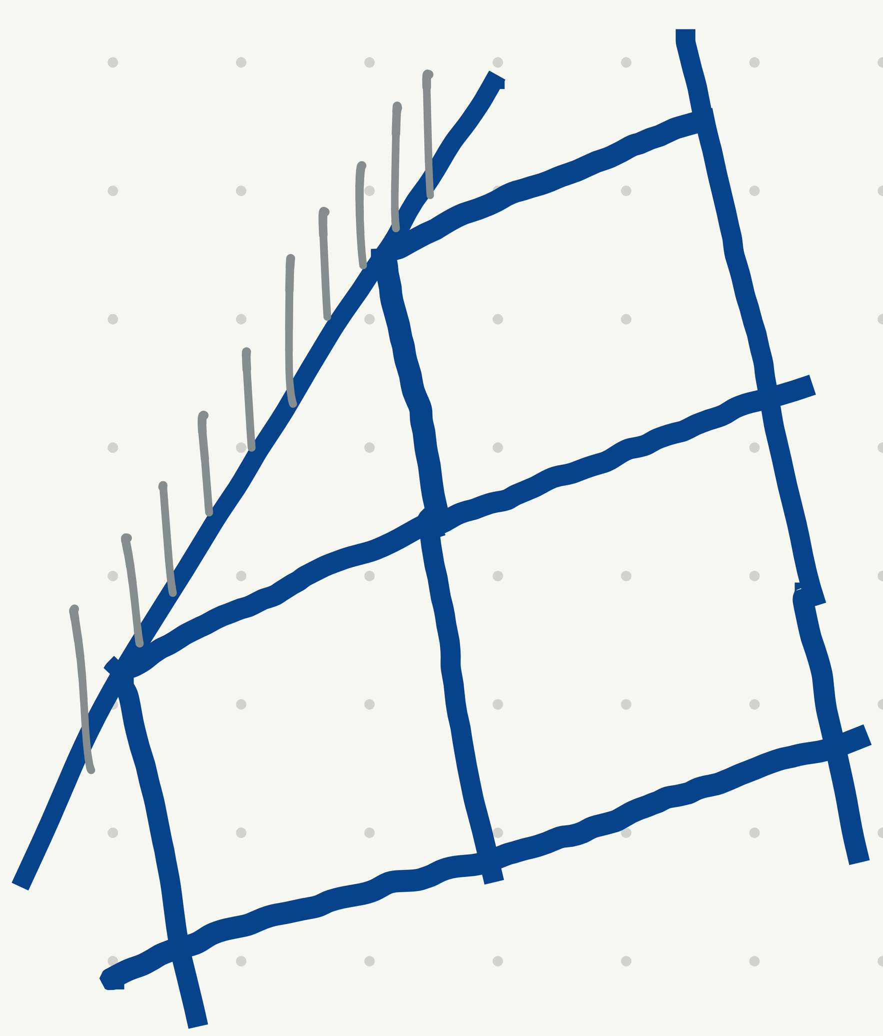
one gets a systematic way to compute the boundary Lagrangian ℓ.

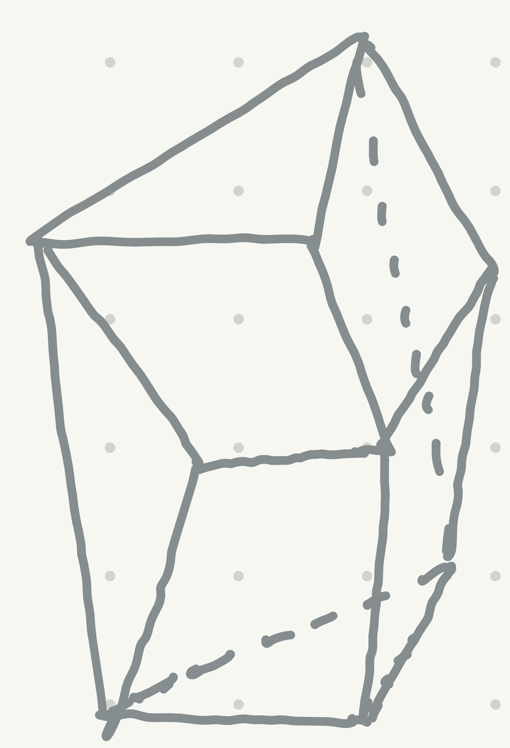
- The set of E-L eqs will produce both $Q=0$ and $q=0$.

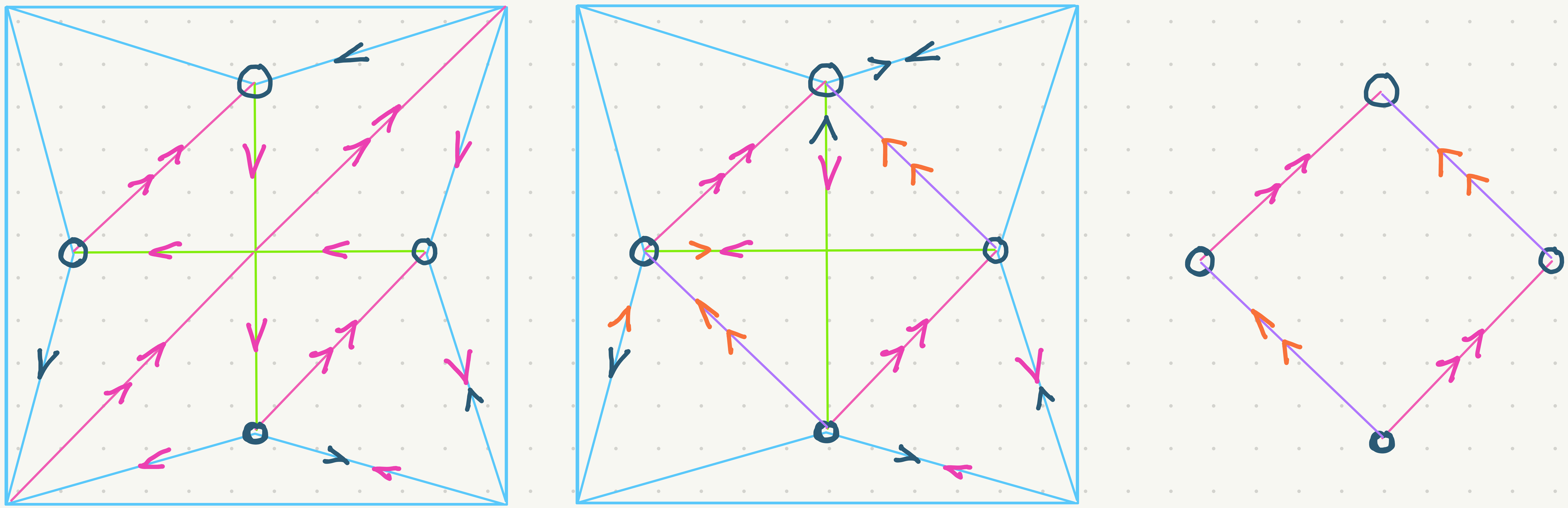
Boundary closure relation:



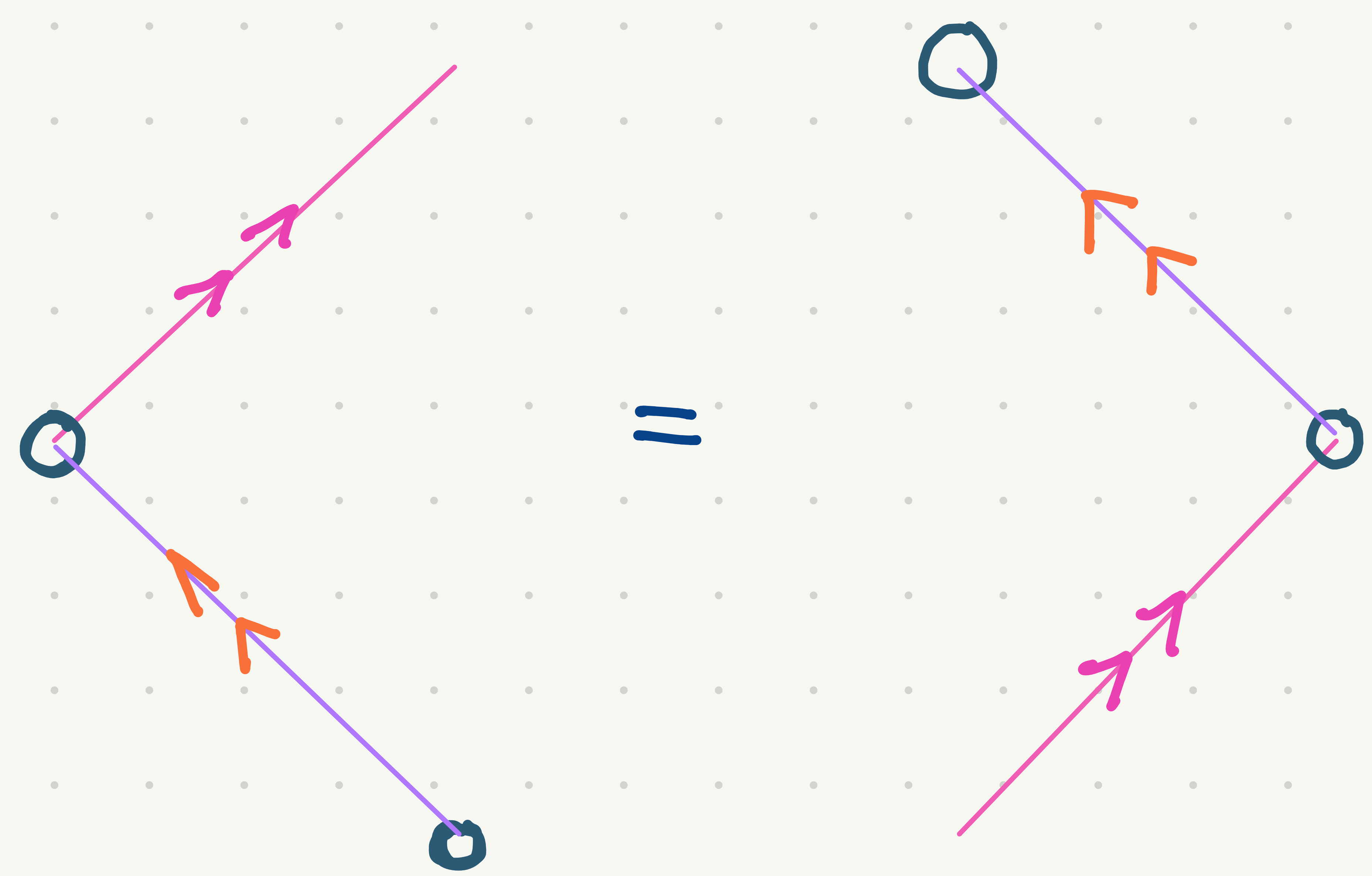
$$0 = S[x;\mathcal{D}] = \begin{aligned} &-\ell(x,x_i;a_i)+\ell(x,x_j;a_j)+\mathcal{L}(x,x_i,x_j;a_i,a_j)\\ &+\mathcal{L}(x_i,x'_i,x_{ij};a'_i,a_j)+\mathcal{L}(x_j,x_{ij},x'_j;a_i,a'_j)\\ &+\ell(x'_j,x'_{ji};a_i)-\ell(x'_i,x'_{ij};a_j)+\mathcal{L}(x_{ij},x'_{ij},x'_{ji};a'_i,a'_j)\end{aligned}.$$







or



(THIS MOVE IS A BOUNDARY VERSION OF THE STAR-TRIANGLE RELATION.

AN INTERRABLE BOUNDARY VARIATIONAL PRINCIPLE:

THE ABOVE SET OF E-L EQUATIONS ARE CONSEQUENCES OF THE CLOSURE RELATION

<u>IN PARTICULAR:</u> 1) GIVEN $\mathcal{L}$ and ℓ, ONLY THREE OF THEM ARE FUNDAMENTAL (THE REST CAN BE SEEN AS CONSEQUENCES)

2) THE SET OF E-L EQUATIONS BY VARYING x IS A CONSEQUENCES, HENCE THE BULK AND BOUNDARY EQs.

Outline

SITUATIONS WHERE YOU MIGHT NEED A BOUNDARY