On Shafarevich problems for hyper-Kähler varieties

Geometry of Hyper-Kähler Varieties

Haitao Zou

2023/09/08

Table of Contents

Hyper-Kähler variety

- Let $K \subset \mathbb{C}$ be a field.

Hyper-Kähler variety

- Let $K \subset \mathbb{C}$ be a field.

Definition

A *hyper-Kähler variety* X is a smooth projective variety over K such that

- $\pi_1^{\mathrm{et}}(X_{\bar{K}}) = \{1\}$.
- $\mathrm{H}^0(X_{\bar{K}}, \Omega^2_{X_{\bar{K}}})$ is generated a nowhere degenerate closed 2-form $\sigma\colon \mathcal{O}_X \to \Omega^2_X$.

Hyper-Kähler variety

1 Introduction

- Let $K \subset \mathbb{C}$ be a field.

Definition

A *hyper-Kähler variety* X is a smooth projective variety over K such that

- $\pi_1^{\mathrm{et}}(X_{\bar{K}}) = \{1\}$.
- $\mathrm{H}^0(X_{\bar{K}}, \Omega^2_{X_{\bar{K}}})$ is generated a nowhere degenerate closed 2-form $\sigma \colon \mathcal{O}_X \to \Omega^2_X$.

Remark

- X is HK if and only if $X_{\mathbb{C}}$ is complex hyper-Kähler variety.

- Let $K \subset \mathbb{C}$ be a field.

Definition

A *hyper-Kähler variety* X is a smooth projective variety over K such that

- $\pi_1^{\text{et}}(X_{\bar{K}}) = \{1\}$.
- $\mathrm{H}^0(X_{\bar{K}}, \Omega^2_{X_{\bar{K}}})$ is generated a nowhere degenerate closed 2-form $\sigma \colon \mathcal{O}_X \to \Omega^2_X$.

Remark

- X is HK if and only if $X_{\mathbb{C}}$ is complex hyper-Kähler variety.
- For HKs of known types, its deformation is independent of the embedding $K \hookrightarrow \mathbb{C}$.

Finiteness of HKs over number fields

- Let K be a number field, and fixed an embedding $K \hookrightarrow \mathbb{C}$.

Shafarevich problem

$\mathcal{M}$ is a set of hyper-Kähler varieties defined over K. Is the following set finite?

$$\mathrm{Shaf}_{\mathcal{M}}(K, S) = \left\{ \begin{array}{c} K\text{-isomorphism classes of varieties in} \\ \mathcal{M} \text{ defined over } K \\ (\star) \text{ with good reduction outside } S \end{array} \right\}$$

- Let K be a number field, and fixed an embedding $K \hookrightarrow \mathbb{C}$.

Shafarevich problem

$\mathcal{M}$ is a set of hyper-Kähler varieties defined over K. Is the following set finite?

$$\mathrm{Shaf}_{\mathcal{M}}(K,S) = \left\{ \begin{array}{c} K\text{-isomorphism classes of varieties in} \\ \mathcal{M} \text{ defined over } K \\ (\star) \text{ with good reduction outside } S \end{array} \right\}$$

- $(\star)$ is called the Shafarevich condition, which corresponds to the existence of integral points "inside" Shimura variety.

Finiteness of HKs over number fields

- Let K be a number field, and fixed an embedding $K \hookrightarrow \mathbb{C}$.

Shafarevich problem

$\mathcal{M}$ is a set of hyper-Kähler varieties defined over K. Is the following set finite?

$$\mathrm{Shaf}_{\mathcal{M}}(K, S) = \left\{ \begin{array}{c} K\text{-isomorphism classes of varieties in} \\ \mathcal{M} \text{ defined over } K \\ (\star) \text{ with good reduction outside } S \end{array} \right\}$$

- $(\star)$ is called the Shafarevich condition, which corresponds to the existence of integral points "inside" Shimura variety.
- the Shafarevich condition can be replaced by other weaker conditions, e.g., cohomological Shafarevich condition

Shafarevich conjecture for curves

Theorem (Faltings, '83)

Let S be a finite set of places of a number field K, the Shafarevich set

$$\left\{ C \;\middle|\; \begin{array}{l} \text{Curve } C \text{ has genus } g \text{ defined over } K, \\ \text{with } \underline{\text{good reduction}} \text{ outside } S \end{array} \right\} / \cong_F .$$

is finite.

Shafarevich conjecture for curves

Theorem (Faltings, '83)

Let S be a finite set of places of a number field K, the Shafarevich set

$$\left\{ C \;\middle|\; \begin{array}{l} \text{Curve } C \text{ has genus } g \text{ defined over } K, \\ \text{with } \underline{\text{good reduction}} \text{ outside } S \end{array} \right\} / \cong_F .$$

is finite.

- Mordell Conjecture:

$$\#C(K) < \infty \quad \text{if } g(C) \geq 2$$

  is a direct consequence from the Shafarevich conjecture by Kodaira–Paršhin construction:

$$C(K) \leftrightarrow \{D_p \to C \text{ finite of degree } 2^{2g} \text{ ramified exactly at } p \in C(K)\}$$

Guideline in higher dimensions

- K: a number field

Guideline in higher dimensions

- K: a number field

Bombieri–Lang Conjecture

If smooth projective variety X over K is of general type, then $X(K)$ is not Zariski dense.

Guideline in higher dimensions

- K: a number field

Bombieri–Lang Conjecture

If smooth projective variety X over K is of general type, then $X(K)$ is not Zariski dense.

Remark

- The relationship between finiteness of rational (integral) points in a (quasi-)projective variety and its geometry is predicted by Lang–Vojta conjectures:

Guideline in higher dimensions

- K: a number field

Bombieri–Lang Conjecture

If smooth projective variety X over K is of general type, then $X(K)$ is not Zariski dense.

Remark

- The relationship between finiteness of rational (integral) points in a (quasi-)projective variety and its geometry is predicted by Lang–Vojta conjectures:

hyperbolicity $\Leftrightarrow$ finiteness of rational (integral) points

Guideline in higher dimensions

- K: a number field

Bombieri–Lang Conjecture

If smooth projective variety X over K is of general type, then $X(K)$ is not Zariski dense.

Remark

- The relationship between finiteness of rational (integral) points in a (quasi-)projective variety and its geometry is predicted by Lang–Vojta conjectures:

hyperbolicity $\Leftrightarrow$ finiteness of rational (integral) points

- There are also stacky version Lang–Vojta conjectures.

Examples of known results

- Faltings, Zarhin: Abelian varieties of a fixed dimension g

Examples of known results

- Faltings, Zarhin: Abelian varieties of a fixed dimension g
- André: hyper-Kähler varieties (with $b_2 > 3$) of given dimension with a very ample polarization of bounded degree;

Examples of known results

- Faltings, Zarhin: Abelian varieties of a fixed dimension g
- André: hyper-Kähler varieties (with $b_2 > 3$) of given dimension with a very ample polarization of bounded degree;
- Lawrence–Sawin: hypersurfaces in a fixed abelian variety

Examples of known results

- Faltings, Zarhin: Abelian varieties of a fixed dimension g
- André: hyper-Kähler varieties (with $b_2 > 3$) of given dimension with a very ample polarization of bounded degree;
- Lawrence–Sawin: hypersurfaces in a fixed abelian variety
- Y. She, Takamatsu: unpolarized K3 surfaces

Examples of known results

- Faltings, Zarhin: Abelian varieties of a fixed dimension g
- André: hyper-Kähler varieties (with $b_2 > 3$) of given dimension with a very ample polarization of bounded degree;
- Lawrence–Sawin: hypersurfaces in a fixed abelian variety
- Y. She, Takamatsu: unpolarized K3 surfaces
- Takamatsu: Enrique surfaces

Examples of known results

- Faltings, Zarhin: Abelian varieties of a fixed dimension g
- André: hyper-Kähler varieties (with $b_2 > 3$) of given dimension with a very ample polarization of bounded degree;
- Lawrence–Sawin: hypersurfaces in a fixed abelian variety
- Y. She, Takamatsu: unpolarized K3 surfaces
- Takamatsu: Enrique surfaces
- $\cdots$

Table of Contents

Shafarevich conjecture for HK (after André and She)

- Let M be a geometric deformation type of hyper-Kähler variety.
- Let K be a number field, S a finite set of places of K.

Shafarevich conjecture for HK (after André and She)

- Let M be a geometric deformation type of hyper-Kähler variety.
- Let K be a number field, S a finite set of places of K.

Theorem (Fu–Li–Takamatsu–Zou, '22)

If $b_2(M) \geq 4$, then the following Shafarevich set is finite.

$$\mathrm{Shaf}_M(K, S) = \left\{ X \;\middle|\; \begin{array}{l} X \text{ is hyper-Kähler variety defined over } K \text{, which is} \\ \quad 1.\ \text{geometrically deformation equivalent to } M \\ \quad 2.\ \text{with good reduction outside } S. \end{array} \right\}$$

Kuga–Satake construction

- $\mathrm{HK}_d^\dagger$ be a fixed geometrically connected component of HK_d,

Kuga–Satake construction

- $\mathrm{HK}_d^\dagger$ be a fixed geometrically connected component of HK_d,
- $(X, h) \in \mathrm{HK}_d^\dagger(\mathbb{C})$ a fixed $\mathbb{C}$-point

Kuga–Satake construction

- $\mathrm{HK}_d^\dagger$ be a fixed geometrically connected component of HK_d,
- $(X, h) \in \mathrm{HK}_d^\dagger(\mathbb{C})$ a fixed $\mathbb{C}$-point
- $\Lambda_h := h^\perp \subset \mathrm{H}^2(X, \mathbb{Z})$.

Theorem (Deligne '72, André '96, Rizov '06, Madapusi Pera '15, Bindt '21)

For any integer d > 0, There is following diagram

$$\begin{array}{ccccc} & & & & \mathrm{Sh}(\mathrm{CSpin}(\Lambda_h)) \\ & & & \overset{\mathrm{ad}}{\swarrow} \; \underset{\gamma}{\nearrow} & \downarrow \mathrm{sp} \\ \mathrm{HK}_d^\dagger & \xrightarrow{\mathcal{P}} & \mathrm{Sh}(\mathrm{SO}(\Lambda_h)) & & \mathrm{Sh}(\mathrm{GSp}(V)) \cong \mathcal{A}_g \end{array}$$

in which γ is defined over a number field E and others are defined over $\mathbb{Q}$.

- Strictly speaking, here $\mathrm{HK}_d^\dagger$ should be replaced by a double-covering.

Main idea of proof

- Let $\mathrm{HK}_d^{\dagger}$ be a fixed geometrically connected component of HK_d.

Main idea of proof

- Let $\mathrm{HK}_d^{\dagger}$ be a fixed geometrically connected component of HK_d.

Uniform Kuga–Satake for HK

There are integers n, N, and a quasi-finite morphism

$$\mathrm{HK}_{d,n}^{\dagger} \to A_{g,N}$$

defined over a number field E, with E, n, N, g independent with d.

Main idea of proof

- Let $\mathrm{HK}_d^\dagger$ be a fixed geometrically connected component of HK_d.

Uniform Kuga–Satake for HK

There are integers n, N, and a quasi-finite morphism

$$\mathrm{HK}_{d,n}^\dagger \to A_{g,N}$$

defined over a number field E, with E, n, N, g independent with d.

Finiteness of Picard lattices

- Serre–Tate's good reduction theorem for abelian varieties
- Faltings' finiteness theorem for abelian varieties
- If (X, h) and (X', h') have same image in $A_{g,N}$, then $T(X_\mathbb{C}) \simeq T(X'_\mathbb{C})$ Hodge isometry.

Main idea of proof

- Let $\mathrm{HK}_d^\dagger$ be a fixed geometrically connected component of HK_d.

Uniform Kuga–Satake for HK

There are integers n, N, and a quasi-finite morphism

$$\mathrm{HK}_{d,n}^\dagger \to A_{g,N}$$

defined over a number field E, with E, n, N, g independent with d.

Finiteness of Picard lattices

- Serre–Tate's good reduction theorem for abelian varieties
- Faltings' finiteness theorem for abelian varieties
- If (X, h) and (X', h') have same image in $A_{g,N}$, then $T(X_{\mathbb{C}}) \simeq T(X'_{\mathbb{C}})$ Hodge isometry.

Birational Geometry of HK

- the elements in Shafarevich set form a finite set up to K-birational replacements;
- Kawamata–Morrison's Cone Conjecture over K
- bounded square of exceptional classes

Cohomological Shafarevich conjectures for HK

Cohomological Shafarevich conjectures for HK

- “good reduction outside S ”$\Rightarrow$ ”$\mathrm{H}^*_{\mathrm{et}}(X_{\bar{K}}, \mathbb{Q}_\ell)$ are unramified as Galois modules”

Cohomological Shafarevich conjectures for HK

- “good reduction outside S ”$\Rightarrow$ ”$\mathrm{H}^*_{\mathrm{et}}(X_{\bar{K}}, \mathbb{Q}_\ell)$ are unramified as Galois modules”
- We can replace the Shafarevich condition by cohomological Shafarevich conditions.

Cohomological Shafarevich conjectures for HK

- "good reduction outside S "$\Rightarrow$ "$\mathrm{H}^*_{\text{et}}(X_{\bar{K}}, \mathbb{Q}_\ell)$ are unramified as Galois modules"
- We can replace the Shafarevich condition by cohomological Shafarevich conditions.

Theorem (Fu–Li–Takamatsu–Zou, '22)

If $M = K3^{[n]}$-type, generalized Kummer, OG6 or OG10, then the following cohomological Shafarevich set is finite

$$\mathrm{Shaf}_M^{hom}(K, S) = \left\{ X \,\middle|\, \begin{array}{l} X \text{ is hyper-Kähler variety defined over } K \text{, which is} \\ \quad 1.\ \text{geometrically deformation-equivalent to } M \\ \quad 2.\ \mathrm{H}^*_{\text{et}}(X_{\bar{K}}, \mathbb{Q}_\ell) \text{ are unramified outside } S \text{ as} \\ \quad\quad \text{Galois modules.} \end{array} \right\}$$

Cohomological Shafarevich conjectures for HK

- “good reduction outside S ”$\Rightarrow$ ”$\mathrm{H}^*_{\mathrm{et}}(X_{\bar{K}}, \mathbb{Q}_\ell)$ are unramified as Galois modules”
- We can replace the Shafarevich condition by cohomological Shafarevich conditions.

Theorem (Fu–Li–Takamatsu–Zou, ’22)

If $M = \mathrm{K3}^{[n]}$-type, generalized Kummer, OG6 or OG10, then the following cohomological Shafarevich set is finite

$$\mathrm{Shaf}_M^{hom}(K, S) = \left\{ X \;\middle|\; \begin{array}{l} X \text{ is hyper-Kähler variety defined over } K \text{, which is} \\ \quad 1.\ \text{geometrically deformation-equivalent to } M \\ \quad 2.\ \mathrm{H}^*_{\mathrm{et}}(X_{\bar{K}}, \mathbb{Q}_\ell) \text{ are unramified outside } S \text{ as} \\ \qquad \text{Galois modules.} \end{array} \right\}$$

- ▶ Liedtke–Matsumoto, Yang: For K3 surfaces, at prime p big enough (e.g., $p > 36$):

unramifiedness $\Leftrightarrow$ reduction with at worst ADE singularities

Cohomological Shafarevich condition

K3 surface (Takamatsu, ’18)

$\mathrm{H}^2_{\mathrm{et}}(X_{\bar{K}}, \mathbb{Q}_\ell)$ being unramified.

K3$^{[n]}$-type, OG10

$\mathrm{H}^2_{\mathrm{et}}(X_{\bar{K}}, \mathbb{Q}_\ell)$ being unramified.

generalized Kummer, OG6

$\mathrm{H}^*_{\mathrm{et}}(X_{\bar{K}}, \mathbb{Q}_\ell)$ are all unramified.

Cohomological Shafarevich condition

K3 surface (Takamatsu, '18)	$K3^{[n]}$-type, OG10	generalized Kummer, OG6
$H^2_{et}(X_{\bar{K}}, \mathbb{Q}_\ell)$ being unramified.	$H^2_{et}(X_{\bar{K}}, \mathbb{Q}_\ell)$ being unramified.	$H^*_{et}(X_{\bar{K}}, \mathbb{Q}_\ell)$ are all unramified.

- It depends on whether $\mathrm{Aut}(X_{\bar{K}})$ acts faithfully on $\bigoplus_{i \in I} H^i_{et}(X_{\bar{K}}, \mathbb{Q}_\ell)$ for some index set I.

Cohomological Shafarevich condition

K3 surface (Takamatsu, '18)	$K3^{[n]}$-type, OG10	generalized Kummer, OG6
$\mathrm{H}^2_{\mathrm{et}}(X_{\bar{K}}, \mathbb{Q}_\ell)$ being unramified.	$\mathrm{H}^2_{\mathrm{et}}(X_{\bar{K}}, \mathbb{Q}_\ell)$ being unramified.	$\mathrm{H}^*_{\mathrm{et}}(X_{\bar{K}}, \mathbb{Q}_\ell)$ are all unramified.

- It depends on whether $\mathrm{Aut}(X_{\bar{K}})$ acts faithfully on $\oplus_{i \in I} \mathrm{H}^i_{\mathrm{et}}(X_{\bar{K}}, \mathbb{Q}_\ell)$ for some index set I.
- For generalized Kummer and OG6, $\mathrm{Aut}(X_{\bar{K}})$ does NOT act faithfully on H^2.

Cohomological Shafarevich condition

K3 surface (Takamatsu, '18)	$K3^{[n]}$-type, OG10	generalized Kummer, OG6
$H^2_{et}(X_{\bar{K}}, \mathbb{Q}_\ell)$ being unramified.	$H^2_{et}(X_{\bar{K}}, \mathbb{Q}_\ell)$ being unramified.	$H^*_{et}(X_{\bar{K}}, \mathbb{Q}_\ell)$ are all unramified.

- It depends on whether $\mathrm{Aut}(X_{\bar{K}})$ acts faithfully on $\oplus_{i\in I} H^i_{et}(X_{\bar{K}}, \mathbb{Q}_\ell)$ for some index set I.
- For generalized Kummer and OG6, $\mathrm{Aut}(X_{\bar{K}})$ does NOT act faithfully on H^2.
- For any hyper-Kähler variety in dimension 4, cohomological Shafarevich conjecture holds ($H^*_{et}(X_{\bar{K}}, \mathbb{Q}_\ell)$ being unramified), by recent work of Cheng Jiang and Wenfei Liu.

Theorem (Fu–Li–Takamatsu–Zou)

- *Let k be an algebraically closed field in characteristic 0;*
- *$(C, 0)$ a pointed curve over k (i.e., $0 \in C$ a fixed closed point)*

The following set is finite.

$$\mathrm{Shaf}\,((C,0),X) = \left\{ f\colon \mathfrak{X} \to C \;\middle|\; \begin{array}{l} f \text{ is smooth proper family of hyper-} \\ \text{Kähler varieties over } k \text{ such that } \mathfrak{X}_0 = \\ f^{-1}(0) \cong X \end{array} \right\}$$

- $\mathfrak{X}$ can only be an algebraic space.

Recent progress: pointed Shafarevich conjecture

Theorem (Fu–Li–Takamatsu–Zou)

- *Let k be an algebraically closed field in characteristic 0;*
- *$(C, 0)$ a pointed curve over k (i.e., $0 \in C$ a fixed closed point)*

The following set is finite.

$$\mathrm{Shaf}\left((C,0),X\right) = \left\{ f\colon \mathfrak{X} \to C \;\middle|\; \begin{array}{l} f \text{ is smooth proper family of hyper-} \\ \text{Kähler varieties over } k \text{ such that } \mathfrak{X}_0 = \\ f^{-1}(0) \cong X \end{array} \right\}$$

- $\mathfrak{X}$ can only be an algebraic space.
- For pointed Shafarevich conjecture for polarized HK (i.e., f is projective), this is a direct consequence of the hyperbolicity of the moduli stack of polarized HKs.

Further remarks

Further remarks

- Does cohomological Shafarevich conjecture hold for all HK? It is sufficient to show $\mathrm{Aut}(X)$ acts on the cohomology ring $\mathrm{H}^*(X)$ trivially.

Further remarks

- Does cohomological Shafarevich conjecture hold for all HK? It is sufficient to show $\mathrm{Aut}(X)$ acts on the cohomology ring $\mathrm{H}^*(X)$ trivially.
- For higher dimensional HK variety X, $\mathrm{H}^*_{\mathrm{et}}(X_{\bar{K}}, \mathbb{Q}_\ell)$ being unramified $\Rightarrow$?

Further remarks

- Does cohomological Shafarevich conjecture hold for all HK? It is sufficient to show $\mathrm{Aut}(X)$ acts on the cohomology ring $\mathrm{H}^*(X)$ trivially.
- For higher dimensional HK variety X, $\mathrm{H}^*_{\mathrm{et}}(X_{\bar{K}}, \mathbb{Q}_\ell)$ being unramified $\Rightarrow$?
- If derived equivalences preserves deformation type of HK varieties, then

cohomological Shafarevich conjecture $\Rightarrow$ finiteness of FM partners over K.

On Shafarevich problems for hyper-Kähler varieties

Thank you for listening!

Any questions?