# On the full Kostant-Toda lattice and the flag varieties

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# The Tri-diagonal Toda Lattice

The classical finite open Toda lattice is a Hamiltonian system with

$$H(p,q)=\frac{1}{2}\sum_{k=1}^{n}p_k^2+\sum_{k=1}^{n-1}e^{-(q_{k+1}-q_k)}$$

$$\begin{cases}\frac{dq_k}{dt}=\frac{\partial H}{\partial p_k}\\ \frac{dp_k}{dt}=-\frac{\partial H}{\partial q_k}\end{cases}\Rightarrow\begin{cases}\frac{dq_k}{dt}=p_k\\ \frac{dp_k}{dt}=-e^{-(q_{k+1}-q_k)}+e^{-(q_k-q_{k-1})}\end{cases}\tag{1}$$

where $k=1,\ldots,n$ and $e^{-(q_1-q_0)}=0=e^{-(q_{n+1}-q_n)}$ with $q_0=-\infty, q_{n+1}=\infty$.

# Flaschka's change of variable for Kostant-Toda lattice

Making the change of variables

$$\begin{cases} a_k = -p_k & k = 1, \dots, n \\ b_k = e^{-(q_{k+1} - q_k)} & k = 1, \dots, n-1. \end{cases}$$

we obtain the Kostant-Toda (KT) lattice from (1)

$$\frac{d}{dt} L(t) = [(L)_{\geq 0}, L] = [L, (L)_{<0}], \tag{2}$$

where

$$L = \begin{pmatrix} a_1 & 1 & & & \\ b_1 & a_2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & b_{n-2} & a_{n-1} & 1 \\ & & & b_{n-1} & a_n \end{pmatrix}, \ (L)_{\geq 0} = \begin{pmatrix} a_1 & 1 & & & \\ 0 & a_2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 0 & a_{n-1} & 1 \\ & & & 0 & a_n \end{pmatrix}$$

# Kostant-Toda lattice on simple Lie algebras

Some notations:
Lie algebra: $\mathfrak{g}, \mathfrak{h}, \mathfrak{n}_\pm, \mathfrak{b}_\pm, \Pi = (\alpha_1, \ldots, \alpha_l), \Delta_\pm, \omega_i, \{H_{\alpha_i}, X_\alpha, Y_\alpha\}$
Lie groups: $G, H, N_\pm, B_\pm, \mathcal{W}$
Let

$$\mathcal{J} = \{L = \sum_{j=1}^{l} X_{\alpha_j} + a_j H_{\alpha_j} + b_j Y_{\alpha_j} | a_j, b_j \in \mathbb{C}\}$$

be the Jacobi variety. Then the KT lattice is defined as:

$$\frac{d}{dt}L = [\Pi_{\mathfrak{b}_+}L, L] \Leftrightarrow \begin{cases} \frac{d}{dt}a_j = b_j \\ \frac{d}{dt}b_j = -b_j \sum_{k=1}^{l} C_{jk} a_k \end{cases}$$

where $(C_{ij})$ is the Cartan matrix. It is Hamiltonian with $H = \frac{1}{2}\mathrm{tr}L^2$.

# Integrability and Kostant's solution

The solution of Kostant-Toda lattice can be written down by the factorization method:

$$\exp(tL_0) = n_-(t)b_+(t), \qquad \text{where } n_-(t) \in N_- \text{ and } b_+(t) \in B_+,$$

where $L_0 = L(0)$, then

$$L(t) = \mathrm{Ad}(n_-(t)^{-1})(L_0) = \mathrm{Ad}(b_+(t))(L_0). \tag{3}$$

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Let $(\rho^{\omega_j}, V^{\omega_j}), 1 \leq j \leq l$ be the fundamental modules with highest weight vector $v^{\omega_j}$, and we define the $j$-th $\tau$-function as

$$\tau_j(t) = \langle v^{\omega_j}, \exp(tL_0) \cdot v^{\omega_j} \rangle \tag{4}$$

Then the solution is given by

$$\begin{aligned} a_j(t) &= \frac{d}{dt} \ln \tau_j(t) \\ b_j(t) &= b_j^\circ \prod_{k=1}^{l} \tau_k(t)^{-C_{jk}} = \frac{d^2}{dt^2} \ln \tau_j(t). \end{aligned}$$

# Toda lattice as coadjoint action of $B$

According to (3), Toda lattice should be viewed as the coadjoint action of a Borel subgroup $B_+ \subset G$ on $\mathfrak{b}_+^*$. For example, the classical QR and LU decompositions of $SL_n$ identify $\mathfrak{b}_+^*$ with the linear space of symmetric matrices and the affine space of lower Hessenberg matrices which correspond to the symmetric and the Kostant-Toda lattice, respectively. Now to study the generic coadjoint orbits we consider the so-called full Kostant-Toda (f-KT) lattice with Lax matrix $L_{\mathfrak{g}} = \sum_{i=1}^{l} X_i + \sum_{i=1}^{l} a_i(t) H_i + \sum_{\alpha \in \Delta_+} b_\alpha(t) Y_\alpha$, e.g. in type $A$

$$L = \begin{pmatrix} a_1 & 1 & 0 & \cdots & 0 \\ b_{2,1} & a_2 - a_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ b_{l,1} & b_{l,2} & \cdots & \cdots & 1 \\ b_{l+1,1} & b_{l+1,2} & \cdots & \cdots & -a_l \end{pmatrix}.$$

# The full Kostant-Toda (f-KT) lattice

The f-KT lattice can be defined as

$$\frac{d}{dt}L(t) = [\Pi_{\mathfrak{b}_+}L, L] = [L, \Pi_{\mathfrak{n}_-}L], \tag{5}$$

The factorization method still works, that is, let

$$\exp(tL_0) = n_-(t)b_+(t), \quad n_-(t) \in N_-, \; b_+(t) \in B_+, \tag{6}$$

then

$$L(t) = \mathrm{Ad}(n_-(t)^{-1})(L_0) = \mathrm{Ad}(b_+(t))(L_0).$$

Note that the Chevalley invariants are still constants of motion, but they do not provide enough first integrals for the integrability of f-KT lattice. The Toda flows stay on the iso-spectral variety defined as

$$\mathcal{F}_\Lambda := \{L \in \mathcal{H} | L \text{ has Chevalley invariants } \Lambda = (I_1, \ldots, I_l)\}.$$

# Integrability: The chopping method

As it turns out the Toda flows on generic coadjoint orbits in simple Lie algebras are still completely integrable. The additional constants of motion can be obtained from the chopping method.

**Theorem (DLNT, EFS, GS)**

*For $k = 0, \ldots, [(n-1)/2]$, where $n = l + 1$ in type A, denote by $(L - \lambda Id)_k$ the result of removing the first $k$ rows and the last $k$ columns from $(L - \lambda Id)$ (we call this the $k$-chop of $L$), and let $\lambda_{rk}, r = 1, \ldots, n-2k$, denote the roots of*

$$\tilde{Q}_k(L, \lambda) := \det(L - \lambda Id)_k = E_{0k}(\lambda^{n-2k} + I_{1k}\lambda^{n-2k-1} + \cdots + I_{n-2k,k}).$$

*Then $\lambda_{rk}$'s (equivalently $I_{rk}$'s) are constants of motion.*

# $\tau$-functions and their geometric structure

The $\tau$-functions can be similarly defined as

$$\tau_j(t) = \langle v^{\omega_j}, \exp(tL_0) \cdot v^{\omega_j} \rangle,$$

then

**Proposition**

*For $t$ small enough, we have*

$$a_i(t) = \frac{d}{dt} \ln \tau_i(t), \qquad 1 \leq i \leq l. \tag{7}$$

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Let $e = \sum_{i=1}^{l} X_i \in \mathfrak{g}$. According to a theorem of Kostant, there exists an $l$-dimensional linear subspace $\mathfrak{s} \subset \mathfrak{b}_-$ such that elements in the affine subspace $e + \mathfrak{s}$ are regular. The map

$$\begin{array}{rcl} N_- \times (e + \mathfrak{s}) & \to & e + \mathfrak{b}_- \\ (n, X) & \mapsto & \mathrm{Ad}_n X, \end{array}$$

is an isomorphism of affine varieties.

# Companion embedding and the flag varieties

Fix a choice of $\mathfrak{s}$, then for any $L \in \mathcal{F}_\Lambda$ we have $L = u^{-1} C_\Lambda u$, $C_\Lambda \in e + \mathfrak{s}$ and we use this $\mathfrak{s}$ to embed $\mathcal{F}_\Lambda$ into the flag variety $G/B_+$:

$$\begin{array}{rcl} c_\Lambda : \mathcal{F}_\Lambda & \rightarrow & G/B_+ \\ L & \mapsto & uB_+ . \end{array}$$

## Proposition

*Let $L_0 = u_0^{-1} C_\Lambda u_0$, then f-KT flow is linearized on the flag variety*

$$\begin{array}{ccl} L_0 & \xrightarrow{\quad c_\Lambda \quad} & u_0 B_+ \\ \Big\downarrow {\scriptstyle Ad_{u(t)^{-1}}} & & \quad\Big\downarrow \\ L(t) & \xrightarrow{c_\Lambda} & \left\{ \begin{array}{l} u_0 u(t) B^+ \\ = u_0 \exp(tL_0) B^+ \\ = \exp(tC_\Lambda) u_0 B_+ \end{array} \right. \end{array}$$

# Painlevé divisor and singular solutions

When there exists $t_* \in \mathbb{C}$ such that the LU factorization (6) could not be performed, we have

$$\exp(t_* L_0) = u_* \dot{w}_* b_*,$$

where $w_* \in \mathcal{W}$. This means that the f-KT flow hits the boundary of a Bruhat cell which happens when there exists $1 \le k \le l$ such that $\tau_k(t_*) = 0$ and the solution becomes singular at $t = t_*$. The set of $t_*$ where some of the $\tau_k$ vanish is called the Painlevé divisor. Setting $t \to t + t_*$, then

$$\tau_k(t; w_*) = \langle v^{\omega_k}, u_0^{-1} e^{tC_\Lambda} u_0 u_* \dot{w}_* b_* v^{\omega_k} \rangle.$$

Note that

$$b_* v^{\omega_k} = d_k v^{\omega_k},$$

where $d_k \in \mathbb{C}$ is a constant, we obtain

$$\tau_k(t; w_*) = d_k \langle v^{\omega_k}, e^{tC_\Lambda} u \dot{w}_* v^{\omega_k} \rangle \qquad 1 \le k \le l, \tag{8}$$

where $u = u_0 u_* \in N_-$.

# The main problem and the strategy

Note that $N_-B_+$ from Kostant's theorem only represents the big cell in the Bruhat decomposition of $G$, and we would like to know what kinds of singularities the f-KT flow may develop which is equivalent to the

**Problem**

*For which $u \in N_-$ and $w_* \in \mathcal{W}$, with $\tau$-functions defined by* (8)*, the diagonal elements given by* (7) *satisfy f-KT lattice?*

Our strategy is to perform a complete local analysis which is known as the Kowalevski-Painlevé analysis to obtain the singular information of all possible Laurent series solutions for f-KT lattice. Kowalevski-Painlevé analysis is a nonlinear version of the classical Frobenius method solving linear ordinary differential equations.

# The Kowalevski-Painlevé analysis

Assume solutions of (5) have the following form

$$a_i(t) = \sum_{k=0}^{\infty} a_{ik} t^{\delta_i + k} \quad 1 \leq i \leq \ell, \qquad b_\alpha(t) = \sum_{k=0}^{\infty} b_{\alpha k} t^{\gamma_\alpha + k} \quad \alpha \in \Delta_+.$$

Kowalevski-Painlevé analysis takes the following three steps:

(1) Identify the leading singularities and leading coefficients, i.e. $\delta_i, \gamma_\alpha$ and $a_{i0}, b_{\alpha 0}$.

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(2) Find the resonances. Substituting Laurent series in step (1) into (5) and find the coefficients recursively, the resonances are the the places where the iteration procedure to uniquely solve the higher order coefficients fail and new free parameters to be introduced.

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(3) Check the compatibility and convergence of the Laurent series. Check the compatibility conditions at the resonance levels and show that the resulting formal Laurent series have a positive convergent radius.

# Local analysis example: $\mathfrak{so}_5(\mathbb{C})$

Consider the Lax equation $\frac{dL}{dt} = [B, L]$ with
$L = X_2 + X_1 + a_2 H_2 + a_1 H_1 + b_2 Y_2 + b_1 Y_1 + c_1 Y_{\alpha_1+\alpha_2} + d_1 Y_{2\alpha_1+\alpha_2}$

$$L = \begin{pmatrix} a_2 & 1 & 0 & 0 & 0 \\ b_2 & 2a_1 - a_2 & 1 & 0 & 0 \\ 2c_1 & 2b_1 & 0 & 1 & 0 \\ 4d_1 & 0 & 2b_1 & a_2 - 2a_1 & 1 \\ 0 & 4d_1 & -2c_1 & b_2 & -a_2 \end{pmatrix} \quad \text{and} \quad B = (L)_{\geq 0}.$$

Explicitly, the differential equations read as:

$$\begin{aligned} &\frac{d}{dt}a_2 = b_2 \qquad \frac{d}{dt}a_1 = b_1 \\ &\frac{d}{dt}b_2 = (2a_1 - 2a_2)b_2 + 2c_1 \qquad \frac{d}{dt}b_1 = (a_2 - 2a_1)b_1 - c_1 \\ &\frac{d}{dt}c_1 = -a_2 c_1 + 2d_1 \qquad \frac{d}{dt}d_1 = -2a_1 d_1. \end{aligned} \tag{9}$$

# $\mathfrak{so}_5(\mathbb{C})$: the indicial equations

- The Laurent series solutions have the following form (can be proved by balancing the leading singularity):

$$a_i(t) = \sum_{k=0}^{\infty} a_{ik} t^{k-1}, \qquad b_i(t) = \sum_{k=0}^{\infty} b_{ik} t^{k-2},$$

$$c_1(t) = \sum_{k=0}^{\infty} c_{1k} t^{k-3}, \qquad d_1(t) = \sum_{k=0}^{\infty} d_{1k} t^{k-4}.$$

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- Plugging them into (9), and comparing the powers of $t$, we get the following indicial equations ($k = 0$):

$$\begin{aligned}
&-a_{20} = b_{20}, \qquad -a_{10} = b_{10}, \\
&-2b_{20} = (2a_{10} - 2a_{20})b_{20} + 2c_{10}, \\
&-2b_{10} = (a_{20} - 2a_{10})b_{10} - c_{10}, \\
&-3c_{10} = -a_{20}c_{10} + 2d_{10}, \qquad -4d_{10} = -2a_{10}d_{10}.
\end{aligned} \tag{10}$$

# $\mathfrak{so}_5(\mathbb{C})$: the indicial equations

- Eliminating $b_{i0}, c_{10}, d_{10}$ from equation (10), we obtain the following two equations for $a_{i0}$:

$$\begin{aligned} &a_{20}^2 - 2a_{20}a_{10} + 2a_{10}^2 - 2a_{10} - a_{20} = 0 \\ &a_{20}(a_{20} - 3)(a_{10} - 2)(1 + a_{10} - a_{20}) = 0 \end{aligned} \tag{11}$$

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$$a_{20}(a_{20}-3)(a_{10}-2)(1+a_{10}-a_{20}) = 0$$

- Remarkably, solutions of (11) can be read through Weyl group action on coroots (tridiagonal:Flaschka-Haine, Casian-Kodama noticed some special cases and connections): Let

$$\sum_{\check{\alpha}\in\Phi_w^+} \check{\alpha} = \sum_{j=1}^{n} \check{q}_j\check{\alpha}_j,$$

then $a_{j0} = \check{q}_j$ gives a solution to (11).

# $\mathfrak{so}_5(\mathbb{C})$: solutions to indicial equations

More explicitly, in the current case ($\mathfrak{so}_5(\mathbb{C})$), all the solutions are given by

$$\sum_{\check{\alpha}\in\Phi^+_e}\check{\alpha}=0;\qquad \sum_{\check{\alpha}\in\Phi^+_{s_1^B}}\check{\alpha}=\check{\alpha}_1;$$

$$\sum_{\check{\alpha}\in\Phi^+_{s_2^B}}\check{\alpha}=\check{\alpha}_2;\qquad \sum_{\check{\alpha}\in\Phi^+_{w_{21}}}\check{\alpha}=\check{\alpha}_2+2\check{\alpha}_1;$$

$$\sum_{\check{\alpha}\in\Phi^+_{w_{12}}}\check{\alpha}=3\check{\alpha}_2+\check{\alpha}_1;\qquad \sum_{\check{\alpha}\in\Phi^+_{w_{121}}}\check{\alpha}=3\check{\alpha}_2+3\check{\alpha}_1;$$

$$\sum_{\check{\alpha}\in\Phi^+_{w_{212}}}\check{\alpha}=4\check{\alpha}_2+2\check{\alpha}_1;\qquad \sum_{\check{\alpha}\in\Phi^+_{w_0}}\check{\alpha}=4\check{\alpha}_2+3\check{\alpha}_1.$$

# $\mathfrak{so}_5(\mathbb{C})$: the higher order terms

For $k \geq 1$, $a_{ik}, b_{\alpha k}$ can be obtained from the following iterative procedure:

$$(k-1)a_{2k} - b_{2k} = 0, \qquad (k-1)a_{1k} - b_{1k} = 0,$$

$$(k-2-2a_{10}+2a_{20})b_{2k} + 2b_{20}a_{2k} - 2b_{20}a_{1k} - 2c_{1k} = 2\sum_{i=1}^{k-1} b_{2i}(a_{1,k-i} - a_{2,k-i}),$$

$$(k-2-a_{20}+2a_{10})b_{1k} - b_{10}a_{2k} + 2b_{10}a_{1k} + c_{1k} = \sum_{i=1}^{k-1} b_{1i}(a_{2,k-i} - 2a_{1,k-i}),$$

$$(k-3+a_{20})c_{1k} + c_{10}a_{2k} - 2d_{1k} = \sum_{i=1}^{k-1} a_{2i}c_{1,k-i},$$

$$(k-4+2a_{10})d_{1k} + 2d_{10}a_{1k} = -2\sum_{i=1}^{k-1} a_{1i}d_{1,k-i}.$$

# $\mathfrak{so}_5(\mathbb{C})$: Kowalevski matrix

This system can be put in the form

$$(k\mathrm{Id} - \mathcal{K})\begin{pmatrix} a_{1k} \\ \vdots \\ a_{lk} \\ b_{\alpha_1 k} \\ \vdots \end{pmatrix} = \vec{R}(a_{1i}, \cdots, a_{li}, b_{\alpha_1, i}, \cdots) \quad \text{with } i < k.$$

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$$\begin{aligned}\det(k\mathrm{Id}-\mathcal{K}) =&(k-2)(k-4)(k-1+2a_{20}-2a_{10})\\ &(k-1-a_{20}+2a_{10})(k-2+a_{20})(k-3+2a_{10}),\end{aligned}$$

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$$\begin{aligned} \det(k\mathrm{Id} - \mathcal{K}) =&(k-2)(k-4)(k-1+2a_{20}-2a_{10}) \\ &(k-1-a_{20}+2a_{10})(k-2+a_{20})(k-3+2a_{10}), \end{aligned}$$

Note that $2, 4$: degree of Chevalley invariants;
$E_{\alpha_2} = 1 - 2a_{20} + 2a_{10}$, $E_{\alpha_1} = 1 + a_{20} - 2a_{10}$, $E_{\alpha_3} = 2 - a_{20}$,
$E_{\alpha_4} = 3 - 2a_{10}$: type “X" eigenvalues (here $\alpha_3 = \alpha_1 + \alpha_2$ and $\alpha_4 = 2\alpha_1 + \alpha_2$).

# $\mathfrak{so}_5(\mathbb{C})$: type X eigenvalues

| Level.case | $E_{\alpha_2}$ | $E_{\alpha_1}$ | $E_{\alpha_3}$ | $E_{\alpha_4}$ | $\mathfrak{W}$ | $\ell(w)$ | # of parameters |
|---|---|---|---|---|---|---|---|
| 0.1 | 1 | 1 | 2 | 3 | $e$ | 0 | 6 |
| 1.1 | 3 | $-1$ | 2 | 1 | $s_1$ | 1 | 5 |
| 1.2 | $-1$ | 2 | 1 | 3 | $s_2$ | 1 | 5 |
| 2.1 | 3 | $-2$ | 1 | $-1$ | $s_2 s_1$ | 2 | 4 |
| 2.2 | $-3$ | 2 | $-1$ | 1 | $s_1 s_2$ | 2 | 4 |
| 3.1 | 1 | $-2$ | $-1$ | $-3$ | $s_1 s_2 s_1$ | 3 | 3 |
| 3.2 | $-3$ | 1 | $-2$ | $-1$ | $s_2 s_1 s_2$ | 3 | 3 |
| 4.1 | $-1$ | $-1$ | $-2$ | $-3$ | $s_2 s_1 s_2 s_1$ | 4 | 2 |

The number of free parameters are exactly $l + \ell(w_0) - \ell(w)$, where $\ell(w_0) - \ell(w)$ is the dimension of the Schubert cell.

# Local analysis for f-KT equation: general setting

The above formula for solutions of indicial equations of Toda lattice are true in all Lie algebras.

- The indicial equations ($k = 0$) have $|\mathcal{W}|$ many solutions, and they can be obtained from (R.J. Marsh and K. Rietsch)

$$\sum_{\check{\alpha}\in\Phi_w^+} \check{\alpha} = \sum_{j=1}^{n} \check{q}_j\check{\alpha}_j, \quad \check{q}_j \in \mathbb{N}_0, \text{ or } a_{i0}^w = \sum_{\check{\alpha}\in\Phi_w^-} \langle \omega_i, w^{-1}\check{\alpha}\rangle$$

# Kowalevski-Painlevé analysis: eigenvalues of $\mathcal{K}$

The higher order terms are recursively determined by

$$(k\mathrm{Id} - \mathcal{K}_w)\begin{pmatrix} a_{1k} \\ \vdots \\ a_{lk} \\ b_{\alpha_1 k} \\ \vdots \end{pmatrix} = \vec{R}(a_{1i}, \cdots, a_{li}, b_{\alpha_1, i}, \cdots) \quad \text{with } i < k.$$

- The eigenvalues of $\mathcal{K}_w$: $l$ of them are degrees of Chevalley invariants, the others are given by

$$E_\alpha^w = L(w\alpha) \tag{12}$$

where $L(\alpha)$ is the height of root $\alpha \in \Delta_+$.

# Kowalevski-Painlevé analysis: moduli

- All the compatibility conditions for Laurent series solutions of f-KT lattice at the resonant levels are automatically satisfied and the Laurent series solutions are convergent (majorant method).

# Kowalevski-Painlevé analysis: moduli

- All the compatibility conditions for Laurent series solutions of f-KT lattice at the resonant levels are automatically satisfied and the Laurent series solutions are convergent (majorant method).
- The number of non-trivial free parameters in Laurent series solution corresponding to $w \in \mathcal{W}$ equals the dimension of the corresponding Bruhat cell.

### Theorem

*For any $\Lambda \in \mathbb{C}^l$, the compactification of $c_\Lambda(F_\Lambda)$ is $G/B_+$. All the Laurent series solutions of f-KT lattice are parameterized by $G/B_+ \times \mathbb{C}^l$, where $G/B_+$ is the flag variety and $\mathbb{C}^l$ parametrizes the data for spectral parameters.*

Thank you!