Birational geometry on moduli space of polarized K3 surfaces of low genus

Geometry of hyperKähler varieties

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I. Introduction

Moduli space of curves

- The (coarse) moduli space of smooth curves of genus $g \geq 2$

$$\mathscr{M}_g = \Big\{ \Sigma_g :$$

$$\Big\} / \cong$$

is a non-compact scheme of dimension $3g - 3$.

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- **Mumford, Giesker:**

$$\overline{\mathscr{M}}_g \cong \mathtt{Chow}_{g,n\geq 5} /\!\!/ \mathrm{SL}((2n-1)(g-1)) \cong \mathtt{Hilb}_{g,n\geq 5} /\!\!/ \mathrm{SL}((2n-1)(g-1))$$

  where
  - $\mathtt{Hilb}_{g,n}$: Hilbert scheme of n-canonically embedded curves of genus g
  - $\mathtt{Chow}_{g,n}$: Chow variety of n-canonically embedded curves of genus g.

Other compactifications

Allowing worse singularities

- $\overline{\mathscr{M}}_g^{ps} = \mathscr{M}_g \cup \left\{ \text{allowing cusp, no elliptic tails} \right\}$
- $\overline{\mathscr{M}}_g^{cs} = \mathscr{M}_g \cup \left\{ \text{allowing tacnode, no elliptic bridges} \right\}$
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Varying GIT models (Schubert, Hassett-Heyon)

- $\mathtt{Chow}_{g,3} /\!\!/ \mathrm{SL}(5g-5) \cong \mathtt{Chow}_{g,4} /\!\!/ \mathrm{SL}(7g-7) \cong \overline{\mathscr{M}}_g^{ps}$
- $\mathtt{Chow}_{g,2} /\!\!/ \mathrm{SL}(3g-3) \cong \overline{\mathscr{M}}_g^{cs}$
- $\mathtt{Hilb}_{g,2} /\!\!/ \mathrm{SL}(7g-7) \cong \overline{\mathscr{M}}_g^{hs}$

Their relations via LMMP

Hassett-Keel Program (Hassett-Heyon)

- **BCHM**: Let δ_g be the boundary divisor of $\overline{\mathscr{M}}_g$. The log canonical model
$$\overline{\mathscr{M}}_g(\alpha) := \mathbf{Proj} \bigoplus_{m \geq 0} \mathrm{H}^0(\overline{\mathscr{M}}_g, m(K_{\overline{\mathscr{M}}_g} + \alpha\delta_g))$$
exists for $\alpha \in [0, 1]$.
- $\overline{\mathscr{M}}_g(\alpha > \frac{9}{11}) \cong \overline{\mathscr{M}}_g$
- $\overline{\mathscr{M}}_g(\frac{7}{10} < \alpha \leq \frac{9}{11}) \cong \overline{\mathscr{M}}_g^{ps}$ and $\overline{\mathscr{M}}_g(1) \to \overline{\mathscr{M}}_g(\frac{9}{11})$ is a divisorial contraction
- $\overline{\mathscr{M}}_g(\frac{7}{10}) \cong \mathtt{Chow}_2 /\!\!/ \mathrm{SL}(3g-3)$ and $\overline{\mathscr{M}}_g(\frac{7}{10} - \epsilon) \cong \mathtt{Hilb}_2 /\!\!/ \mathrm{SL}(3g-3)$ and there is a flip

$$\overline{\mathscr{M}}_g(\tfrac{7}{10} - \epsilon) \longrightarrow \overline{\mathscr{M}}_g(\tfrac{7}{10}) \longleftarrow \overline{\mathscr{M}}_g(\tfrac{7}{10} + \epsilon)$$

2. Moduli of projective K3 surfaces

K3 surfaces

Definition

- A **K3 surface** S over $\mathbb{C}$ is a smooth compact surface satisfying
$$\omega_S \cong \mathcal{O}_S \text{ and } \mathrm{H}^1(S, \mathcal{O}_S) = 0.$$
- A **polarized K3 surface of genus** g is a pair (S, L), where L is an ample line bundle with $L^2 = 2g - 2 > 0$.

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Example (Mukai models)

- $g = 2$: $S \xrightarrow{2:1} \mathbb{P}^2$ double cover branched over a smooth sextic.
- $g = 3$: $S \hookrightarrow \mathbb{P}^3$ a smooth quartic surface
- $g = 4$: $S = Q \cap C \subseteq \mathbb{P}^4$ a smooth complete intersections of a quadric and a cubic.
- $g = 5$: $S = Q_1 \cap Q_2 \cap Q_3$ is smooth complete intersection of three quadric in $\mathbb{P}^5$.
- $12 \geq g \geq 6, g \neq 11$: smooth complete intersections in a homogenous space

Moduli space of polarized smooth K3 surfaces

- For $g \geq 2$, let

$$\mathscr{F}_g^\circ = \Big\{(S, L) \text{ primitively polarized K3 with } L^2 = 2g-2\Big\} / \cong$$

  be the (coarse) moduli space of primitively polarized smooth K3 surfaces of genus g.
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A natural partial compactification

- Allowing ADE singularities: $\mathcal{F}_g = \mathcal{F}_g^\circ \cup \Delta_g$ where

$$\Delta_g = \Big\{(S, L) | \, L \text{ ample with } L^2 = 2g-2, \text{ S has isolated ADE singularities}\Big\}.$$

- $\mathscr{F}_g$ is "almost" a projective scheme.

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GIT compactifications

- **Viehweg**: $\mathscr{F}_g^\circ \subseteq \mathbf{Hilb}_{g,n} /\!\!/ \mathrm{SL}(N)$ for n sufficiently large
- **Donaldson**: $\mathscr{F}_g^\circ \subseteq \mathbf{Chow}_{g,n} /\!\!/ \mathrm{SL}(N)$ for n sufficiently large

GIT compactifications of $\mathscr{F}_g^\circ$

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Open problem: do they also compactify $\mathscr{F}_g$?

global Torelli theorem

- **Pjateckiĭ-Šapiro, Šhafarevič:**

$$\mathscr{F}_g \cong \mathrm{Sh}(G)$$

is a connected Shimura variety associated to an orthogonal group G with $G(\mathbb{R}) = \mathrm{O}(2, 19)$.

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- **Looijenga:** there is a semitoric compactification

$$\overline{\mathscr{F}}_g^{\mathscr{D}} \cong \mathbf{Proj}\ \mathrm{R}(\mathscr{F}_g - \mathscr{D}, \lambda|_{\mathscr{F}_g - \mathscr{D}}),$$

where $\mathscr{D}$ is a union of Shimura subvarieties of codimension 1.

Slc stable pairs moduli spaces

- **Kollár-Shepherd-Barron, Alexeev:** The (coarse) moduli space of K_S-trivial slc pairs

$$\overline{\mathscr{P}}_g = \{(S, \epsilon C) \mid C \in |nL|\}/\cong$$

is a projective scheme. It admits a forgetful rational map $\overline{\mathscr{P}}_g \dashrightarrow \mathscr{F}_g$.

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K-stable moduli spaces

- **Ascher-Devleming-Liu** : for $c \in (0, \frac{1}{2})$, the good moduli space of K-stable pairs

$$\overline{\mathscr{K}}_{h,c} = \Big\{(X, cS) \text{ is K-polystable with Hilbert polynomial } h\Big\}/\cong$$

is a projective scheme of finite type.

New modular compactifications

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Guiding Problem: Carrying out the birational morphisms between various compactifications with modular interpretations.

A motivated example: $g = 2$

Set

- $\overline{\mathscr{F}}_2^{Mukai} = |\mathcal{O}_{\mathbb{P}^2}(6)| /\!\!/ \mathrm{SL}(3)$
- $\overline{\mathscr{K}}_{6,c}$: the moduli space of K-polystable log Fano surface pairs smoothable to $(\mathbb{P}^2, cC)$ where $C \in |\mathcal{O}_{\mathbb{P}^2}(6)|$
- $\overline{\mathscr{P}}_2$: moduli space of KSBA stable degree 2 K3 pairs.

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Shah, Laza, Ascher-Devleming-Liu

- There is a diagram

$$\begin{array}{ccc} \widehat{\mathscr{F}_2} \cong \overline{\mathscr{K}}_{6,(\frac{1}{4},\frac{1}{2}-\epsilon)} & \overset{h}{\dashleftarrow} & \overline{\mathscr{P}}_2 \\ \downarrow \pi & \searrow & \downarrow \\ \overline{\mathscr{F}}_2^{Mukai} \cong \overline{\mathscr{K}}_{6,(0,\frac{1}{4}]} & \dashrightarrow & \mathscr{F}_2^* \end{array}$$

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- $\widehat{\mathscr{F}_2} \to \mathscr{F}_2^*$ is a $\mathbb{Q}$-Carterization map and $\widehat{\mathscr{F}_2} \to \mathscr{F}_2^*$ contracts the unigonal loci $\mathscr{D}_{1,1}$.
- $\widehat{\mathscr{F}_2} \to \overline{\mathscr{F}}_2^{\mathit{Mukai}}$ is the Kirwan partial desingularization of $\overline{\mathscr{F}}_2^{\mathit{Mukai}}$

From LMMP point of view

All the compactifications can be constructed through a unified way.

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Then we have

- $\overline{\mathscr{F}}_2(0) \cong \mathscr{F}_2^*$, $\overline{\mathscr{F}}_2(\alpha) \cong \widehat{\mathscr{F}_2}$ for $\alpha \in (0,1)$, $\overline{\mathscr{F}}_2(1) \cong \overline{\mathscr{F}}_2^{\mathit{Mukai}}$
- $\overline{\mathscr{F}}_2(\epsilon) \to \overline{\mathscr{F}}_2(1)$ contracts the strict transform of $\mathscr{D}_{1,1}$ to a point.

The case with Mukai models

❏ **Mukai's GIT compactification** $\overline{\mathscr{F}}_g^{Mukai}$

g	Mukai model	$\overline{\mathscr{F}}_g^{Mukai}$
3	quartic surface	$\mathbb{P}^{34} /\!\!/ \mathrm{SL}(4)$
4	cubic hypersurface on Q	$\mathbb{P}^{29} /\!\!/ \mathrm{SO}(5)$
5	c.i. of three quadratics in $\mathbb{P}^5$	$\mathrm{Gr}(3,21) /\!\!/ \mathrm{SL}(6)$
6	quadric hypersurface on $\mathbf{F}_5$	$\mathbb{P}^{22} /\!\!/ \mathrm{PSL}(2)$
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Idea: Run MMP with scaling on $\mathscr{F}_g^*$.

Noether-Lefschetz divisors

- $\mathscr{D}_{d,h}$: parametrizing $(S, L) \in \mathscr{F}_g$ whose $\mathrm{Pic}(S)$ contains a primitive lattice

$$\begin{pmatrix} 2g-2 & d \\ d & 2h-2 \end{pmatrix}$$

Before construction: divisors on $\mathscr{F}_g^*$

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Examples

- ❑ $\mathscr{D}_{0,0}$ = nodal loci
- ❑ $\mathscr{D}_{1,1}$ = unigonal loci
- ❑ $\mathscr{D}_{2,1}$ = hyperelliptic loci, i.e. the loci where $S \to |L|$ is $2:1$.

Greer-Li-Tian: the Picard group of $\mathscr{F}_g^*$ for $g \leq 10$ is given by

g	$\dim \mathrm{Pic}_{\mathbb{Q}}(\mathscr{F}_g)$	generators (besides λ)
2	2	$\mathscr{D}_{1,1}$
3	3	$\mathscr{D}_{1,1}$,$\mathscr{D}_{2,1}$
4	4	$\mathscr{D}_{1,1}$, $\mathscr{D}_{2,1}$,$\mathscr{D}_{3,1}$
5	4	$\mathscr{D}_{1,1}$, $\mathscr{D}_{2,1}$,$\mathscr{D}_{3,1}$
6	6	$\mathscr{D}_{1,1}$, $\mathscr{D}_{2,1}$, $\mathscr{D}_{3,1}$, $\mathscr{D}_{5,2}$,$\mathscr{D}_{4,1}$
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10	9	$\mathscr{D}_{1,1}$, $\mathscr{D}_{2,1}$, $\mathscr{D}_{3,1}$, $\mathscr{D}_{4,1}$, $\mathscr{D}_{7,2}$, $\mathscr{D}_{9,3}$, $\mathscr{D}_{8,2}$, $\mathscr{D}_{5,1}$,

General construction

Interpolating models: consider

$$\overline{\mathscr{F}}_g(\alpha) = \mathbf{Proj}\ \mathrm{R}(\mathscr{F}_g^*, \lambda + \mathbf{B}(\alpha))$$

with α varying in $[0,1] \cap \mathbb{Q}$, where $\mathbf{B}(\alpha) = \sum_{i=1}^{\rho} a_i(\alpha)\mathscr{D}_{d_i,h_i}$ with $\mathscr{D}_{d_i,h_i}$ given in the previous Table, $a_i(\alpha)$ are linear functionals.

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Expectation: there is a chamber structure on α and the birational maps

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1. **existence of** $\overline{\mathscr{F}}_g(\alpha)$: finite generation is missing.

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Problem:

1. **existence of $\overline{\mathscr{F}}_g(\alpha)$**: finite generation is missing.
2. **computation of walls**: high dimensional varieties have very complicated degenerations

A short cut (least wall crossings)

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- **(Semitoric part)** $\overline{\mathscr{F}}_g(0)$ is Looijenga's semitoric compactification and the birational map

$$\mathscr{F}_g^* \dashrightarrow \overline{\mathscr{F}}_g(0)$$

  factors through a $\mathbb{Q}$-factorialization, a series of flips, and divisorial contractions.

- **(GIT part)** $\overline{\mathscr{F}}_g(\alpha)$ is a VGIT and $\overline{\mathscr{F}}_g^{Mukai} \cong \overline{\mathscr{F}}_g(1)$. The parameter $\alpha \in \mathbb{Q} \cap [0,1]$ admits a chamber structure with finite many walls $0 < \alpha_0 < \cdots < \alpha_m < 1$, i.e.

$$\overline{\mathscr{F}}_g(\alpha_{n-1}, \alpha_n) \xrightarrow{\pi^-} \overline{\mathscr{F}}_g(\alpha_n) \xleftarrow{\pi^+} \overline{\mathscr{F}}_g(\alpha_n, \alpha_{n+1})$$

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- The centers are proper transforms of Shimura subvarieties lying in Looijenga's stratification of $\sum \mathscr{D}_{d_i,h_i}$.

Looijenga's stratification

- $\mathrm{Sh}(G) = \Gamma\backslash \mathrm{D}$, with $G = O(V)$ an orthogonal group.
- $\mathscr{D} = \Gamma\backslash \sum_{v \in A} v^{\perp}$ is a union of Shimura subvarieties $\mathscr{D}$ of codimension one.

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Connection to HKL

- Potential center: the stratification $\mathbf{B}^{(d)}$ of the support of $\mathbf{B}(\alpha)$.
- Potential walls: solution α_0 of the linear equation $1 - f(\alpha) = 0$, where

$$\lambda + \mathbf{B}(\alpha)|_{\mathbf{B}^{(d)}} = (1 - f(\alpha))\lambda + \mathbf{B}^{(d+1)}(\alpha)$$

  and $\mathbf{B}^{(d+1)}(\alpha)$ is extremal or not effective at α_0.

An ideal situation

The birational map

$$\mathbf{Proj}\ \mathrm{R}(\mathrm{Sh}(G), \lambda) \dashrightarrow \mathbf{Proj}\ \mathrm{R}(\mathrm{Sh}(G), \lambda + \mathbf{B}(\alpha))$$

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Example

- the birational map $\mathrm{Sh}(G)^* \dashrightarrow \overline{\mathrm{Sh}(G)}^{\mathscr{D}}$ is in the ideal situation.
- **Laza-O'Grady, Ascher-Devleming-Liu**:

$$\mathscr{F}_3^* \dashrightarrow \overline{\mathscr{F}}_3(\beta) = \mathbf{Proj}\ \mathrm{R}(\mathscr{F}_3^*, \lambda + \beta(\mathscr{D}_{1,1} + \mathscr{D}_{2,1}))$$

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Remark. In HKL, this never happen when $g > 3$. The centers will become much more complicated.

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Understanding of effective cones

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- **Greer-Laza-Li-Si-Tian**: $\mathscr{D}_{d,h}$ is extremal in $\mathrm{Eff}(\mathscr{F}_g^*)$ if

$$\frac{15}{8}(g-1) \geq d^2 - 4(g-1)(h-1).$$

List of generators of $\mathrm{Eff}(\mathscr{F}_g^*)^{\mathrm{NL}}$ for $g \leq 10$

g	$\dim \mathrm{Pic}_{\mathbb{Q}}(\mathscr{F}_g)$	Generators
2	2	$\mathscr{D}_{0,0}$, $\mathscr{D}_{1,1}$
3	3	$\mathscr{D}_{0,0}$ $\mathscr{D}_{1,1}$,$\mathscr{D}_{2,1}$
4	4	$\mathscr{D}_{0,0}$ $\mathscr{D}_{1,1}$, $\mathscr{D}_{2,1}$,$\mathscr{D}_{3,1}$
5	4	$\mathscr{D}_{0,0}$, $\mathscr{D}_{1,1}$, $\mathscr{D}_{2,1}$,$\mathscr{D}_{3,1}$
6	6	$\mathscr{D}_{0,0}$, $\mathscr{D}_{1,1}$, $\mathscr{D}_{2,1}$, $\mathscr{D}_{3,1}$, $\mathscr{D}_{5,2}$,$\mathscr{D}_{4,1}$
7	7	$\mathscr{D}_{0,0}$, $\mathscr{D}_{1,1}$, $\mathscr{D}_{2,1}$, $\mathscr{D}_{3,1}$, $\mathscr{D}_{5,2}$, $\mathscr{D}_{6,2}$, $\mathscr{D}_{4,1}$
8	7	$\mathscr{D}_{0,0}$, $\mathscr{D}_{1,1}$, $\mathscr{D}_{2,1}$, $\mathscr{D}_{3,1}$, $\mathscr{D}_{6,2}$, $\mathscr{D}_{7,2}$, $\mathscr{D}_{4,1}$
9	8	$\mathscr{D}_{0,0}$, $\mathscr{D}_{1,1}$, $\mathscr{D}_{2,1}$, $\mathscr{D}_{3,1}$, $\mathscr{D}_{6,2}$, $\mathscr{D}_{7,2}$, $\mathscr{D}_{4,1}$, $\mathscr{D}_{5,1}$
10	9	$\mathscr{D}_{0,0}$, $\mathscr{D}_{1,1}$, $\mathscr{D}_{2,1}$, $\mathscr{D}_{3,1}$, $\mathscr{D}_{4,1}$, $\mathscr{D}_{7,2}$, $\mathscr{D}_{9,3}$, $\mathscr{D}_{8,2}$, $\mathscr{D}_{5,1}$,

Remark. The blue ones are extremal in $\mathrm{Eff}(\mathscr{F}_g^*)$.

IV. HKL for $\mathscr{F}_4$

Projective models

- For $(S, L) \in \mathscr{F}_4$, the image $S \to |L|$ is a complete intersection of a quadric and a cubic in $\mathbb{P}^4$ iff (S, L) is not lying in $\mathscr{D}_{1,1}$, $\mathscr{D}_{2,1}$.
- The image $S \to |L|$ is a complete intersection of a smooth quadric Q and a cubic in $\mathbb{P}^4$ iff (S, L) is not lying in $\mathscr{D}_{1,1}$, $\mathscr{D}_{2,1}$ and $\mathscr{D}_{3,1}$.

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Parametrization space

- $(2, 3)$-complete intersections are parametrized by an open subset of a projective bundle $\mathbb{P}(E) \to |\mathcal{O}_{\mathbb{P}^4}(2)| = \mathbb{P}^{14}$, where

  $$0 \to p_*(\mathcal{I}_{\mathcal{Q}} \otimes q^*\mathcal{O}_{\mathbb{P}^4}(3)) \to p_*(q^*\mathcal{O}_{\mathbb{P}^4}(3)) \to E \to 0,$$

  $\mathcal{Q}$ is the universal quadric with projections $p : \mathcal{Q} \to |\mathcal{O}_{\mathbb{P}^4}(2)|$ and $q : \mathcal{Q} \to \mathbb{P}^4$.
- **(Fix Q)** cubic hypersurfaces on Q are parametrized by $|\mathcal{O}_Q(3)|$.

GIT compactifications

As $(2,3)$-complete intersection

- **GIT quotient of $\mathbb{P}(E)$**: $\mathbb{P}(E)/\!\!/_t \mathrm{SL}(5)$ the GIT w.r.t. the linearization

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As nodal cubic fourfolds

- $(2,3)$-complete intersection (q, f) in $\mathbb{P}^4 \Leftrightarrow$ a nodal cubic $\{x_5 q + f = 0\}$ in $\mathbb{P}^5$.
- This gives a non-reductive GIT model

$$\Delta_0/\!\!/G$$

where $\Delta_0 \subseteq \Delta$ paramterizing cubics which are singular at $p = [0, \ldots, 0, 1]$ and $G \leq \mathrm{SL}(6)$ is the stabilizer of p.

Boundary stratum

❑ **Li-Tian**: $\overline{\mathscr{F}}_4^{Mukai} - \mathscr{F}_4$ consists of 9 irreducible components parametrizing singular c.i. as below:

1. (dim = 6) two simple elliptic singularities of type $\tilde{E}_6$
2. (dim = 2) two simple elliptic singularities of type $\tilde{E}_8$, whose projective tangent cone meeting the surface along lines.
3. (dim = 11) a simple elliptic singularities of type $\widetilde{E}_7$
4. (dim = 8) a simple elliptic singularity of type $\widetilde{E}_8$, whose projective tangent cone meeting the surface along points.
5. (dim = 11) a line
6. (dim = 7) a conic
7. (dim = 3) a twisted cubic
8. (dim = 2) a rational curve of degree 4
9. (dim = 7) an elliptic curve of degree 4

❑ **Stark**: the boundary $\mathscr{F}_4^* - \mathscr{F}_4$ consists of 10 modular curves meeting at a point.

Conjecture A

Set $\mathbf{B}(\alpha) = \mathscr{D}_{1,1} + \mathscr{D}_{2,1} + \alpha\mathscr{D}_{3,1}$ and

$$\mathscr{F}_4(\alpha) := \mathbf{Proj}\ \mathrm{R}(\mathscr{F}_4^*, \lambda + \mathbf{B}(\alpha)).$$

Then

- **(Existence)** $\mathrm{R}(\mathscr{F}_4^*, \lambda + \mathbf{B}(\alpha))$ is finitely generated for $\alpha \in \mathbb{Q} \cap [0, 1]$.
- the walls of the Mori chamber decomposition of the cone

$$\left\{\lambda + \mathbf{B}(\alpha) \mid \alpha \in \mathbb{Q}, \alpha > 0\right\}$$

  are located at the following critical values

$$\mathbf{Wall} = \left\{0, \frac{1}{28}, \frac{1}{16}, \frac{1}{14}, \frac{1}{12}, \frac{1}{10}, \frac{1}{8}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1\right\}.$$

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Remark: $\frac{1}{9}$ is missing.

HKL conjecture

❑ **(Tower structure)** The centers of $\mathscr{F}_4(\alpha_n - \epsilon) \to \mathscr{F}_4(\alpha_n)$ forms a descending towers of Shimura subvarieties in $\overline{\mathscr{F}}_6^*$

➥ A_n-tower:

$$\mathscr{D}_{3,1} = \mathrm{Sh}(\Lambda_{A_2}) \supset \ldots \supset \mathrm{Sh}(\Lambda_{A_5}) \supset \mathrm{Sh}(\Lambda_{A_6'}) \supset \mathrm{Sh}(\Lambda_{A_7'}) \cup \mathrm{Sh}(\Lambda_{A_7''})$$
$$\supset \mathrm{Sh}(\Lambda_{A_8'}) \cup \mathrm{Sh}(\Lambda_{A_8''}) \supset \mathrm{Sh}(\Lambda_{A_9'}) \supset \mathrm{Sh}(\Lambda_{A_{10}'})$$

➥ D_n-tower:

$$\mathrm{Sh}(\Lambda_{D_4}) \supset \mathrm{Sh}(\Lambda_{D_5}) \supset \mathrm{Sh}(\Lambda_{D_6}) \supset \mathrm{Sh}(\Lambda_{D_7}) \supset \mathrm{Sh}(\Lambda_{D_8'}) \supset \mathrm{Sh}(\Lambda_{D_9'})$$

➥ E_n-tower:

$$\mathrm{Sh}(\Lambda_{E_6}) \supset \mathrm{Sh}(\Lambda_{E_7}) \supset \mathrm{Sh}(\Lambda_{E_8})$$

where $\Lambda_{A_n} = (E_6 \oplus A_n)^{\perp}$, $\Lambda_{D_n} = (E_6 \oplus D_n)^{\perp}$, $\Lambda_{E_n} = (E_6 \oplus E_n)^{\perp} \subseteq \mathrm{U}^{\oplus 2} \oplus E_8^{\oplus 3}$.

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Remark. $\mathrm{Sh}(\Lambda_{A_n})$, $\mathrm{Sh}(\Lambda_{D_n})$, $\mathrm{Sh}(\Lambda_{E_n})$ and $\mathrm{Sh}(\Lambda_{A_{10}'})$ are irreducible components of $\mathscr{D}_{3,1}^{(n)}$.

Generic member in $\mathscr{D}_{3,1}^{(\bullet)}$

- A_n-tower: $S = Q \cap Y$ with $\mathrm{rank}(Q) = 4$ and S has an A_{n-1} singularity at the vertex of Q.
- D_n-tower: $S = Q \cap Y$ with $\mathrm{rank}(Q) = 3$ and S has a D_{n-2} and an A_1 singularity in $\mathrm{Sing}(Q)$.
- E_n-tower: $S = Q \cap Y$ with $\mathrm{rank}(Q) = 3$ and S has an A_5 (resp. D_6, E_7) singularity in $\mathrm{Sing}(Q)$.

Modular interpretation of $\mathscr{D}_{3,1}^{(\bullet)}$ and modifications

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Generic member in $\mathrm{Sh}(\Lambda_{A'_\bullet})$

- $(S, \mathcal{O}_S(1)) \in \mathrm{Sh}(\Lambda_{A_n})$ and contains a special line passing through the vertex of Q

Main result

Theorem (Greer-Laza-Li-Si-Tian)

- $\overline{\mathscr{F}}_4(1-\epsilon) \cong \overline{\mathscr{F}}_4^{Mukai}$
- $\overline{\mathscr{F}}_4(\frac{1}{10}) \cong \mathtt{Chow}_{6,2}(\mathbb{P}^4) /\!\!/ \mathrm{SL}(5)$.
- $\overline{\mathscr{F}}_4(0) \cong \Delta^0 /\!\!/ G$ is a Looijenga compactification.
- $\overline{\mathscr{F}}_4(\alpha)$ exists when $\alpha \geq 1/10$ or $\alpha = 0$ and there is an isomorphism

$$\mathbb{P}(E) /\!\!/_t \mathrm{SL}(5) \cong \mathbf{Proj}\ \mathrm{R}(\mathscr{D}_{1,1} + \frac{4+t}{5t}\mathscr{D}_{2,1} + \frac{1-t}{5t}\mathscr{D}_{3,1}) \cong \overline{\mathscr{F}}_4(\frac{1-t}{5t})$$

  for $0 < t \leq \frac{2}{3}$.

Moreover, the conjecture holds when $\overline{\mathscr{F}_4}(\alpha)$ exists.

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Moreover, the conjecture holds when $\overline{\mathscr{F}_4}(\alpha)$ exists.

The proof makes use of variational GIT, but there is a purely arithmetic explanation.

A glimpse of wall crossings for A_n-tower

Using our arithmetic algorithm, one can compute the restriction of $\lambda + \mathbf{B}(\alpha)$ to $\mathrm{Sh}(\Lambda_{A_n})$ as below

$$\lambda + \mathbf{B}(\alpha)|_{\mathrm{Sh}(\Lambda_{A_n})} = (1-(n-1)\alpha)\lambda + \alpha\,\mathrm{Sh}(\Lambda_{A_{n+1}}) + (1+4s)\mathscr{D}_{hyper} + \alpha(n-1)\,\mathrm{Sh}(\Lambda_{D_{n+1}}) + \alpha\frac{(n-2)(n-1)}{2}\,\mathrm{Sh}(\Lambda_{E_{n+1}}).$$

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- $\mathscr{D}_{hyper}$, $\mathrm{Sh}(\Lambda_{D_{n+1}})$ and $\mathrm{Sh}(\Lambda_{E_{n+1}})$ are birationally contractible on $\mathrm{Sh}(\Lambda_{A_n})$.
- $\mathrm{Sh}(\Lambda_{A_{n+1}})$ is birationally contractible when $n \leq 5$. At $\alpha = \frac{1}{n-1}$, it will contract $\mathrm{Sh}(\Lambda_{A_n})$.
- However, $\mathrm{Sh}(\Lambda_{A_{n+1}})$ is movable when $n > 5$. Indeed, $\mathrm{Sh}(\Lambda_{A_6})$ will be also contracted at $\alpha = \frac{1}{4}$ (instead of $\frac{1}{5}$). This is essentially the reason why there are modifications $\mathrm{Sh}(\Lambda_{A'_6})$ from $\alpha = \frac{1}{5}$.

Conjecture B (GLLST)

Let $\overline{\mathscr{K}}_4(c)$ be the good moduli space of K-semistable Fano pairs (X, cS) smoothing to (Q, cS).

- For $c \in (0,1] \cap \mathbb{Q}$, there is an isomorphism

$$\overline{\mathscr{K}}_4(c) \cong \mathbf{Proj}\ \mathrm{R}(\mathscr{F}_4^*, \lambda + \frac{1-c}{8c}\mathscr{D}_{3,1} + \frac{1-c}{c}\mathscr{D}_{2,1} + \frac{5(1-c)}{2c}\mathscr{D}_{1,1}).$$

  with $\overline{\mathscr{K}}_4(c) \cong \overline{\mathscr{F}}_4(\frac{1-c}{8c})$ for $c \leq \frac{1}{2}$ and $\overline{\mathscr{K}}_4(1) \cong \mathscr{F}_4^*$.

- the walls of $\overline{\mathscr{K}}_4(c)$ are

$$\left\{\frac{5}{7}\right\} \bigcup \left\{\frac{11+n}{27+n}, 1 \leq n \leq 5\right\} \bigcup \left\{\frac{3+n}{11+n}, 6 \leq n \leq 11, n \neq 10\right\} \bigcup$$
$$\left\{\frac{36+n}{52+n}, n = 1,3,4,7\right\} \bigcup \left\{\frac{7}{9}, \frac{2}{3}, \frac{7}{11}, \frac{3}{5}, \frac{5}{9}, \frac{1}{2}, \frac{7}{15}, \frac{3}{7}, \frac{5}{13}, \frac{1}{3}, \frac{3}{11}, \frac{1}{5}, \frac{1}{9}\right\}.$$

❏ **(Arithmetic side)** The arithmetic method works for moduli space of **lattice polarized K3 surfaces**, **hyper-Kähler manifolds** and cetain high dimensional **log Fano pairs**. It is relatively easier with the aid of computer.

Further Remarks

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❏ **Pan-Si-Wu, J. Zhao**: moduli space of hyperelliptic K3 surfaces, log del Pezzol, Hassett-Keel on $\overline{\mathscr{M}}_6$ surfaces.

Thanks!