ABV-packets and generic representations

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Based on joint work with Cunningham, Dijols, Fiori
and with Hazeltine, Liu, Lo

Outline

- F: a p-adic field; W_F its Weil group;
- G: a connected quasi-split reductive group over a p-adic field F;
- $\widehat{G}$: complex dual group of G and ${}^LG = \widehat{G} \rtimes W_F$ the L-group of G;
- A local Langlands parameter is a conjugacy class of group homomorphisms $\phi : W_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^LG$ satisfying certain properties.
- Equivalently, a local Langlands parameter is a conjugacy class of pairs (λ, N) where $\lambda : W_F \to {}^LG$ and $N \in \widehat{\mathfrak{g}}$ is a nilpotent element such that $\mathrm{Ad}(\lambda(w))N = |w|N$.
- λ is called the infinitesimal parameter of ϕ.

Arthur parameter

- An Arthur parameter is a homomorphism

$$\psi : W_F \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G$$

satisfying certain properties; (among others, one requires that $\psi|_{W_F}$ is bounded.)

- Associated with ψ, we consider the corresponding local parameter $\phi_\psi : W_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G$ by

$$\phi_\psi(w, x) = \psi\left(w, x, \begin{pmatrix} |w|^{1/2} & \\ & |w|^{-1/2} \end{pmatrix}\right).$$

- A local Langlands parameter ϕ is of Arthur type if $\phi = \phi_\psi$ for some ψ.

$$\Phi_{temp} \subset \Phi_{Arthur} \subset \Phi(G).$$

Arthur packet

- For classical groups, Arthur, Mok, et.al., constructed an Arthur packet $\Pi_\psi(G) \subset \mathrm{Irr}^{\mathrm{pure}}(G)$ and a group homomorphism $\Pi_\psi(G) \to \widehat{A_\psi} : \pi \mapsto \langle\ ,\pi\rangle_\psi$.
- Π_ψ should classify local components of L^2-automorphic representation (generalized Ramanujan).
- $\Pi_{\phi_\psi} \subset \Pi_\psi$ and $\Pi_\psi \to \widehat{A}_\psi$ is compatible with $\Pi_{\phi_\psi} \to \widehat{A}_{\phi_\psi}$.
- $$\Theta_\psi := \sum_{\pi \in \Pi_\psi} \langle s_\psi, \pi\rangle_\psi \Theta_\pi$$
  is stable.
- Each representation in Π_ψ is unitary.

- Π_ψ is characterized by endoscopic character identity.

Problem

Find alternative description of Π_ψ; or even better, generalize it to other groups and L-parameters of non-Arthur type.

- Moeglin, Bin Xu, Atobe gave quite concrete description of Π_ψ for $\mathrm{Sp}_{2n}, \mathrm{SO}_{2n+1}\dots$
- Adams, Barbasch and Vogan proposed a geometric construction of Π_ψ for any real reductive group in 1992. Adams, Arancibia, and Mezo proved the equivalence of those ABV-packets and Arthur packets in 2021.
- Inspired by ABV's work, Cunningham et.al proposed a geometric construction of Π_ψ for p-adic reductive groups.

Construction of ABV

- Let $\phi : W_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G$ be a local Langlands parameter and $\lambda_\phi : W_F \to {}^L G$ be its infinitesimal parameter.
- Fix a $\lambda : W_F \to {}^L G$, the set $\{\phi \in \Phi(G) : \lambda_\phi = \lambda\}$ is parametrized by

$$V_\lambda := \{x \in \widehat{\mathfrak{g}} : \mathrm{Ad}(\lambda(w))x = |w|x, \forall w \in W_F\}$$
$$= \left\{x \in \widehat{\mathfrak{g}}^{I_F} : \mathrm{Ad}(\lambda(\mathrm{Fr})) = q^{-1}x\right\}.$$

- Let $H_\lambda = \left\{g \in \widehat{G} : g\lambda(w) = \lambda(w)g, \forall w \in W_F\right\}$. Then H_λ acts on V_λ by conjugation and

$$\{\phi \in \Phi(G) : \lambda_\phi = \lambda\} / equivalence \cong V_\lambda /\!\!/ H_\lambda.$$

- V_λ is called **Vogan variety** of λ.

- The action of H_λ on V_λ has finite number of orbits, a unique closed orbit (zero orbit C_0) and a unique open orbit (denoted by C^o).
- Fix notation $\{\phi : \lambda_\phi = \lambda\} \ni \phi \leftrightarrow C_\phi \in V_\lambda /\!\!/ H_\lambda$ and $\phi_C \leftrightarrow C$.
- For an orbit C, let $A_C = \pi_0(Z_{H_\lambda}(x))$ for any $x \in C$.
- One has $A_C \cong A_{\phi_C} := \pi_0(Z_{\widehat{G}}(\phi))$.
- Characters of A_C classifies H_λ-equivariant local systems on C, namely, $\mathrm{Rep}(A_C) \cong \mathrm{Loc}_{H_\lambda}(C)$.
- Associated with each C, one can define a group A_C^{ABV}, whose characters classifies H-equivariant local system on some space Λ_C^{sreg}.
- If ϕ_C is of Arthur type, say $\phi_C = \phi_\psi$, then $A_C^{\mathrm{ABV}} = A_\psi = \pi_0(Z_{\widehat{G}}(\psi))$.

- Let Rep_λ be the category of smooth admissible representations of $G(F)$ and its pure inner forms with infinitesimal character λ. Then

$$\mathrm{Rep}_\lambda^{\textit{simple}} \cong \coprod_{\phi:\lambda_\phi=\lambda} \Pi_\phi^{\mathrm{pure}}(G).$$

- LLC gives a bijection

$$\iota : \Pi_\phi^{\textit{pure}} \to \widehat{A_\phi},$$

and thus

$$\iota : \mathrm{Rep}_\lambda^{\textit{simple}} \to \coprod_{\phi:\lambda_\phi=\lambda} \widehat{A_\phi}.$$

- ι might not be unique; it depends on Whittaker datum.

- Let $\mathrm{Per}_{H_\lambda}(V_\lambda)$ be the category of H_λ-equivariant perverse sheaves on V_λ.
- For an orbit C and a local system $\mathcal{L} \in \mathrm{Loc}_{H_\lambda}(V_\lambda)$, one can associate a simple perverse sheaf $\mathcal{IC}(C, \mathcal{L}) \in \mathrm{Per}_{H_\lambda}(V_\lambda)$.
- There is a bijection

$$\coprod_{C \in V_\lambda /\!\!/ H_\lambda} \widehat{A_C} \to \coprod_{C \in V_\lambda /\!\!/ H_\lambda} \mathrm{Loc}_{H_\lambda}(C)^{simple} \to \mathrm{Per}_{H_\lambda}(V_\lambda)^{simple}$$

- Since $\phi \leftrightarrow C_\phi$ defines a bijection $\{\phi : \lambda_\phi = \lambda\} / \textit{equivalence} \to V_\lambda /\!\!/ H_\lambda$ and $A_\phi \cong A_{C_\phi}$, there is a bijection

$$\mathcal{P}_\iota : \mathrm{Rep}_\lambda^{simple} \to \mathrm{Per}_{H_\lambda}(V_\lambda)^{simple}.$$

ABV-packets

- For each orbit $C \in V_\lambda /\!\!/ H_\lambda$, Cunningham et.al. defined a functor

$$\mathrm{NEv}_C : \mathrm{Per}_{H_\lambda}(V_\lambda) \to \mathrm{Loc}_{H_\lambda}(\Lambda_C^{sreg}) \cong \widehat{A_C^{\mathrm{ABV}}}.$$

Definition of ABV-packets

$$\Pi_{\phi,\iota}^{\mathrm{ABV}}(G) = \left\{ \pi \in \mathrm{Rep}_\lambda^{simple} : \mathrm{NEv}_{C_\phi}(\mathcal{P}_\iota(\pi)) \neq 0 \right\}.$$

- There is a natural map

$$\Pi_{\phi,\iota}^{\mathrm{ABV}}(G) \to \widehat{A_\phi^{\mathrm{ABV}}} : \pi \mapsto \langle\ ,\pi\rangle_\phi^{\mathrm{ABV}}.$$

ABV-packets

Basic expectations

- $\Pi^{\mathrm{ABV}}_{\phi,\iota}(G)$ is independent of ι.
- For classical group G and Langlands parameter of Arthur type $\phi = \phi_\psi$, one should have $\Pi_\psi(G) = \Pi^{\mathrm{ABV}}_\phi(G)$, and the map $\Pi^{\mathrm{ABV}}_\phi \to \widehat{A^{\mathrm{ABV}}_\phi}$ should be the same as $\Pi_\psi \to \widehat{A_\psi}$ of Arthur, i.e., $\langle\ ,\pi\rangle^{\mathrm{ABV}}_{\phi_\psi} = \langle\ ,\pi\rangle_\psi$.
- For general G and ϕ, Π^{ABV}_ϕ should behave pretty much like Arthur packets.

Remark

Recently, Cunningham-Ray established the above conjecture for GL_n.

- Given a reductive group G of adjoint type, Kazhdan-Lusztig and Lusztig established LLC for unipotent representations, which are parametrized by unramified local Langlands parameters.
- This was generalized to arbitrary group by Solleveld recently.
- In particular, we know the LLC for unipotent representations of the exceptional group G_2.
- Recently, LLC for G_2 was established by Aubert-Xu, and by Gan-Savin independently.
- Joint with Cunningham and Fiori, we explicated the ABV-packets for unipotent representations of G_2.

Theorem (Cunningham-Fiori-Z)

For each $\phi \in \Phi_{ur}(G_2)$, there exists an explicit finite set Π_ϕ^{ABV} of irreducible unipotent representations of $G_2(F)$ and a map $\Pi_\phi^{\mathrm{ABV}} \to \widehat{A_\phi^{\mathrm{ABV}}}$ such that they behave pretty much like the Arthur packets.

Remark

Gan-Gurevich-Jiang, Gan-Gurevich, Alonso-He-Ray-Roset constructed A-packets of G_2 using exceptional theta correspondence.

Here are some examples about the last sentence.

- We have a commutative diagram

$$\begin{array}{ccc} \Pi_\phi & \xrightarrow{\text{LLC}} & \widehat{A_\phi} \\ \downarrow & & \downarrow \\ \Pi_\phi^{\text{ABV}} & \longrightarrow & \widehat{A_\phi^{\text{ABV}}} \end{array}$$

- If ϕ is open, then the map $\Pi_\phi^{\text{ABV}} \to \widehat{A_\phi^{\text{ABV}}}$ is a bijection.
- A representation π is unramified (spherical), if and only if ϕ is closed and $\langle\ ,\pi\rangle$ is trivial on A_ϕ^{ABV}.

- For each $\phi \in \Phi_{ur}(G_2(F))$, consider the distribution

$$\Theta_\phi = \sum_{\pi \in \Pi_\phi^{\mathrm{ABV}}} (-1)^{\dim(C_\phi) - \dim(C_\pi)} \langle 1, \pi \rangle \Theta_\pi.$$

  If Θ_ϕ is stable for all elliptic $\phi \in \Phi_{ur}(G_2(F))$, then Θ_ϕ is stable for all $\phi \in \Phi_{ur}(G_2(F))$.
- For each $\phi \in \Phi_{ur}(G_2(F))$, there exists a unique $\hat{\phi} \in \Phi_{ur}(G_2(F))$ such that the Aubert involution defines a bijection

$$\Pi_\phi^{\mathrm{ABV}} \to \Pi_{\hat{\phi}}^{\mathrm{ABV}}.$$

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- For each $\phi \in \Phi_{ur}(G_2(F))$, there exists a unique $\hat{\phi} \in \Phi_{ur}(G_2(F))$ such that the Aubert involution defines a bijection

$$\Pi_\phi^{\mathrm{ABV}} \to \Pi_{\hat{\phi}}^{\mathrm{ABV}}.$$

Remark

One should be able to extend the above construction to general $\phi \in \Phi(G_2)$ after the recent proof of LLC for G_2.

One explicit example

Let $\phi : W_F \times \mathrm{SL}_2(\mathbb{C}) \to \widehat{G_2}$ be the unramified local Langlands parameter corresponding to the subregular unipotent conjugacy class and let $\lambda = \lambda_\phi$.

- $V_\lambda = \{a_0x^3 + a_1x^2y + a_2xy^2 + a_3y^3 : a_i \in \mathbb{C}\}$;
- $H_\lambda = \mathrm{GL}_2(\mathbb{C})$;
- The action of H_λ on V_λ is a twisting of Sym^3;
- There are 4 orbits; denoted by $\phi_0, \phi_1, \phi_2, \phi_3$, all of which are of Arthur type, say given by ψ_i.

Gan-Gurevich-Jiang gave a construction of Π_{ψ_i} using theta correspondence.

Proposition (Cunningham-Fiori-Z)

The ABV-packets $\Pi^{\mathrm{ABV}}_{\phi_i}$ agree with Π_{ψ_i} constructed by Gan-Gurevich-Jiang.

Assuming G is quasi-split.

Definition

An L-parameter ϕ is called generic if Π_ϕ contains a generic representation.

Conjecture: Gross-Prasad, Rallis

A parameter ϕ is generic iff $L(s, \phi, \mathrm{Ad})$ is regular at $s = 1$.

This conjecture was checked for many different groups by various authors. Gan-Ichino proved the above conjecture under certain assumptions, which were known to be true for classical groups.

Proposition (Cunningham, Dijois, Fiori, Z.)

The parameter ϕ is open in V_λ with $\lambda = \lambda_\phi$ iff $L(s, \phi, \mathrm{Ad})$ is regular at $s = 1$.

Definition

A parameter ϕ is called open if ϕ is open in V_λ.

Corollary (Geometric description of generic parameters)

Assuming the conjecture of Gross-Prasad, Rallis. Then ϕ is generic iff ϕ is open.

Assuming $\phi = \phi^o$ is open, one could fix the LLC such that $\pi(\phi^o, 1)$ is the generic representation (w.r.t a fixed Whittaker datum), where $1 \in \widehat{A}_{\phi^o}$ is the trivial character.

Conjecture of Gross-Prasad, Rallis is equivalent to:

Gross-Prasad, Rallis' conjecture

Π_ϕ contains a generic representation iff ϕ is open.

A generalization of the above is the following

Conjecture

Let ϕ be an L-parameter. Then Π_ϕ^{ABV} contains a generic representation iff ϕ is open.

The above conjecture is indeed a generalization of that of GP, Rallis, because $\Pi_\phi \subset \Pi_\phi^{\mathrm{ABV}}$ for general ϕ and $\Pi_\phi = \Pi_\phi^{\mathrm{ABV}}$ if ϕ is open.

Theorem (Cunningham, Dijois, Fiori, Z.)

If G is a quasi-split classical group, ϕ is an L-parameter of G, then Π_ϕ^{ABV} contains a generic representation iff ϕ is open.

Note that if ϕ is open, then $\Pi_\phi = \Pi_\phi^{\mathrm{ABV}}$ contains a generic representation by the conjecture of GP, Rallis. On the contrary, suppose that Π_ϕ^{ABV} contains a generic representation π^o. It is known that $\pi^o \in \Pi_{C^o}$ and it corresponds to the trivial character of C^o. Thus the above statement is equivalent to

$$\mathrm{NEv}_C(\mathbf{1}_{C^o}) \neq 0 \text{ iff } C = C^o.$$

Tempered parameters

- Recall that an L-parameter $\phi : W_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^LG$ is called tempered if $\phi|_{W_F}$ is bounded.
- If ϕ is tempered, then Π_ϕ consists of tempered representations.
- If ϕ is tempered, then ϕ is of Arthur type.

Proposition

An L-parameter is tempered iff ϕ is both of Arthur type and open.

Conjecture (Shahidi, 1990)

If ϕ is tempered, then Π_ϕ contains a generic representation.

The above conjecture has been checked by many authors. The following is an enhanced version.

Conjecture (Enhanced Shahidi's conjecture, 2021)

If G is a quasi-split classical group such that Arthur packets can be defined. Let ψ be an Arthur parameter, then Π_ψ contains a generic representation iff ψ is tempered, i.e., $\psi = \phi_\psi$ for a tempered L-parameter ϕ.

- Liu-Shahidi proved the enhanced Shahidi's conjecture for quasi-split classical groups under certain assumptions.
- Hazeltine-Liu-Lo proved the Enhanced Shahidi's conjecture for split SO_{2n+1} and Sp_{2n}.
- The Enhanced Shahidi's conjecture for quasi-split classical groups follows from the Vogan conjecture, namely, if $\psi = \phi_\psi$, then $\Pi_\phi^{ABV} = \Pi_\psi$. In this sense, the above conjecture for generic ABV-packets is a generalization of the Enhanced Shahidi's conjecture.

A new framework to solve the Enhanced Shahidi's conjecture

- For an Arthur parameter $\psi : W_F \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \to {}^LG$, define $\widehat{\psi}$ by $\widehat{\psi}(w, x, y) = \psi(w, y, x)$.
- For a representation π of $G(F)$, let $\widehat{\pi}$ be the Aubert dual of π.

Theorem (Hazeltine, Liu, Lo, Z.)

The Enhanced Shahidi's conjecture follows from the following 3 assumptions:

1. *if $\pi \in \Pi_\psi$, then $\overline{C_{\phi_\pi}} \supset C_{\phi_\psi}$;*
2. *for any Arthur parameter ψ, one has $\Pi_{\widehat{\psi}} = \{\widehat{\pi} : \pi \in \Pi_\psi\}$;*
3. *for any generic representation π^o, one has $\widehat{\pi^o} \in \Pi_{\phi_0}$, where ϕ_0 is the zero orbit in V_λ for $\lambda = \lambda_{\phi_{\pi^o}}$.*

- Assumption (2) for SO_n and Sp_{2n} was proved by Bin Xu.
- Assuming that generic parameters are open parameters, assumption (3) for SO_n and Sp_{2n} was checked by Arthur.
- Assumption (1) appeared as a conjecture in one of Bin Xu's paper. It reveals a very important geometric property of Arthur packets.

Proof of the above theorem: Given π, consider $\Psi(\pi) = \{\psi : \pi \in \Pi_\psi\}$. Assumption (1) says that if $\pi \in \Pi_{\phi_0}$, $\Psi(\pi) = \{\psi_0\}$ with $\phi_0 = \phi_{\psi_0}$. Assumption (2) says that there is a bijection $\Psi(\pi) \to \Psi(\widehat{\pi})$ by sending ψ to $\widehat{\psi}$. In particular, there is a bijection $\Psi(\pi^o) \to \Psi(\widehat{\pi}^o)$. Assumption (3) says that $\widehat{\pi^o} \in \Pi_{\phi_0}$, which implies that $\Psi(\pi^o)$ is a singleton, which must be $\widehat{\psi}_0$. The above discussion shows that $\widehat{\psi}_0$ is tempered.

Theorem (Hazeltine, Liu, Lo, Z.)

For split SO_{2n+1} and Sp_{2n}, the above assumption (1) is true. Thus the Enhanced Shahidi's conjecture is true for split SO_{2n+1} and Sp_{2n}.

The proof uses the explicit construction of Arthur packets of these groups due to Mœglin, Bin Xu, and then further refined by Atobe, Hazeltine-Liu-Lo.

Conjecture (Kazhdan-Lusztig)

For each $\pi \in (\mathrm{Rep}_\lambda(G))^{\mathrm{simple}}$, there exists a perverse sheaf $\mathcal{M}_\pi \in Per_H(V)$ such that for each pair π, π', we have

$$\mathrm{multiplicity}(\pi', M_\pi) = \mathrm{multiplicity}(\mathcal{M}_\pi, \mathcal{P}(\pi')).$$

Here M_π is the standard module of π.

Fix a λ. Consider an orbit C and the open orbit C^o. The geometric multiplicity

$$\mathrm{multiplicity}(\mathcal{M}_{\pi(\phi_C,1)}, \mathcal{P}(\pi(\phi_{C^o}, 1)))$$

could be easily determined.

Proposition

- $\mathrm{multiplicity}(\mathcal{M}_{\pi(\phi_C,1)}, \mathcal{P}(\pi(\phi_{C^o}, 1))) = 1$;
- *For any* $\rho \in \widehat{A_{\phi_{C^o}}}$, $\mathcal{M}_{\pi(\phi_{C^o},\rho)} = \mathcal{P}(\pi(\phi_{C^o}, \rho))$.

Corollary

Assuming Kazhdan-Lusztig. Let π^o *be a generic representation with L-parameter* ϕ^o.

- *If* ϕ *is an L-parameter with* $\lambda_\phi = \lambda_{\phi^o}$, *then*

$$\langle M_{\pi(\phi,1)}, \pi^o \rangle = 1.$$

- *For any* $\pi \in \Pi_{\phi^o}$, M_π *is irreducible.*

The last one is a generalization of Casselman-Shahidi's standard module conjecture, which was proved by Heiermann-Opdam. This generalized version was proved by Heiermann unconditionally.

Central character

A consequence of the first statement of the above corollary is

Corollary

Assuming Kazhdan-Lusztig. We have

$$\omega_{\pi(\phi,1)} = \omega_{M_{\pi(\phi,1)}} = \omega_{\pi^o}.$$

For any ϕ, it is expected that $\omega_{\pi_1} = \omega_{\pi_2}$ for any $\pi_1, \pi_2 \in \Pi_\phi$. Denote this character by ω_ϕ.

Corollary

- *If $\lambda_{\phi_1} = \lambda_{\phi_2} = \lambda$, we have $\omega_{\phi_1} = \omega_{\phi_2}$, which is denoted by ω_λ.*
- *For any $\pi_1, \pi_2 \in \mathrm{Rep}_\lambda$, we should have $\omega_{\pi_1} = \omega_{\pi_2} = \omega_\lambda$. In particular, representations in Π_ϕ^{ABV} have the same central character.*

Thank you!