# Liouville correspondences between the integrable systems and their dual integrable systems

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(joint work with Jing Kang, Xiaochuan Liu and Peter J. Olver)

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- The KdV equation

$$Q_\tau + Q_{yyy} + 6\,QQ_y = 0$$

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- The CH equation

$$m_t + 2u_x m + um_x = 0, \qquad m = u - u_{xx}$$

(Fokas, Fuchssteiner, 1981; Camassa-Holm, 1993)

**The modified KdV (mKdV) hierarchy**

- The (focusing) mKdV equation

$$Q_\tau + Q_{yyy} - 6\,Q^2 Q_y = 0$$

**The modified Camassa-Holm (mCH) hierarchy**

- The mCH equation

$$m_t + \left((u^2 - u_x^2)\,m\right)_x = 0, \qquad m = u - u_{xx}$$

(Fokas, 1995; Olver, Rosenau, 1996; Fuchssteiner, 1996)

**The Novikov hierarchy**

- The Novikov equation

$$m_t = 3uu_x m + u^2 m_x, \qquad m = u - u_{xx}$$

(Novikov, 1999)

**The Sawada-Kotera (SK) hierarchy**

- The SK equation

$$Q_\tau + Q_{yyyyy} + 5(QQ_{yy})_y + 5Q^2 Q_y = 0$$

(Sawada, Kotera, 1974)

**The Kaup-Kupershmidt (KK) hierarchy**

- The KK equation

$$P_\tau + P_{yyyyy} + 20PP_{yyy} + 50P_yP_{yy} + 80P^2P_y = 0$$

**The Degasperis-Procesi (DP) hierarchy**

- The DP equation

$$n_t = 3v_xn + vn_x, \qquad n = v - v_{xx}$$

**The 2CH hierarchy**

- The 2CH system

$$\begin{aligned} & m_t + 2u_x m + u m_x + \rho\rho_x = 0, \qquad m = u - u_{xx}, \\ & \rho_t + (\rho u)_x = 0, \end{aligned} \tag{1}$$

(Olver, Rosenau, 1996)

**The two-component integrable hierarchy**

- The A2CH system

$$\begin{aligned} & P_\tau(\tau, y) = \rho_y, \qquad Q_\tau(\tau, y) = \frac{1}{2}\rho P_y(\tau, y) + \rho_y P(\tau, y), \\ & \rho_{yyy} + 2\rho_y Q(\tau, y) + 2(\rho Q(\tau, y))_y = 0. \end{aligned} \tag{2}$$

**The Geng-Xue hierarchy**

- The Geng-Xue system

$$\begin{aligned} &m_t + 3vu_x m + uvm_x = 0, \quad m = u - u_{xx}, \\ &n_t + 3uv_x n + uvn_x = 0, \quad n = v - v_{xx}. \end{aligned} \tag{3}$$

(Geng, Xue, 2009)

**The dDWW hierarchy**

- The dDWW system

$$\begin{aligned} &\rho_t = ((\rho + v)\, u)_x\,, \quad \rho = v - v_x, \\ &\gamma_t = (\gamma\, u + 2v)_x\,, \quad \gamma = u + u_x, \end{aligned} \tag{4}$$

(Kang, Liu, Olver, Qu, 2020)

# Introduction

**The hierarchy of $1+n$-KdV system**

- The $1+n$-KdV system

$$w_t = w_{xxx} + \frac{3}{2}\left(w^2 + \langle \mathbf{u}, \mathbf{u}\rangle\right)_x, \qquad (5)$$
$$\mathbf{u}_t = \mathbf{u}_{xxx} + 3\left(w\mathbf{u}\right)_x.$$

**The hierarchy of $(1+n)$-component CH system**

- The $(1+n)$-component CH system

$$\rho_t + 2w_x\rho + w\rho_x + \langle \mathbf{u}, \mathbf{m}\rangle_x + \langle \mathbf{u}_x, \mathbf{m}\rangle = 0,$$
$$\mathbf{m}_t + 2w_x\mathbf{m} + w\mathbf{m}_x + 2\rho\mathbf{u}_x + \rho_x\mathbf{u} + \Pi(\mathbf{u}, \mathbf{u}_x)\mathbf{m} = 0, \qquad (6)$$
$$\rho = w - w_{xx}, \quad \mathbf{m} = \mathbf{u} - \mathbf{u}_{xx}.$$

(Kang, Liu, Qu, 2022)

**The Camassa-Holm (CH) type equations ( CH, mCH, etc. . . . )**

- Support nonlinear dispersion
- Describe wave-breaking phenomena for appropriate initial data
- Possess a notable variety of non-smooth soliton-like solutions
  - peakon, multi-peakon, compacton solutions, . . .

**The mCH equation**

- Physical background
- Geometric derivation
- Cubic nonlinearity
- New features: wave breaking and multi-peakon dynamics

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**Some results on the mCH equation**

- Derivation of mCH (Fokas 1995; Fuchsseteiner, 1996; Olver, Rosenau, 1996)
- Integrability of mCH (Olver, Rosenau, 1996; Schiff, 1996; Qiao, 2006; Hone, Wang, 2008; Maruno, 2013; Chang, Szmigielski, 2016; Xia, Zhou, Qiao, 2016; Wang, Liu, Mao, 2020)
- Well-posedness of solutions to Cauchy problem (Gui, Liu, Olver, Qu, 2013)
- Wave breaking phenomena (Gui, Liu, Olver, Qu, 2013; Chen, Liu, Qu, Zhang, 2015-2017)
- Stability of single peakons and periodic peakons (Qu, Liu, Liu, 2013, 2014)
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**Geometric formulation of the mCH equations**

Consider the Euclidean-invariant plane curve flow for $C \subseteq \mathbb{R}^2$

$$\frac{\partial C}{\partial t} = f\,\mathbf{n} + g\,\mathbf{t}, \tag{7}$$

where **t** and **n** are the Euclidean tangent and normal vectors, while the normal and tangent velocities, $f$ and $g$, are arbitrary Euclidean differential invariants, meaning that they depend on the curvature and its derivatives with respect to the arc-length $s$ of the curve $C$. If the flow is intrinsic, meaning that it preserves arc length, if and only if

$$g_s - \kappa f = 0.$$

The curvature invariant satisfies

$$\kappa_t = \mathfrak{R}[f], \quad \text{where} \quad \mathfrak{R} = \partial_s^2 + \kappa^2 + \kappa_s \partial_s^{-1} \kappa$$

is the recursion operator of the mKdV equation

$$\kappa_t = \kappa_{sss} + \frac{3}{2}\kappa^2\kappa_s,$$

which is equivalent to the mKdV flow with $f = \kappa_s$, $g = \frac{1}{2}\kappa^2$ (Goldstein, Petrich, 1992).

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which is equivalent to the mKdV flow with $f = \kappa_s$, $g = \frac{1}{2}\kappa^2$ (Goldstein, Petrich, 1992).

In particular, if we set $f = -2u_s$, $\kappa = m \equiv u - u_{ss}$, then

$$g = -(u^2 - u_s^2) + b,$$

where $b$ is a constant. Therefore, $u(t, s)$ satisfies the equation

$$m_t + \left((u^2 - u_s^2)m\right)_s + (b+2)u_{sss} - bu_s = 0.$$

Setting $x = s + (b+2)t$, it becomes

$$m_t + \left((u^2 - u_x^2)m\right)_x + 2u_x = 0, \quad m = u - u_{xx},$$

which is equivalent, up to rescaling, to the mCH equation. The preceding derivation implies that the mCH equation can be regarded as a Euclidean-invariant version of the CH equation, just as the mKdV equation is a Euclidean-invariant counterpart to the KdV equation from the viewpoint of curve flows in Klein geometries.
(Gui, Liu, Olver, Qu, 2013)

# Introduction

**Tri-Hamiltonian duality method**

- Olver, Rosenau (1996); Fuchssteiner (1996)
- **KEY Issue:**

  The most bi-Hamiltonian integrable soliton equations actually support a compatible trio of Hamiltonian structures through a particular scaling argument.
- Several CH-type equations were obtained from the classical integrable equationns (Olver, Rosenau, 1996)
  - the **KdV** equation $\longleftrightarrow$ the **CH** equation
  - the **mKdV** equation $\longleftrightarrow$ the **mCH** equation
  - the **Ito** equation $\longleftrightarrow$ the two-component **CH** equation
  - the **Schrödinger** equation $\longleftrightarrow$ the Fokas-Lenells equation

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**Motivation**

**It is anticipated that the original soliton equations should be related to their dual counterparts in a certain manner.**

- (Fokas, Fuchssteiner, 1981; Fuchssteiner, 1996):
  - The CH equation $\longleftrightarrow$ The first negative flow of the KdV hierarchy
- The link between the shallow water integrable systems and the negative flows of the classical soliton hierarchies by the Reciprocal-type transformations
  - Two-component Camassa-Holm system $\longleftrightarrow$ The first negative flow of the AKNS hierarchy
  - The Degasperis-Procesi equation $\longleftrightarrow$ a negative flow in the Kaup-Kupershmidt hierarchy
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**Questions**

- Is it possible to establish the correspondence between their respective hierarchies?
- Is it possible to relate the conservation laws between their respective hierarchies?
- Is there generalized Miura transformation relating CH and mCH equations and their hierarchies?
- Is there generalized Miura transformation relating DP and Novikov equations and their hierarchies?
- How about the multi-component integrable systems?

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- (McKean, 2003; Lenells, 2004):
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- **Key ingredients:**
  - The tri-Hamiltonian dual structure of the constituent Hamiltonian operators
  - The relationship between the corresponding Hamiltonian operators under the Liouville transformations

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# The correspondence between the mCH and the mKdV hierarchies

**A Liouville transformation between the isospectral problems of the mCH and the mKdV equations**

- The mCH equation

$$m_t + \left((u^2 - u_x^2)\, m\right)_x = 0, \qquad m = u - u_{xx} \tag{8}$$

- **The isospectral problems** (Schiff, 1996; Qiao, 2006):

$$\boldsymbol{\Psi}_x = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2}\lambda m \\ -\frac{1}{2}\lambda m & \frac{1}{2} \end{pmatrix} \boldsymbol{\Psi}, \qquad \boldsymbol{\Psi} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{9}$$

$$\boldsymbol{\Psi}_t = \begin{pmatrix} \lambda^{-2} + \frac{1}{2}(u^2 - u_x^2) & -\lambda^{-1}(u - u_x) - \frac{1}{2}\lambda m(u^2 - u_x^2) \\ \lambda^{-1}(u + u_x) + \frac{1}{2}\lambda m(u^2 - u_x^2) & -\lambda^{-2} - \frac{1}{2}(u^2 - u_x^2) \end{pmatrix} \boldsymbol{\Psi}$$

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$$Q_\tau + Q_{yyy} - 6\,Q^2 Q_y = 0 \tag{10}$$

- **The isospectral problems:**

$$\mathbf{\Phi}_y = \begin{pmatrix} -\mu & Q \\ -Q & \mu \end{pmatrix} \mathbf{\Phi}, \qquad \mathbf{\Phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \tag{11}$$

$$\mathbf{\Phi}_\tau = \begin{pmatrix} -4\mu^3 - 2\mu Q^2 & 4\mu^2 Q + 2Q^3 - 2\mu Q_y + Q_{yy} \\ -4\mu^2 Q - 2Q^3 - 2\mu Q_y - Q_{yy} & 4\mu^3 + 2\mu Q^2 \end{pmatrix} \mathbf{\Phi}$$

- $\partial_\tau(\mathbf{\Phi}_y) = \partial_y(\mathbf{\Phi}_\tau) \quad \Rightarrow \quad$ the mKdV equation (10)

# The correspondence between the mCH and mKdV hierarchies

**A Liouville transformation between the isospectral problems of the mCH and mKdV equations**

- The Liouville transformation (Kang, Liu, Olver, Qu, 2016)

$$\mathbf{\Phi} = \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} \mathbf{\Psi}, \qquad y = \int^x m(\xi)\, \mathrm{d}\xi \tag{12}$$

  will convert the isospectral problem (9) into the isospectral problem (11), with

$$Q = \frac{1}{2m} \qquad \text{and} \qquad \lambda = -2\mu.$$

- The following coordinate transformations

$$y = \int^x m(t,\xi)\, \mathrm{d}\xi, \qquad \tau = t, \quad Q(\tau, y) = \frac{1}{2m(t,x)}. \tag{13}$$

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# The correspondence between the mCH and mKdV hierarchies

### The mCH hierarchy

- The mCH equation written in the **bi-Hamiltonian** form (Olver, Rosenau, 1996)

$$m_t = \mathcal{K}\frac{\delta \mathcal{H}_1}{\delta m} = \mathcal{J}\frac{\delta \mathcal{H}_2}{\delta m}, \quad m = u - u_{xx} \tag{14}$$

  - ◇ A pair of compatible Hamiltonian operators

$$\mathcal{K} = -\partial_x \, m \, \partial_x^{-1} \, m \, \partial_x \qquad \text{and} \qquad \mathcal{J} = -\left(\partial_x - \partial_x^3\right)$$

  - ◇ The corresponding Hamiltonian functionals

$$\mathcal{H}_1[m] = \int \left(u^2 + u_x^2\right) \mathrm{d}x, \quad \mathcal{H}_2[m] = \frac{1}{4}\int \left(u^4 + 2u^2u_x^2 - \frac{1}{3}u_x^4\right) \mathrm{d}x \tag{15}$$

- Recursion operator : $\mathcal{R} = \mathcal{K}\mathcal{J}^{-1}$

# The correspondence between the mCH and mKdV hierarchies

**The mCH hierarchy**

- **The positive flows**

$$m_t = K_n = \mathcal{K}\frac{\delta \mathcal{H}_{n-1}}{\delta m} = \mathcal{J}\frac{\delta \mathcal{H}_n}{\delta m} = \left(\mathcal{K}\mathcal{J}^{-1}\right)^{n-1}(-2\, m_x), \qquad n = 1, 2, \dots \tag{16}$$

  - ◇ The seed equation: $m_t = K_1[m] = -2m_x, \qquad \text{with} \qquad \mathcal{H}_0[m] = \int m\,\mathrm{d}x$
  - ◇ The mCH equation: $m_t = K_2 = -\left((u^2 - u_x^2)\, m\right)_x = \mathcal{R}K_1[m]$

- **The negative flows**

$$m_t = K_{-n} = \mathcal{K}\frac{\delta \mathcal{H}_{-(n+1)}}{\delta m} = \mathcal{J}\frac{\delta \mathcal{H}_{-n}}{\delta m} = -\left(\mathcal{J}\mathcal{K}^{-1}\right)^{n-1}\mathcal{J}\frac{1}{m^2}, \qquad n = 1, 2, \dots \tag{17}$$

  - ◇ **The Casimir equation:**

$$m_t = K_{-1} = \mathcal{J}\frac{\delta \mathcal{H}_{-1}}{\delta m} = \mathcal{J}\frac{\delta \mathcal{H}_C}{\delta m} = \left(\frac{1}{m^2}\right)_x - \left(\frac{1}{m^2}\right)_{xxx} \tag{18}$$

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# The correspondence between the mCH and mKdV hierarchies

**The mKdV hierarchy**

- The positive flows

$$Q_\tau = \bar{K}_n = \bar{\mathcal{K}}\,\frac{\delta\bar{\mathcal{H}}_{n-1}}{\delta Q} = \bar{\mathcal{J}}\,\frac{\delta\bar{\mathcal{H}}_n}{\delta Q} = -\left(\bar{\mathcal{K}}\bar{\mathcal{J}}^{-1}\right)^{n-1}(4\,Q_y), \qquad n = 1, 2, \ldots. \tag{19}$$

◇ A pair of compatible Hamiltonian operators:

$$\bar{\mathcal{K}} = \frac{1}{4}\partial_y^3 - \partial_y Q \partial_y^{-1} Q \partial_y, \qquad \bar{\mathcal{J}} = \partial_y$$

◇ Recursion operator : $\bar{\mathcal{R}} = \bar{\mathcal{K}}\,\bar{\mathcal{J}}^{-1}$

- The negative flows

$$Q_\tau = \bar{K}_{-n} = \bar{\mathcal{K}}\,\frac{\delta\bar{\mathcal{H}}_{-(n+1)}}{\delta Q} = \bar{\mathcal{J}}\,\frac{\delta\bar{\mathcal{H}}_{-n}}{\delta Q} \iff \bar{\mathcal{R}}^n Q_\tau = 0, \qquad n = 1, 2, \ldots. \tag{20}$$

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**REMARK on the negative flows of the mKdV hierarchy**

$$\left(\bar{\mathcal{K}}\bar{\mathcal{J}}^{-1}\right)^{n} Q_{\tau}=0 \quad \Longrightarrow \quad \left(\frac{1}{4}\partial_y - Q\partial_y^{-1}Q\right)\left(\bar{\mathcal{K}}\bar{\mathcal{J}}^{-1}\right)^{n-1} Q_{\tau}=\bar{C}_{-n}, \qquad n=1,2,\ldots \tag{21}$$

**Case 1.** $\bar{C}_{-n}=0, \qquad n=1,2,\ldots$

- $n=1$

$$Q_{\tau}=\left(\frac{1}{4}\partial_y - Q\,\partial_y^{-1}Q\right)^{-1}0=\sin(2\,\partial_y^{-1}Q) \tag{22}$$

- ◇ **The sine-Gordon equation:** $U_{y\tau}=\sin(2\,U), \quad (U=\partial_y^{-1}Q)$
- ◇ The corresponding Casimir functional

$$\bar{\mathcal{H}}_S=-\frac{1}{2}\int\cos(2\,\partial_y^{-1}Q)\,dy, \qquad \frac{\delta\bar{\mathcal{H}}_S}{\delta Q}=-\partial_y^{-1}\sin(2\,\partial_y^{-1}Q) \tag{23}$$

# The correspondence between the mCH and mKdV hierarchies

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# The correspondence between the mCH and mKdV hierarchies

**REMARK for the negative flows of the mKdV hierarchy**

**Case 1.** $\bar{C}_{-n} = 0, \qquad n = 1, 2, \dots$

- $n \geq 1$

$$Q_\tau = \left(\bar{\mathcal{J}}\bar{\mathcal{K}}^{-1}\right)^{n-1} \sin(2\,\partial_y^{-1} Q), \qquad n = 1, 2, \dots \tag{24}$$

◇ $\widetilde{R}^{n-1} U_\tau = \sin(2\,U), \quad (U = \partial_y^{-1} Q) \qquad n = 1, 2, \dots$

◇ $\widetilde{R} = \frac{1}{4}\partial_y^2 - U_y^2 + U_y \partial_y^{-1} U_{yy}$

$---$ the recursion operator of the sine-Gordon equation

◇ $U_\tau + \widetilde{R}^{n-1}(4\,U_y) = 0, \quad \text{for} \quad n = 1, 2, \dots$

$---$ the positive flows in the potential mKdV hierarchy

- $n = 2$, **the potential mKdV equation:** $U_\tau + U_{yyy} + 2\,U_y^3 = 0$

**Case 2.** $\bar{C}_{-n} \neq 0, \qquad n = 1, 2, \dots$

$$\left(\frac{1}{4}\partial_y - Q\partial_y^{-1} Q\right)\left(\bar{\mathcal{K}}\bar{\mathcal{J}}^{-1}\right)^{n-1} Q_\tau = \bar{C}_{-n}, \quad \bar{C}_{-n} \neq 0, \qquad n = 1, 2, \dots \tag{25}$$

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# The correspondence between the mCH and mKdV hierarchies

**Theorem**

*Under the transformations*

$$Q(\tau, y) = \frac{1}{2m(t,x)}, \quad y = \int^{x} m(t,\xi)\,\mathrm{d}\xi, \quad \tau = t, \tag{26}$$

*for each* $l \in \mathbb{Z}$*, the* $(\mathrm{mCH})_{l+1}$ *equation is related to the* $(\mathrm{mKdV})_{-l}$ *equation. More precisely, for each integer* $n \geq 0$, **(i).** *m solves the equation*

$$m_t + \left(\mathcal{K}\mathcal{J}^{-1}\right)^n (2\, m_x) = 0, \qquad n = 0, 1, \ldots . \tag{27}$$

*if and only if Q satisfies* $Q_\tau = 0$ *for* $n = 0$ *or*

$$\left(\frac{1}{4}\partial_y - Q\partial_y^{-1} Q\right)\left(\bar{\mathcal{K}}\bar{\mathcal{J}}^{-1}\right)^{n-1} Q_\tau = \bar{C}_{-n}, \qquad \bar{C}_{-n} = 1/(-4)^n, \qquad n = 1, 2, \ldots ; \tag{28}$$

# The correspondence between the mCH and mKdV hierarchies

**Theorem**

**(Continued)**

**(ii).** *For $n \geq 1$, $m$ is a solution of the following rescaled version of* (17),

$$m_t = K_{-n} = \frac{(-1)^{n+1}}{2^{2n-1}} \left(\mathcal{J}\mathcal{K}^{-1}\right)^{n-1} \mathcal{J} \frac{1}{m^2}, \quad n = 1, 2, \ldots, \tag{29}$$

*if and only if $Q$ satisfies the equation*

$$Q_\tau + \left(\bar{\mathcal{K}}\bar{\mathcal{J}}^{-1}\right)^n (4\, Q_y) = 0, \qquad n = 0, 1, \ldots, \tag{30}$$

*In addition, for $n = 0$, the corresponding equation $m_t = 0$ is equivalent to $Q_\tau + 4Q_y = 0$. (Kang, Liu, Olver, Qu, 2016)*

- $(\mathrm{mCH})_n$, $(\mathrm{mCH})_{-n}$, $(\mathrm{mKdV})_n$, $(\mathrm{mKdV})_{-n}$, $---$ the $n$-th equation in the positive and negative directions of the mCH and mKdV hierarchies

# The correspondence between the mCH and mKdV hierarchies

**KET Issue for the proof of the theorem**

- The relations between the respective recursion operators admitted by the two hierarchies

**Lemma**

*Let* $\mathcal{K}$, $\mathcal{J}$ *be the two compatible Hamiltonian operators* (21) *for the mCH equation* (8), *and* $\bar{\mathcal{K}}$, $\bar{\mathcal{J}}$ *the two of compatible Hamiltonian operators* (23) *for the mKdV equation* (10). *Assume* $m(t,x)$ *and* $Q(\tau,y)$ *be related by the transformations* (26).

**THEN**, *for each integer* $n \geq 0$, *the following formulae hold:*

**(i).** $\left(\mathcal{K}\mathcal{J}^{-1}\right)^n\left(1-\partial_x^2\right) = \frac{1}{(-4)^n}\left(1+\frac{Q_y}{4Q^3}\partial_y - \frac{1}{4Q^2}\partial_y^2\right)\left(\bar{\mathcal{J}}\bar{\mathcal{K}}^{-1}\right)^n;$

**(ii).** $\partial_x\left(\mathcal{K}^{-1}\mathcal{J}\right)^n\partial_x^{-1} = (-4)^n\left(\bar{\mathcal{K}}\bar{\mathcal{J}}^{-1}\right)^n;$

**(iii).** $(1-\partial_x^2)\left(\mathcal{K}^{-1}\mathcal{J}\right)^n = -(-4)^n\frac{1}{Q}\left(\frac{1}{4}\partial_y - Q\partial_y^{-1}Q\right)\left(\bar{\mathcal{K}}\bar{\mathcal{J}}^{-1}\right)^n\frac{1}{Q}\partial_y.$

- The reciprocal relation which adheres to the conservative structure of the mCH flows

# The correspondence between the mCH and mKdV hierarchies

**KET Issue for the proof of the theorem**

- The relations between the respective recursion operators admitted by the two hierarchies

**Lemma**

*Let* $\mathcal{K}, \mathcal{J}$ *be the two compatible Hamiltonian operators* (21) *for the mCH equation* (8), *and* $\bar{\mathcal{K}}, \bar{\mathcal{J}}$ *the two of compatible Hamiltonian operators* (23) *for the mKdV equation* (10). *Assume* $m(t,x)$ *and* $Q(\tau, y)$ *be related by the transformations* (26).

**THEN**, *for each integer* $n \geq 0$, *the following formulae hold:*

**(i).** $$\left(\mathcal{K}\mathcal{J}^{-1}\right)^n \left(1-\partial_x^2\right) = \frac{1}{(-4)^n}\left(1+\frac{Q_y}{4Q^3}\partial_y - \frac{1}{4Q^2}\partial_y^2\right)\left(\bar{\mathcal{J}}\bar{\mathcal{K}}^{-1}\right)^n;$$

**(ii).** $$\partial_x\left(\mathcal{K}^{-1}\mathcal{J}\right)^n \partial_x^{-1} = (-4)^n\left(\bar{\mathcal{K}}\bar{\mathcal{J}}^{-1}\right)^n;$$

**(iii).** $$(1-\partial_x^2)\left(\mathcal{K}^{-1}\mathcal{J}\right)^n = -(-4)^n \frac{1}{Q}\left(\frac{1}{4}\partial_y - Q\partial_y^{-1}Q\right)\left(\bar{\mathcal{K}}\bar{\mathcal{J}}^{-1}\right)^n \frac{1}{Q}\partial_y.$$

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# The correspondence between the Hamiltonian conservation laws of the mCH and mKdV equations

**An infinite hierarchy of Hamiltonian conservation laws of the bi-Hamiltonian system**

- **The mCH equation**

$$\mathcal{K}\,\frac{\delta \mathcal{H}_{n-1}}{\delta m} = \mathcal{J}\,\frac{\delta \mathcal{H}_n}{\delta m}, \qquad n \in \mathbb{Z}, \tag{31}$$

  ⋄ $\mathcal{K} = -\partial_x\, m\, \partial_x^{-1}\, m\, \partial_x, \qquad \mathcal{J} = -\left(\partial_x - \partial_x^3\right)$

- **The mKdV equation**

$$\bar{\mathcal{K}}\,\frac{\delta \bar{\mathcal{H}}_{n-1}}{\delta Q} = \bar{\mathcal{J}}\,\frac{\delta \bar{\mathcal{H}}_n}{\delta Q}, \qquad n \in \mathbb{Z} \tag{32}$$

  ⋄ $\bar{\mathcal{K}} = -\frac{1}{4}\partial_y^3 + \partial_y Q \partial_y^{-1} Q \partial_y, \qquad \bar{\mathcal{J}} = -\partial_y$

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# The correspondence between the Hamiltonian conservation laws of the mCH and mKdV equations

**The relationship between the variational derivatives of $\delta\mathcal{H}_n/\delta m$ and $\delta\bar{\mathcal{H}}_n/\delta Q$**

**Lemma**

*Let $\{\mathcal{H}_n\}$ and $\{\bar{\mathcal{H}}_n\}$ be the hierarchies of conserved functionals determined by the recursive formulae* (31) *and* (32)*, respectively.* **THEN** *their corresponding variational derivatives satisfy the relation*

$$\frac{\delta\mathcal{H}_{-n}}{\delta m} = (-1)^{n-1}2^{2n-1}\bar{\mathcal{J}}^{-1}Q\bar{\mathcal{J}}\frac{\delta\bar{\mathcal{H}}_n}{\delta Q}, \qquad 0 \neq n \in \mathbb{Z}. \tag{33}$$

**The change of the variational derivative under the Liouville transformations**

**Lemma**

*Let $m(t,x)$ and $Q(\tau,y)$ be related by the transformations* (26)*. If $\mathcal{H}(m) = \bar{\mathcal{H}}(Q)$,* **THEN**

$$\frac{\delta\mathcal{H}}{\delta m} = -\frac{1}{Q}\left(\frac{1}{4}\bar{\mathcal{J}}^2 - \bar{\mathcal{J}}^{-1}\bar{\mathcal{K}}\right)\frac{\delta\bar{\mathcal{H}}}{\delta Q},$$

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**Theorem**

*For any non-zero integer n, each Hamiltonian conserved functional* $\bar{\mathcal{H}}_n(Q)$ *of the mKdV equation in* (32) *yields the Hamiltonian conservation law* $\mathcal{H}_{-n}(m)$ *of the mCH equation in* (31), *under the Liouville transformations* (26), *according to the following identity*

$$\mathcal{H}_{-n}(m) = (-1)^n 2^{2n-1}\,\bar{\mathcal{H}}_n(Q), \qquad 0 \neq n \in \mathbb{Z}. \tag{34}$$

*(Kang, Liu, Olver, Qu, 2016)*

**REMARK**

- A direct application of relation (34) is to derive another Casimir functional, in addition to the Hamitonial functional $\bar{\mathcal{H}}_S$ of the sine-Gordon equation, for the Hamiltonian operator $\bar{\mathcal{K}}$.

$$\diamond \quad \mathcal{H}_1[m] = \int \left(u^2 + u_x^2\right) \mathrm{d}x \qquad \text{and}$$

$$\Downarrow$$

$$\bar{\mathcal{H}}_{-1}(Q) = -8\,\bar{\mathcal{H}}_C(Q), \qquad \text{where} \quad \bar{\mathcal{H}}_C(Q) = \int m\left(1 - \partial_x^2\right)^{-1} m\,\mathrm{d}x$$

# The correspondence between the Hamiltonian conservation laws of the mCH and mKdV equations

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# The transformation mapping the mCH equation into the CH equation

**Motivation**

$$
\begin{array}{ccc}
\textbf{KdV} & \overset{\text{Miura T.}}{\Longleftarrow\!=\!=\!=\!=\!=\!=\!=\!=\!\Longrightarrow} & \textbf{mKdV} \\
\Big\downarrow \text{dual} & & \text{dual} \Big\downarrow \\
\textbf{CH} & \Longleftarrow\!=\!=\!=\!=\!=\!=\!=\!=\!\Longrightarrow & \textbf{mCH}
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**The mKdV hierarchy and the KdV hierarchy**

- **mKdV** $\overset{}{\underset{\text{Miura T.}}{\Longleftarrow\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!\Longrightarrow}}$ **KdV**
  - **The KdV equation:** $P_\tau + P_{yyy} - 6\,PP_y = 0$
  - **The mKdV equation:** $Q_\tau + Q_{yyy} - 6\,Q^2 Q_y = 0$
  - **The Miura transformation:** $\mathcal{B}(P,Q) \equiv P - Q^2 + Q_y = 0$
- Fokas and Fuchssteiner (1981):

$$(\textbf{mKdV})_n \underset{\text{Miura T. (31)}}{\Longleftarrow\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!\Longrightarrow} (\textbf{KdV})_n \qquad n \in \mathbb{Z}^+$$

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# The transformation mapping the mCH equation into the CH equation

**The mKdV hierarchy and the KdV hierarchy**

- **(mKdV)**$_{-1}$ $\underset{\text{Miura T. (31)}}{=====================\Rightarrow}$ **(KdV)** $_{-1}$

**Proposition 3.1.** Assume that $Q$ satisfies the first negative flow of the mKdV hierarchy

$$\left(\bar{\mathcal{K}}\bar{\mathcal{J}}^{-1}\right) Q_\tau = 0.$$

**THEN** $P = Q^2 - Q_y$ satisfies the first negative flow of the KdV hierarchy

$$\left(\bar{\mathcal{L}}\bar{\mathcal{D}}^{-1}\right) P_\tau = 0,$$

where $\bar{\mathcal{L}} = \frac{1}{4}\partial_y^3 - \frac{1}{2}\left(P\partial_y + \partial_y P\right)$ and $\bar{\mathcal{D}} = \partial_y$ are the compatible bi-Hamiltonian operators admitted by the KdV hierarchy.

# The transformation mapping the mCH equation into the CH equation

**The map from the mCH equation to the CH equation**

- The mCH equation: $m_t + \left((u^2 - u_x^2)\, m\right)_x = 0, \quad m = u - u_{xx}$
- The CH equation: $\rho_t + 2v_x\rho + v\rho_x = 0, \qquad \rho = v - v_{xx}$
- The Liouville transformation ( **mCH** $\leftrightarrow$ **mKdV** )
- The Liouville transformation ( **CH** $\leftrightarrow$ **KdV** )

Miura T.

**(mKdV)**$_{-1}$ $=====================\Rightarrow$ **(KdV)**$_{-1}$

$\updownarrow$ Liouville T. Liouville T. $\updownarrow$

**mCH** **CH**

$=========================\Rightarrow$

L.T. + M.T. + L.T.

# The transformation mapping the mCH equation into the CH equation

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Miura T.

**(mKdV)**$_{-1}$ $==================\Longrightarrow$ **(KdV)**$_{-1}$

$\updownarrow$ Liouville T.      Liouville T. $\updownarrow$

**mCH**      **CH**

$======================\Longrightarrow$

L.T. + M.T. + L.T.

# The transformation mapping the mCH equation into the CH equation

**Theorem**

*Assume $m(t,x)$ is the solution of the mCH equation (34).* **THEN**, *$\rho(t,x)$ satisfies the CH equation (35), where $\rho(t,x)$ is determined by the relation*

$$P(\tau,y) = \frac{1}{\rho(t,x)}\left(\frac{1}{4} - \frac{\left(\rho(t,x)^{-1/4}\right)_{xx}}{\rho(t,x)^{-1/4}}\right), \qquad y = \int^{x} \sqrt{\rho(t,\xi)}\,\mathrm{d}\xi, \qquad \rho = v - v_{xx}, \tag{36}$$

*with $P(\tau,y) = Q^2(\tau,y) - Q_y(\tau,y)$ and $Q(\tau,y)$ defined by*

$$Q(\tau,y) = \frac{1}{2m(t,x)}, \quad y = \int^{x} m(t,\xi)\,\mathrm{d}\xi, \quad \tau = t. \tag{37}$$

*(Kang, Liu, Olver, Qu, 2016)*

# The correspondence between the Novikov and SK hierarchies

**A Liouville transformation between the isospectral problems of the Novikov and SK equations**

- The Novikov equation

$$m_t = u^2 m_x + 3uu_x m, \qquad m = u - u_{xx} \tag{38}$$

- **The isospectral problems** (Novikov, 2009):

$$\Psi_x = \begin{pmatrix} 0 & \lambda m & 1 \\ 0 & 0 & \lambda m \\ 1 & 0 & 0 \end{pmatrix} \Psi, \qquad \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} \tag{39}$$

$$\Psi_t = \begin{pmatrix} \frac{1}{3\lambda^2} - uu_x & \frac{u_x}{\lambda} - \lambda u^2 m & u_x^2 \\ \frac{u}{\lambda} & -\frac{2}{3\lambda^2} & -\frac{u_x}{\lambda} - \lambda u^2 m \\ -u^2 & \frac{u}{\lambda} & \frac{1}{3\lambda^2} + uu_x \end{pmatrix} \Psi$$

- $\partial_t(\Psi_x) = \partial_x(\Psi_t) \quad \Rightarrow \quad$ Novikov equation (38)

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Note that (38) is equivalent to the equation by setting $\Psi = \psi_2$

$$\Psi_{xxx} = 2m^{-1}m_x\Psi_{xx} + (m^{-1}m_{xx} - 2m^{-2}m_x^2 + 1)\Psi_x + \lambda^2 m^2\Psi, \quad (40)$$

which can be converted into

$$\Phi_{yyy} + Q\Phi_y = \mu\Phi, \quad (41)$$

by the reciprocal transformation

$$dy = m^{\frac{2}{3}}dx + m^{\frac{2}{3}}u^2 dt, \quad d\tau = dt,$$

with

$$\Phi = \Psi, \quad \mu = \lambda^2, \quad \text{and} \quad Q = -\frac{1}{3}m^{-\frac{7}{3}}m_{xx} + \frac{4}{9}m^{-\frac{10}{3}}m_x^2 - m^{-\frac{4}{3}}.$$

The time part for the isospectral problem becomes

$$\Phi_\tau - \frac{1}{\mu}(V\Phi_{yy} - V_y\Phi_y) + \frac{2}{3\mu}\Phi = 0, \quad \text{with} \quad V = um^{\frac{1}{3}}. \quad (42)$$

It is easy to see (42) is equivalent to

$$\Phi_\tau + \frac{1}{3\mu}(W\Phi_{yy} - W_y\Phi_y) = 0 \tag{43}$$

after gauging $\Phi$ by a factor, and setting $W = -3V$. The compatibility condition $\Phi_{yyyt} = \Phi_{tyyy}$ gives the first equation in the negative SK heirarchy (Gordoa, Pickering, 2002)

$$Q_\tau = W_y, \quad W_{yy} + QW = T, \quad T_y = 0. \tag{44}$$

# The correspondence between the Novikov and SK hierarchies

**A Liouville transformation between the isospectral problems of the Novikov and SK equations**

- The SK equation

$$Q_t + Q_{yyyyy} + 5(QQ_{yy})_y + 5Q^2Q_y = 0 \qquad (45)$$

- **The isospectral problems** (Kaup, 1980)

$$\Phi_y = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \mu & -Q & 0 \end{pmatrix} \Phi, \qquad \Phi = \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{pmatrix} \qquad (46)$$

$$\Phi_t = \begin{pmatrix} 6\mu Q & Q_{yy} - Q^2 & 9\mu - 3Q_y \\ 3\mu(Q_y + 3\mu) & Q_{yyy} + QQ_y - 3\mu Q & -2Q_{yy} - Q^2 \\ \mu(Q_{yy} - Q^2) & Q_{yyyy} + 3QQ_{yy} + Q_y^2 + Q^3 + 9\mu^2 & -Q_{yyy} - QQ_y - 3\mu Q \end{pmatrix} \Phi$$

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# The correspondence between Novikov and SK hierarchies

**A Liouville transformation between the isospectral problems of the Novikov and SK equations**

- The coordinate transformation

$$\Phi = \Psi, \qquad y = \int^x m^{\frac{2}{3}}(t,\xi)\,\mathrm{d}\xi \tag{47}$$

will convert the isospectral problem (39) into the isospectral problem (46), with

$$Q = -\frac{1}{3}m^{-\frac{7}{3}}m_{xx} + \frac{4}{9}m^{-\frac{10}{3}}m_x^2 - m^{-\frac{4}{3}} \qquad \text{and} \qquad \lambda = -2\mu.$$

- The Liouville transformation

$$\begin{aligned} &y = \int^x m^{\frac{2}{3}}(t,\xi)\,\mathrm{d}\xi, \qquad \tau = t, \\ &Q(\tau,y) = -\frac{1}{3}m^{-\frac{7}{3}}m_{xx} + \frac{4}{9}m^{-\frac{10}{3}}m_x^2 - m^{-\frac{4}{3}} = -m^{-1}(1-\partial_x^2)m^{-\frac{1}{3}} \end{aligned} \tag{48}$$

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# The correspondence between the Novikov and SK hierarchies

## The Novikov hierarchy

- The Novikov equation written in the **bi-Hamiltonian** form (Hone, Wang, 2008)

$$m_t = K_1 = \mathcal{K}\frac{\delta \mathcal{H}_0}{\delta m} = \mathcal{J}\frac{\delta \mathcal{H}_1}{\delta m}, \quad m = u - u_{xx} \tag{49}$$

  - ◇ A pair of compatible Hamiltonian operators

$$\mathcal{K} = \frac{1}{2} m^{\frac{1}{3}} \partial_x m^{\frac{2}{3}} (4\partial_x - \partial_x^3)^{-1} m^{\frac{2}{3}} \partial_x m^{\frac{1}{3}} \ \text{ and } \ \mathcal{J} = \left(1-\partial_x^2\right)\frac{1}{m}\partial_x \frac{1}{m}\left(1-\partial_x^2\right) \tag{50}$$

  - ◇ The corresponding Hamiltonian functionals

$$\begin{aligned} \mathcal{H}_0[m] &= 9\int m\,u\,\mathrm{d}x = 9\int \left(u^2 + u_x^2\right)\mathrm{d}x, \\ \mathcal{H}_1[m] &= \frac{1}{6}\int \left(um\partial_x^{-1} m \left(1-\partial_x^2\right)^{-1}\left(u^2 m_x + 3uu_x m\right)\right)\mathrm{d}x \end{aligned} \tag{51}$$

- Recursion operator: $\mathcal{R} = \mathcal{K}\mathcal{J}^{-1}$

# The correspondence between the Novikov and SK hierarchies

**The Novikov hierarchy**

- The positive flows of the Novikov hierarchy

$$m_t = K_n = \left(\mathcal{K}\mathcal{J}^{-1}\right)^{n-1} K_1, \quad n = 1, 2, \dots$$

- The negative flows of the Novikov hierarchy
  - ◇ The Hamiltonian operator $\mathcal{K}$ admits the Casimir functional

$$\mathcal{H}_C = \frac{9}{2}\int m^{\frac{2}{3}}\,\mathrm{d}x \qquad \text{with} \qquad \frac{\delta \mathcal{H}_C}{\delta m} = 3m^{-\frac{1}{3}}. \tag{52}$$

  - ◇ The Casimir equation

$$m_t = K_{-1} = \mathcal{J}\frac{\delta \mathcal{H}_{-1}}{\delta m} = 3\mathcal{J}\, m^{-\frac{1}{3}}.$$

  - ◇ The *n*-th negative flow of the Novikov hierarchy

$$m_t = K_{-n} = \left(\mathcal{J}\mathcal{K}^{-1}\right)^{n-1} K_{-1}, \quad n = 1, 2, \dots$$

# The correspondence between the Novikov and SK hierarchies

**Hamiltonian functional $\mathcal{H}_1$**
Note that the conserved Hamiltonian functional $\mathcal{H}_1$ is nonlocal, indeed, one can show that it is equivalent to

$$\mathcal{H}_1[m] = \frac{1}{6}\int \left(u^4 m^2 - u_t m_t\right) \mathrm{d}x. \tag{53}$$

**Proof.**

In fact, using Novikov equation, we can denote $\mathcal{H}_1[m]$ in (51) as

$$\mathcal{H}_1[m] = \frac{1}{6}\int u m \partial_x^{-1}(m u_t)\mathrm{d}x. \tag{54}$$

Since

$$\begin{aligned}
\partial_x^{-1}(mu_t) &= \int_{-\infty}^{x}(u-u_{xx})u_t dx = -(u_x u_t - u u_{xt})(t,x) + \int_{-\infty}^{x} u(u_t - u_{xxt})\mathrm{d}x \\
&= -(u_x u_t - u u_{xt})(t,x) + \int_{-\infty}^{x} u(u^2 m_x + 3uu_x m)dx \\
&= (uu_{xt} - u_x u_t + u^3 m)(t,x).
\end{aligned}$$

□

# The correspondence between Novikov and SK hierarchies

**The Sawada-Kotera hierarchy**

- The SK equation—— the generalized bi-Hamiltonian system (Fuchssteiner, Oevel, 1982)

$$Q_\tau = \bar{K}_1 = \bar{\mathcal{K}} \frac{\delta \bar{\mathcal{H}}_0}{\delta Q} \quad \text{and} \quad \bar{\mathcal{J}} \bar{K}_1 = \frac{\delta \bar{\mathcal{H}}_1}{\delta Q}$$

◇

$$\begin{aligned} \bar{\mathcal{K}} &= -\left(\partial_y^3 + 2(Q\partial_y + \partial_y Q)\right), \\ \bar{\mathcal{J}} &= 2\partial_y^3 + 2(\partial_y^2 Q \partial_y^{-1} + \partial_y^{-1} Q \partial_y^2) + Q^2 \partial_y^{-1} + \partial_y^{-1} Q^2. \end{aligned} \tag{55}$$

◇ The Hamiltonian functionals

$$\bar{\mathcal{H}}_0[Q] = \frac{1}{6} \int \left(Q^3 - 3Q_y^2\right) \mathrm{d}y$$

- Recursion operator: $\bar{\mathcal{R}} = \bar{\mathcal{K}} \bar{\mathcal{J}}$

# The correspondence between Novikov and SK hierarchies

**The SK hierarchy**

$$Q_\tau = \bar{K}_n = \bar{\mathcal{K}} \frac{\delta \bar{\mathcal{H}}_{n-1}}{\delta Q} \quad \text{and} \quad \bar{\mathcal{J}}\, \bar{K}_n = \frac{\delta \bar{\mathcal{H}}_n}{\delta Q}, \qquad n \in \mathbb{Z}. \tag{56}$$

- The positive flows of the SK hierarchy

$$Q_\tau = \bar{K}_n = \left(\bar{\mathcal{K}}\bar{\mathcal{J}}\right)^{n-1} \bar{K}_1, \qquad n = 1, 2, \ldots.$$

- The negative flows of the SK hierarchy
  - ◇ $\bar{\mathcal{J}} \cdot 0 = \frac{\delta \bar{\mathcal{H}}_0}{\delta Q} = \frac{1}{2} Q^2 + Q_{yy}$
  - ◇ The $n$-th negative flow

$$\bar{\mathcal{R}}^n Q_\tau = \left(\bar{\mathcal{K}}\, \bar{\mathcal{J}}\right)^n Q_\tau = 0, \qquad n = 1, 2, \ldots. \tag{57}$$

# The correspondence between Novikov and SK hierarchies

**REMARK**

**Lemma**

*There holds (Chou, Qu, 2004, Physica D)*

$$\bar{\mathcal{R}} = \bar{\mathcal{K}}\bar{\mathcal{J}} = -2\left(\partial_y^4 + 5Q\partial_y^2 + 4Q_y\partial_y + Q_{yy} + 4Q^2 + 2Q_y\partial_y^{-1}Q\right)\left(\partial_y^2 + Q + Q_y\partial_y^{-1}\right). \tag{58}$$

It implies that the equation

$$\left(\partial_y^2 + Q + Q_y\partial_y^{-1}\right)Q_\tau = 0 \tag{59}$$

can be regarded as a reduction of the more general first negative flow $\mathcal{R}Q_\tau = 0$. One can verify that equation (59) is equivalent to

$$\left(Q + \partial_y^2\right)\partial_y^{-1}Q_\tau = C, \tag{60}$$

with $C$ being the integration constant. Obviously, if we set $T = C$ in (44), the corresponding equation is exactly the reduced first negative flow (60).

# The correspondence between the Novikov and SK hierarchies

**Theorem**

*Under the transformations* (48)*, for each* $n \in \mathbb{Z}$*, the* $(\mathrm{Novikov})_n$ *equation is mapped into the equation* $(\mathrm{SK})_{-n}$ *equation, and conversely.*

The proof of this theorem relies on the following two Lemmas.

**Lemma**

*Let* $m(t,x)$ *and* $Q(\tau,y)$ *be related by the transformations* (48)*, then the following operator identities hold:*

$$m^{-1}\,(1-\partial_x^2)\,m^{-\frac{1}{3}} = -(Q+\partial_y^2);$$
$$m^{-1}\,\mathcal{J}\,m^{-\frac{1}{3}} = \frac{1}{2}\partial_y\,\bar{\mathcal{J}}\,\partial_y;$$
$$m^{-\frac{4}{3}}\,(4\partial_x-\partial_x^3)\,m^{-\frac{2}{3}} = \bar{\mathcal{K}}.$$

# The correspondence between Novikov and SK hierarchies

**KET Issue for the proof of the theorem**

- The relations between the respective recursion operators admitted by the two hierarchies

**Lemma**

*Let* $\mathcal{K}$, $\mathcal{J}$ *be the two compatible Hamiltonian operators* (50) *for Novikov equation* (38), $\bar{\mathcal{K}}$ *and* $\bar{\mathcal{J}}$ *be the Hamiltonian operator and symplectic operator* (55) *of SK equation* (45), *respectively. Assume* $m(t, x)$ *and* $Q(\tau, y)$ *be related by the transformations* (48).

**THEN**, *the relation*

$$m^{-1}\left(\mathcal{J}\mathcal{K}^{-1}\right)^{n} m = \partial_y \left(\bar{\mathcal{J}}\bar{\mathcal{K}}\right)^{n} \partial_y^{-1} \tag{61}$$

*holds for each integer* $n \geq 1$.

# The correspondence between the Hamiltonian conservation laws of Novikov and SK equations

An infinite hierarchy of Hamiltonian conservation laws of the bi-Hamiltonian system

- The Novikov hierarchy:

$$\mathcal{K}\frac{\delta \mathcal{H}_{n-1}}{\delta m} = \mathcal{J}\frac{\delta \mathcal{H}_n}{\delta m}, \qquad n \in \mathbb{Z}. \tag{62}$$

- The SK hierarchy:

$$\bar{\mathcal{J}}\bar{\mathcal{K}}\frac{\delta \bar{\mathcal{H}}_{n-1}}{\delta Q} = \frac{\delta \bar{\mathcal{H}}_n}{\delta Q}, \qquad n \in \mathbb{Z} \tag{63}$$

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# The correspondence between the Hamiltonian conservation laws of the Novikov and SK equations

**The relationship between the variational derivatives of $\delta\mathcal{H}_n/\delta m$ and $\delta\bar{\mathcal{H}}_n/\delta Q$**

**Lemma**

*Let $\{\mathcal{H}_n\}$ and $\{\bar{\mathcal{H}}_n\}$ be the hierarchies of Hamiltonian conserved functionals of the Novikov equation and SK equation, respectively.* **THEN**, *for each $n \in \mathbb{Z}$, their corresponding variational derivatives satisfy the relation*

$$\frac{\delta\bar{\mathcal{H}}_n}{\delta Q} = \frac{1}{3}\partial_x^{-1} m^{-\frac{1}{3}} \mathcal{K} \frac{\delta\mathcal{H}_{-(n+2)}}{\delta m}. \tag{64}$$

**The change of the variational derivative under the Liouville transformations**

**Lemma**

*Let $m(t,x)$ and $Q(\tau,y)$ be related by the transformations (48). If $\mathcal{H}(m) = \bar{\mathcal{H}}(Q)$,* **THEN**

$$\frac{\delta\mathcal{H}}{\delta m} = \frac{1}{3} m^{-\frac{1}{3}} \partial_y^{-1} \bar{\mathcal{K}} \frac{\delta\bar{\mathcal{H}}}{\delta Q},$$

*where $\bar{\mathcal{K}}$ is the Hamiltonian operator (55) of the SK equation.*

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*where $\bar{\mathcal{K}}$ is the Hamiltonian operator* (55) *of the SK equation.*

# The correspondence between the Hamiltonian conserved functionals of the Novikov and SK equations

**Theorem**

*For any $n \in \mathbb{Z}$, each Hamiltonian conserved functional $\mathcal{H}_n(m)$ of Novikov equation in (62) is related to the Hamiltonian conservation law $\bar{\mathcal{H}}_{-n}(Q)$ of the SK equation in (63), under the Liouville transformations (48), according to the following identity*

$$\mathcal{H}_n(m) = 18\,\bar{\mathcal{H}}_{-(n+2)}(Q), \qquad n \in \mathbb{Z}. \tag{65}$$

*(Kang, Liu, Olver, Qu, 2017)*

# The correspondence between the DP and KK hierarchies

**A Liouville transformation between the isospectral problems of the DP and KK equations**
**The DP equation**

$$n_t = vn_x + 3v_x n, \qquad n = v - v_{xx} \tag{66}$$

- **The Lax pair** (Degasperis, Procesi, 1996):

$$\Psi_x = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\lambda n & 1 & 0 \end{pmatrix} \Psi, \qquad \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} \tag{67}$$

$$\Psi_t = \begin{pmatrix} v_x & -v & -\lambda^{-1} \\ v & -\lambda^{-1} & -v \\ \lambda vn + v_x & 0 & -\lambda^{-1} - v_x \end{pmatrix} \Psi,$$

- (67) is equivalent to

$$\Psi_{xxx} - \Psi_x + \lambda n \Psi = 0 \tag{68}$$

$$\Psi_t + \lambda^{-1}\Psi_{xx} + v\Psi_x - v_x\Psi = 0.$$

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# The correspondence between the DP and KK hierarchies

**A Liouville transformation between the isospectral problems of the DP and KK equations**

**The KK equation**

$$P_\tau + P_{yyyyy} + 20PP_{yyy} + 50P_yP_{yy} + 80P^2P_y = 0 \quad (69)$$

- The Lax pair for the first negative flow

$$\Phi_{yyy} + 4P\Phi_y + 2P_y\Phi = \mu\Phi \quad (70)$$

and

$$\Phi_\tau + \mu^{-1}\left(U\Phi_{yy} - \frac{1}{2}U_y\Phi_y + \frac{1}{6}(U_{yy} + 16PU)\Phi\right) = 0. \quad (71)$$

- The compatibility condition for (70) and (71), $\Phi_{yyy\tau} = \Phi_{\tau yyy}$

$$\Downarrow$$

$$P_\tau = \frac{3}{4}U_y, \qquad \mathcal{A}\,U = 0, \quad (72)$$

where $\mathcal{A} = \partial_y^5 + 6(\partial_y P\,\partial_y^2 + \partial_y^2 P\,\partial_y) + 4(\partial_y^3 P + P\,\partial_y^3) + 32(\partial_y P^2 + P^2\partial_y)$

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# The correspondence between the DP and KK hierarchies

**A Liouville transformation between the isospectral problems of the DP and KK equations**

- The coordinate transformation

$$\mathrm{d}y = n^{\frac{1}{3}}\mathrm{d}x + n^{\frac{1}{3}}v^2\mathrm{d}t, \qquad \mathrm{d}\tau = \mathrm{d}t, \tag{73}$$

  together with $\Psi = n^{-\frac{1}{3}}\Phi$, $\lambda = -\mu$ and

$$P = \frac{1}{4}\left(\frac{7}{9}n^{-\frac{8}{3}}n_x^2 - \frac{2}{3}n^{-\frac{5}{3}}n_{xx} - n^{-\frac{2}{3}}\right) \tag{74}$$

  convert the isospectral problem (68) into (70).

- The Liouville transformation between DP and KK hierarchy

$$y = \int^x n^{\frac{1}{3}}(t,\xi)\,\mathrm{d}\xi, \qquad \tau = t,$$
$$P = \frac{1}{4}\left(\frac{7}{9}n^{-\frac{8}{3}}n_x^2 - \frac{2}{3}n^{-\frac{5}{3}}n_{xx} - n^{-\frac{2}{3}}\right) = \frac{1}{4}n^{-\frac{1}{2}}\left(4\partial_x^2 - 1\right)n^{-\frac{1}{6}} \tag{75}$$

# The correspondence between the DP and KK hierarchies

**A Liouville transformation between the isospectral problems of the DP and KK equations**

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# The correspondence between the DP and KK hierarchies

**The DP hierarchy**

- The DP equation (66) written in bi-Hamiltonian form (Degasperis, Procesi, 1996)

$$n_t = G_1 = \mathcal{L}\,\frac{\delta \mathcal{E}_0}{\delta n} = \mathcal{D}\,\frac{\delta \mathcal{E}_1}{\delta n}, \quad n = v - v_{xx}, \tag{76}$$

  - ◇ A pair of compatible Hamiltonian operators

$$\mathcal{L} = n^{\frac{2}{3}}\,\partial_x\,n^{\frac{1}{3}}\,(\partial_x - \partial_x^3)^{-1}\,n^{\frac{1}{3}}\,\partial_x\,n^{\frac{2}{3}} \quad \text{and} \quad \mathcal{D} = \partial_x\,(1-\partial_x^2)\,(4-\partial_x^2) \tag{77}$$

(KAM for DP equation, R. Feola, F. Giuliani, M. Procesi, 2019, 2020)

  - ◇ The corresponding Hamiltonian functionals

$$\mathcal{E}_0 = \frac{9}{2}\int n\,\mathrm{d}x \quad \text{and} \quad \mathcal{E}_1 = \frac{1}{6}\int u^3\,\mathrm{d}x.$$

- The recursion operator $\tilde{\mathcal{R}} = \mathcal{L}\mathcal{D}^{-1}$

# The correspondence between the DP and KK hierarchies

**The DP hierarchy**

- The positive flows of the DP hierarchy

$$n_t = G_l = \left(\mathcal{L}\mathcal{D}^{-1}\right)^{l-1} G_1, \quad l = 1, 2, \ldots$$

- The negative flows of the DP hierarchy
  - ◇ The Hamiltonian operator $\mathcal{L}$ admits the Casimir functional

$$\mathcal{E}_C = 18 \int n^{\frac{1}{3}} \, \mathrm{d}x \qquad \text{with variational derivative} \qquad \frac{\delta \mathcal{E}_C}{\delta n} = 6n^{-\frac{2}{3}}. \tag{78}$$

  - ◇ The Casimir equation

$$n_t = G_{-1} = \mathcal{D} \frac{\delta \mathcal{E}_C}{\delta n} = 6\mathcal{D} n^{-\frac{2}{3}}. \tag{79}$$

  - ◇ The $l$-th negative flow of the DP hierarchy

$$n_t = G_{-l} = 6\left(\mathcal{D}\mathcal{L}^{-1}\right)^{l-1} \mathcal{D} n^{-\frac{2}{3}}, \quad l = 1, 2, \ldots. \tag{80}$$

**The KK hierarchy**

- The KK equation—— the generalized bi-Hamiltonian system (Fuchssteiner, Oevel, 1982)

$$P_\tau = \bar{G}_1 = \bar{\mathcal{L}} \frac{\delta \bar{\mathcal{E}}_0}{\delta P} \quad \text{and} \quad \bar{\mathcal{D}} \, \bar{G}_1 = \frac{\delta \bar{\mathcal{E}}_1}{\delta P},$$

$$\begin{aligned} \bar{\mathcal{L}} &= -\left(\partial_y^3 + 2(P\partial_y + \partial_y P)\right), \\ \bar{\mathcal{D}} &= \partial_y^3 + 6(P\,\partial_y + \partial_y P) + 4(\partial_y^2 P\,\partial_y^{-1} + \partial_y^{-1} P\partial_y^2) + 32(P^2\,\partial_y^{-1} + \partial_y^{-1} P^2) \end{aligned} \qquad (81)$$

- Recursion operators: $\hat{\mathcal{R}} = \bar{\mathcal{L}}\bar{\mathcal{D}}$
- The positive flows

$$P_\tau = \bar{G}_n = \left(\bar{\mathcal{L}}\bar{\mathcal{D}}\right)^{l-1} \bar{G}_1, \qquad l = 1, 2, \ldots \qquad (82)$$

- The negative flows

$$(\bar{\mathcal{L}}\bar{\mathcal{D}})^l Q_\tau = 0, \qquad l = 1, 2, \ldots \qquad (83)$$

# The correspondence between the Novikov and SK hierarchies

**Theorem**

*Under the transformations* (75)*, for each* $l \in \mathbb{Z}$*, the* $(\mathrm{DP})_l$ *equation is mapped into the equation* $(\mathrm{KK})_{-l}$ *equation, and conversely.*

The proof of this theorem relies on the following two Lemmas.

**Lemma**

*Let* $n(t,x)$ *and* $P(\tau,y)$ *be related by the transformations* (75)*, then the following identities hold:*

$$n^{-\frac{1}{2}}\,(\frac{1}{4}-\partial_x^2)\,n^{-\frac{1}{6}}=-(P+\partial_y^2);$$
$$n^{-\frac{2}{3}}\,(\partial_x-\partial_x^3)\,n^{-\frac{1}{3}}=\bar{\mathcal{L}};$$
$$n^{-1}\,\mathcal{D}\,n^{-\frac{2}{3}}=\partial_y\,\bar{\mathcal{D}}\,\partial_y.$$

**KET Issue for the proof of the theorem**

- The relations between the respective recursion operators admitted by the two hierarchies

**Lemma**

*Let $\mathcal{L}$, $\mathcal{D}$ be the two compatible Hamiltonian operators* (77) *for DP equation* (66)*, and $\bar{\mathcal{L}}$, $\bar{\mathcal{D}}$ the two of compatible Hamiltonian operators* (81) *for KK equation* (69)*. Assume $n(t,x)$ and $P(\tau,y)$ be related by the transformations* (75)*.*

**THEN***, under the transformations* (75)*, the relation*

$$n^{-1}\left(\mathcal{D}\mathcal{L}^{-1}\right)^{l} n = \partial_y \left(\bar{\mathcal{D}}\bar{\mathcal{L}}\right)^{l} \partial_y^{-1} \tag{84}$$

*holds for each integer $l \geq 1$.*

# The correspondence between the Hamiltonian conservation laws of DP and KK equations

**An infinite hierarchy of Hamiltonian conservation laws of the bi-Hamiltonian system**

- **The DP hierarchy:**

$$\mathcal{L}\frac{\delta\mathcal{E}_{l-1}}{\delta n} = \mathcal{D}\frac{\delta\mathcal{E}_{l}}{\delta n}, \qquad l \in \mathbb{Z} \tag{85}$$

- **The KK hierarchy:**

$$\bar{\mathcal{D}}\,\bar{\mathcal{L}}\frac{\delta\bar{\mathcal{E}}_{l-1}}{\delta P} = \frac{\delta\bar{\mathcal{E}}_{l}}{\delta P}, \qquad l \in \mathbb{Z} \tag{86}$$

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- **The KK hierarchy:**

$$\bar{\mathcal{D}}\,\bar{\mathcal{L}}\frac{\delta\bar{\mathcal{E}}_{l-1}}{\delta P} = \frac{\delta\bar{\mathcal{E}}_l}{\delta P}, \qquad l \in \mathbb{Z} \tag{86}$$

# The correspondence between the Hamiltonian conservation laws of the DP and KK equations

**The relationship between the variational derivatives of $\delta\mathcal{E}_l/\delta n$ and $\delta\bar{\mathcal{E}}_l/\delta P$**

**Lemma**

*Let $\{\mathcal{E}_l\}$ and $\{\bar{\mathcal{E}}_l\}$ be the hierarchies of Hamiltonian conserved functionals of the DP and KK equations, respectively.* **THEN**, *for each $l \in \mathbb{Z}$, their corresponding variational derivatives are related according to the following identity*

$$\frac{\delta\mathcal{E}_l}{\delta n} = 6\mathcal{L}^{-1}\, n\,\partial_y\, \frac{\delta\bar{\mathcal{E}}_{-(l+2)}}{\delta P}. \qquad (87)$$

**The change of the variational derivative under the Liouville transformations**

**Lemma**

*Let $n(t, x)$ and $P(\tau, y)$ be related by the transformations (75). If $\mathcal{E}(n) = \bar{\mathcal{E}}(P)$.* **THEN**

$$\frac{\delta\mathcal{E}}{\delta n} = \frac{1}{6} n^{-\frac{2}{3}} \partial_y^{-1} \bar{\mathcal{L}} \frac{\delta\bar{\mathcal{E}}}{\delta P}, \qquad (88)$$

*where $\bar{\mathcal{L}}$ is the Hamiltonian operator (81) admitted by the KK equation (69).*

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# The correspondence between the Hamiltonian conservation laws of the DP and KK equations

**Theorem**

*Under the Liouville transformations (75), for each $l \in \mathbb{Z}$, the Hamiltonian conserved functional $\bar{\mathcal{E}}_l(P)$ of the KK equation is related to the Hamiltonian conserved functional $\mathcal{E}_l(n)$ of the DP equation, according to the following identity*

$$\mathcal{E}_l(n) = 36\,\bar{\mathcal{E}}_{-(l+2)}(P), \qquad l \in \mathbb{Z}. \tag{89}$$

# The relationship between the Novikov equation and the DP equation

**Motivation**

$$\begin{array}{ccc} \textbf{SK} & \overset{\text{Fordy, Gibbon, 1979 T.}}{\Longleftarrow\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!\Longrightarrow} & \textbf{KK} \\ \Big\downarrow \text{“dual”} & & \text{“dual”} \Big\downarrow \\ \textbf{Novikov} & \Longleftarrow\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!\Longrightarrow & \textbf{DP} \end{array}$$

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$\downarrow$ "dual" $\qquad\qquad\qquad$ "dual" $\downarrow$

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# The relationship between the Novikov equation and the DP equation

**The Novikov hierarchy and the DP hierarchy**

- **SK** $\overset{}{\underset{\text{Forday, Gibbon, Miura T.}}{\Longleftarrow\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!\Longrightarrow}}$ **KK**
  - **The KK equation:** $P_\tau + P_{yyyyy} + 5(PP_{yy})_y + 5P^2P_y = 0$
  - **The SK equation:** $Q_\tau + Q_{yyyyy} + 20QQ_{yyy} + 25Q_yQ_{yy} + 80Q^2Q_y = 0$
  - **The Miura transformations (Forday, Gibbon, 1979):**

$$\begin{aligned} \mathcal{B}_1(P, Q) &\equiv Q - (W_y - W^2) = 0, \\ \mathcal{B}_2(P, Q) &\equiv P + (2W_y + W^2) = 0, \end{aligned} \tag{90}$$

  where $W$ satisfies

$$W_t = W_{yyyyy} - 5(W_yW_{yyy} + W_{yy}^2 + W_y^3 + 4WW_yW_{yy} + W^2W_{yyy} - W^4W_y)$$

- As in (Fokas and Fuchssteiner, 1981):

$$\textbf{(SK)}_n \underset{\text{Miura T. (90)}}{\Longleftarrow\!=\!=\!=\!=\!=\!=\!=\!=\!=\!\Longrightarrow} \textbf{(KK)}_n \qquad n \in \mathbb{Z}^+$$

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# The relationship between the Novikov equation and the DP equation

**The SK hierarchy and the KK hierarchy**

- $\textbf{(SK)}_{-1} \overset{}{\underset{\text{Miura T. (90)}}{==================\Longrightarrow}} \textbf{(KK)}_{-1}$

**Lemma**

*Assume that Q satisfies the first negative flow of the SK hierarchy*

$$\left(\bar{\mathcal{K}}\bar{\mathcal{J}}\right) Q_\tau = 0, \tag{91}$$

*and P satisfies the first negative flow of the KK hierarchy*

$$\left(\bar{\mathcal{L}}\bar{\mathcal{D}}\right) P_\tau = 0, \tag{92}$$

**THEN** *The Miura transformation* (90) *relates the first negative flow of the KK hierarchy and the first negative flow of the SK hierarchy.*

# The relationship between the Novikov equation and the DP equation

- The Novikov equation: $m_t = u^2 m_x + 3uu_x m, \quad m = u - u_{xx}$
- The DP equation: $n_t = vn_x + 3v_x n, \quad n = v - v_{xx}$
- The Liouville transformation ( **Novikov** $\leftrightarrow$ $\mathbf{(SK)_{-1}}$ )
- The Liouville transformation ( **DP** $\leftrightarrow$ $\mathbf{(KK)_{-1}}$ )

$$
\begin{array}{ccc}
\mathbf{(SK)_{-1}} & \overset{\text{Miura T. (90)}}{\Longleftarrow\!=\!=\!=\!=\!=\!=\!=\!=\!=\!\Longrightarrow} & \mathbf{(KK)_{-1}} \\
\Big\updownarrow \text{Liouville T.} & & \text{Liouville T.} \Big\updownarrow \\
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\end{array}$$

# The transformation mapping the Novikov equation and the DP equation

**Theorem**

*Assume $m(t,x)$ is the solution of the Novikov equation.* **THEN**, *$n(t,x)$ satisfies the DP equation, where $n(t,x)$ is determined implicitly by the relation*

$$P(\tau,y) = \frac{1}{4} n^{-\frac{1}{2}}(4\partial_x^2 - 1)n^{-\frac{1}{6}}, \qquad y = \int^x n^{\frac{1}{3}}(t,\xi)\,\mathrm{d}\xi, \qquad n = v - v_{xx}, \tag{93}$$

*with $P(\tau,y)$ determined by $Q(\tau,y)$ via* (90), *and $Q(\tau,y)$ satisfies*

$$Q(\tau,y) = -m^{-1}(1-\partial_x^2)m^{-\frac{1}{3}}, \quad y = \int^x m^{\frac{2}{3}}(t,\xi)\,\mathrm{d}\xi, \quad \tau = t. \tag{94}$$

- **The 2CH hierarchy**
  First, the hierarchy of 2CH system (1) is given by

$$\begin{pmatrix} m \\ \rho \end{pmatrix}_t = \mathcal{K}\delta\mathcal{H}_{n-1}(m,\rho) = \mathcal{J}\delta\mathcal{H}_n(m,\rho), \quad \delta\mathcal{H}_n(m,\rho) = \left(\frac{\delta\mathcal{H}_n}{\delta m}, \frac{\delta\mathcal{H}_n}{\delta\rho}\right)^T, \quad n = 1, 2, \quad (95)$$

  with compatible Hamiltonian operators

$$\mathcal{K} = \begin{pmatrix} m\partial_x + \partial_x m & \rho\partial_x \\ \partial_x\rho & 0 \end{pmatrix}, \qquad \mathcal{J} = \begin{pmatrix} \partial_x - \partial_x^3 & 0 \\ 0 & \partial_x \end{pmatrix}. \quad (96)$$

- **The A2CH hierarchy**
  First, the hierarchy of A2CH system (2) is given by

$$\begin{pmatrix} Q \\ P \end{pmatrix}_{\tau} = \overline{\mathbf{K}}_n = \overline{\mathcal{R}}^{n-1}\, \overline{\mathbf{K}}_1, \quad n = 1, 2, \ldots. \tag{97}$$

with

$$\overline{\mathcal{R}} = \frac{1}{2} \begin{pmatrix} 0 & \partial_y^2 + 4Q + 2Q_y \partial_y^{-1} \\ -4 & 4P + 2P_y \partial_y^{-1} \end{pmatrix}, \quad \overline{\mathbf{K}}_1 = \left(-Q_y, -P_y\right)^T. \tag{98}$$

# The correspondence between the 2-CH and A2CH hierarchies

**A Liouville transformation between the isospectral problems of the 2-CH and A2CH equations**

- The Liouville transformation (Kang, Liu, Olver, Qu, 2020)

$$\begin{aligned} &\Phi = \sqrt{\rho}\,\Psi, \ \ \tau = t, \quad y = \int^x \rho(t,\xi)\,\mathrm{d}\xi, \quad P(\tau,y) = -m(t,x)\,\rho(t,x)^{-2}, \\ &Q(\tau,y) = -\frac{1}{4}\rho(t,x)^{-2} + \frac{3}{4}\rho(t,x)^{-4}\rho_x^2(t,x) - \frac{1}{2}\rho(t,x)^{-3}\rho_{xx}(t,x). \end{aligned} \tag{99}$$

will convert the isospectral problem

$$\Psi_{xx} + \left(-\frac{1}{4} - \lambda m + \lambda^2\rho^2\right)\Psi = 0, \qquad \Psi_t = \left(\frac{1}{2\lambda} - u\right)\Psi_x + \frac{u_x}{2}\Psi, \tag{100}$$

into the isospectral problem

$$\Phi_{yy} + (Q + \lambda P + \lambda^2)\Phi = 0, \quad \Phi_\tau - \frac{1}{2\lambda}\rho\,\Phi_y + \frac{1}{4\lambda}\rho_y\Phi = 0, \tag{101}$$

# The correspondence between the 2CH and 2AKNS hierarchies

**Theorem**

*Under the Liouville transformation* (99)*, for each integer n, the hierarchy* (95) *is mapped into the hierarchy* (97)*.*

**Theorem**

*(Kang, Liu, Olver, Qu, 2020) Under the Liouville transformation* (99)*, for each nonzero integer n, the Hamiltonian functionals* $\mathcal{H}_n(m,\rho)$ *of the 2CH hierarchy* (95) *are related to the Hamiltonian functionals* $\bar{\mathcal{H}}_n(Q,P)$ *of the A2CH hierarchy* (97)*, according to*

$$\mathcal{H}_n(m,\rho) = \bar{\mathcal{H}}_{-n}(Q,P), \qquad 0 \neq n \in \mathbb{Z}.$$

**Remark**

*Similar results hold for the dDWW hierarchy.*

# The correspondence between the $1+n$-KdV and $1+n$-CH hierarchies

- **The $1+n$-CH hierarchy**
  First, the hierarchy of $1+n$-CH system (1) is given by

$$\begin{pmatrix}\rho \\ \mathbf{m}\end{pmatrix}_t = \overline{\mathbf{G}}_i(\rho,\mathbf{m}) = \overline{\mathcal{K}}(\rho,\mathbf{m})\delta\overline{\mathcal{H}}_{i-1}(\rho,\mathbf{m}) = \overline{\mathcal{J}}(\rho,\mathbf{m})\delta\overline{\mathcal{H}}_i(\rho,\mathbf{m}), \quad i \in \mathbb{Z}^+, \quad (102)$$

with compatible Hamiltonian operators

$$\overline{\mathcal{K}}(\rho,\mathbf{m}) = \mathcal{K}_1(\rho,\mathbf{m}) = \begin{pmatrix}\rho\partial_x + \partial_x\rho & \partial_x\mathbf{m}^{\mathrm{T}} + \mathbf{m}^{\mathrm{T}}\partial_x \\ \partial_x\mathbf{m} + \mathbf{m}\partial_x & (\rho\partial_x + \partial_x\rho)\mathbf{I}_n + \sum_{i<j}\mathbf{J}_{i,j}\mathbf{m}\partial_x^{-1}(\mathbf{J}_{i,j}\mathbf{m})^{\mathrm{T}}\end{pmatrix}$$

and

$$\overline{\mathcal{J}}(\rho,\mathbf{m}) = \mathcal{J} - \mathcal{K}_2 = \begin{pmatrix}\partial_x - \partial_x^3 & \mathbf{0}_n^{\mathrm{T}} \\ \mathbf{0}_n & (\partial_x - \partial_x^3)\mathbf{I}_n\end{pmatrix},$$

where the associated Hamiltonian functionals $H_1$ and $H_2$ are

$$\overline{\mathcal{H}}_1 = \frac{1}{2}\int \left(w^2 + w_x^2 + \langle \mathbf{u}, \mathbf{u}\rangle + \langle \mathbf{u}_x, \mathbf{u}_x\rangle\right) \mathrm{d}x$$

and

$$\overline{\mathcal{H}}_2 = \frac{1}{2}\int \left[w\left(w^2 + w_x^2 + \langle \mathbf{u}, \mathbf{u}\rangle + 2\langle \mathbf{m}, \mathbf{u}\rangle - \langle \mathbf{u}_x, \mathbf{u}_x\rangle\right) + \langle \mathbf{u}, \partial_x^{-1}\Pi(\mathbf{u}, \mathbf{u_x})\mathbf{m}\rangle\right] \mathrm{d}x$$

# The correspondence between the $1+n$-KdV and $1+n$-CH hierarchies

- **The $1+n$-KdV hierarchy**
  Next, the hierarchy of $1+n$-KdV system (5) is given by

$$\begin{pmatrix} w \\ \mathbf{u} \end{pmatrix}_t = \mathcal{K}(w,\mathbf{u})\delta\mathcal{H}_1(w,\mathbf{u}) = \mathcal{J}(w,\mathbf{u})\delta\mathcal{H}_2(w,\mathbf{u}), \tag{103}$$

  where $\delta\mathcal{H}_i = (\delta\mathcal{H}_i/\delta w,\, \delta\mathcal{H}_i/\delta u_1,\, \ldots,\, \delta\mathcal{H}_i/\delta u_n)^{\mathrm{T}}$ $(i=1,2)$ and

$$\begin{aligned} \mathcal{K} &= \begin{pmatrix} \partial_x^3 + w\partial_x + \partial_x w & \partial_x \mathbf{u}^{\mathrm{T}} + \mathbf{u}^{\mathrm{T}}\partial_x \\ \partial_x \mathbf{u} + \mathbf{u}\partial_x & (\partial_x^3 + w\partial_x + \partial_x w)\mathbf{I}_n + \sum_{i<j} \mathbf{J}_{i,j}\mathbf{u}\partial_x^{-1}(\mathbf{J}_{i,j}\mathbf{u})^{\mathrm{T}} \end{pmatrix}, \\ \mathcal{J} &= \begin{pmatrix} \partial_x & \mathbf{0}_n^{\mathrm{T}} \\ \mathbf{0}_n & \partial_x \mathbf{I}_n \end{pmatrix} \end{aligned} \tag{104}$$

  $\mathbf{J}_{i,j}$ are anti-symmetric matrices with nonzero entry of $(i,j)$ being one if $i<j$, i.e. $(\mathbf{J}_{i,j})_{kl} = \delta^i_k \delta^l_j - \delta^i_l \delta^j_k$,

where the Hamiltonian functionals $\mathcal{H}_1$ and $\mathcal{H}_2$ are

$$\mathcal{H}_1 = \frac{1}{2}\int \left(w^2 + \langle \mathbf{u}, \mathbf{u}\rangle\right) \mathrm{d}x,$$

$$\mathcal{H}_2 = \frac{1}{2}\int \left(w^3 + 3w\langle \mathbf{u}, \mathbf{u}\rangle - w_x^2 - \langle \mathbf{u}_x, \mathbf{u}_x\rangle\right) \mathrm{d}x.$$

**Theorem**

*The hierarchy* (102) *can be mapped into the hierarchy* (103) *for $n = 2$ by a Liouville transformation.*

(Kang, Liu, Qu, 2022)

# Conclusions and Discussions

- **Applications of Liouville transformations in orbital stability of solitons?**
- **Liouville transformations for discrete systems and their dual systems?**
- **Geometric formulations of Miura transformations (Qu, Wu, 2023)**
- **Geometric formulations of Liouville transformations?**

# Thank You!