Systematic search for singularities in 3D Euler flows

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3D Euler equations on a periodic domain

$$\begin{aligned}
\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u} &= -\nabla p, && (\boldsymbol{x}, t) \in \mathbb{T}^3 \times (0, T], \\
\boldsymbol{\nabla} \cdot \boldsymbol{u} &= 0, && (\boldsymbol{x}, t) \in \mathbb{T}^3 \times (0, T], \\
\boldsymbol{u}|_{t=0} &= \boldsymbol{\eta}, && \boldsymbol{x} \in \mathbb{T}^3.
\end{aligned} \tag{1}$$

- $\boldsymbol{u} = (u_1, u_2, u_3)$ is the velocity field.
- $p = p(\boldsymbol{x}, t)$ is the scalar pressure.
- $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$ is a unit cube.

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\end{aligned} \tag{1}
$$

- Local well-posedness has been established.

Theorem (Kato, 1972)

If $\boldsymbol{\eta} \in H^m(\mathbb{T}^3)$ or $H^m(\mathbb{R}^3)$ for some $m > 5/2$ and satisfies $\boldsymbol{\nabla} \cdot \boldsymbol{\eta} = 0$, then there exists a time $T > 0$ such that (1) has a unique solution $\boldsymbol{u}$ with the initial condition $\boldsymbol{\eta}$ and the solution satisfies $\boldsymbol{u} \in C([0, T]; H^m) \bigcap C^1([0, T]; H^{m-1})$.

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\tag{1}
$$

Open question (global well-posedness)

Does there exist a smooth initial condition $\boldsymbol{\eta} \in H^m$ for $m > 5/2$, such that

$$
\lim_{t \to T^*} ||\boldsymbol{u}(t; \boldsymbol{\eta})||_{H^m} = \infty, \qquad 0 < T^* < \infty?
$$

- The global-wellposedness of solutions of the Navier-Stokes equations is one of the "millennium problems" posed by the Clay Mathematics Institute.

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\end{aligned} \tag{1}$$

Beale-Kato-Majda (BKM) criterion (Beale et al. 1984)

A smooth solution $\boldsymbol{u}$ of (1) develops a singularity at $t = T^*$ if and only if

$$\int_0^{T^*} ||\boldsymbol{\omega}(t)||_{L^\infty}\, dt = \infty, \qquad \boldsymbol{\omega} = \nabla\times\boldsymbol{u}.$$

- There have been various refinements of the BKM criterion (Kozono & Taniuchi 2000, Chae 2001).

3D Euler equations on a periodic domain

- Kerr 1993, Bustamante & Brachet 2012, Hou 2022, Kang et al. 2020, Kang & Protas 2022, etc.

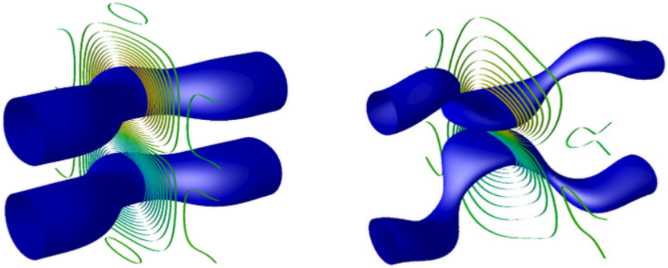

Figure: (Hou & Li 2006) The 3D view of the vortex tube for $t = 0$ and $t = 6$. The tube is the isosurface at 60% of the maximum vorticity. The ribbons on the symmetry plane are the contours at other different values.

3D Euler equations on a periodic domain

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- We choose $m = 3$ and assume $\int_{\mathbb{T}^3} \boldsymbol{u} \, d\boldsymbol{x} = \mathbf{0}$, thus we consider the $\dot{H}^3$-norm.

Optimization problem

- Objective functional

$$\Phi_T(\boldsymbol{\eta}) := \|\boldsymbol{u}(T;\boldsymbol{\eta})\|^2_{\dot{H}^3}, \qquad \boldsymbol{\eta} \in \mathcal{M}_1.$$

$$\mathcal{M}_1 := \left\{ \boldsymbol{\eta} \in G^\sigma \middle| \int_{\mathbb{T}^3} \boldsymbol{\eta}\, d\boldsymbol{x} = \boldsymbol{0}, \boldsymbol{\nabla}\cdot\boldsymbol{\eta} = 0, ||\boldsymbol{\eta}||_{\dot{H}^3} = 1 \right\}.$$

- G^σ (Gevrey space) is endowed with the inner product (Kukavica & Vicol 2009)

$$\langle \boldsymbol{v}, \boldsymbol{u} \rangle_{G^\sigma} = \sum_{\boldsymbol{j}\in\mathbb{Z}^3} (1 + |2\pi\boldsymbol{j}|^2)^3 e^{4\pi\sigma|\boldsymbol{j}|} \hat{\boldsymbol{v}}_{\boldsymbol{j}} \cdot \overline{\hat{\boldsymbol{u}}_{\boldsymbol{j}}}.$$

- Symmetry of the Euler equations: $\lambda\boldsymbol{u}(\boldsymbol{x},\lambda t) \Rightarrow ||\boldsymbol{\eta}||_{\dot{H}^3} = 1$.

Optimization problem

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- Optimization problem: given $T \in \mathbb{R}_+$, find

$$\widetilde{\boldsymbol{\eta}}_T = \underset{\boldsymbol{\eta} \in \mathcal{M}_1}{\operatorname{argmax}}\, \Phi_T(\boldsymbol{\eta}).$$

Evaluation of the gradient

- Objective functional: $\Phi_T(\boldsymbol{\eta}) := \|\boldsymbol{u}(T;\boldsymbol{\eta})\|^2_{\dot{H}^3}$.

$$\begin{aligned}\Phi'_T(\boldsymbol{\eta};\boldsymbol{\eta}') :&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[\Phi_T\left(\boldsymbol{\eta}+\epsilon\boldsymbol{\eta}'\right)-\Phi_T(\boldsymbol{\eta})\right] = \langle\nabla\Phi_T(\boldsymbol{\eta}),\boldsymbol{\eta}'\rangle_{G^\sigma}\\ &= 2\left\langle\boldsymbol{u}(\cdot,T),\boldsymbol{u}'(\cdot,T)\right\rangle_{\dot{H}^3}.\end{aligned}$$

- Linearization of the Euler equations around the solution $(\boldsymbol{u}(t;\boldsymbol{\eta}), p)$

$$\mathcal{L}\begin{bmatrix}\boldsymbol{u}'\\ p'\end{bmatrix} := \begin{bmatrix}\partial_t\boldsymbol{u}' + \boldsymbol{u}'\cdot\boldsymbol{\nabla}\boldsymbol{u} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u}' + \boldsymbol{\nabla}p'\\ \boldsymbol{\nabla}\cdot\boldsymbol{u}'\end{bmatrix} = \begin{bmatrix}\mathbf{0}\\ 0\end{bmatrix},$$

$$\boldsymbol{u}'(0) = \boldsymbol{\eta}'.$$

Evaluation of the gradient

- Objective functional: $\Phi_T(\boldsymbol{\eta}) := \|\boldsymbol{u}(T;\boldsymbol{\eta})\|^2_{\dot{H}^3}$.

$$\Phi_T'(\boldsymbol{\eta};\boldsymbol{\eta}') := \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[\Phi_T\left(\boldsymbol{\eta}+\epsilon\boldsymbol{\eta}'\right)-\Phi_T(\boldsymbol{\eta})\right] = \left\langle \nabla\Phi_T(\boldsymbol{\eta}),\boldsymbol{\eta}'\right\rangle_{G^\sigma}$$
$$= 2\left\langle \boldsymbol{u}(\cdot,T),\boldsymbol{u}'(\cdot,T)\right\rangle_{\dot{H}^3}.$$

- Adjoint states $(\boldsymbol{u}^*,p^*)$

$$\left(\mathcal{L}\begin{bmatrix}\boldsymbol{u}'\\ p'\end{bmatrix},\begin{bmatrix}\boldsymbol{u}^*\\ p^*\end{bmatrix}\right) := \int_0^T\int_{\mathbb{T}^3}\mathcal{L}\begin{bmatrix}\boldsymbol{u}'\\ p'\end{bmatrix}\cdot\begin{bmatrix}\boldsymbol{u}^*\\ p^*\end{bmatrix}\,d\boldsymbol{x}\,dt$$
$$= \left(\begin{bmatrix}\boldsymbol{u}'\\ p'\end{bmatrix},\mathcal{L}^*\begin{bmatrix}\boldsymbol{u}^*\\ p^*\end{bmatrix}\right) + \int_{\mathbb{T}^3}\boldsymbol{u}^*(\boldsymbol{x},T)\cdot\boldsymbol{u}'(\boldsymbol{x},T)\,d\boldsymbol{x}$$
$$- \int_{\mathbb{T}^3}\boldsymbol{u}^*(\boldsymbol{x},0)\cdot\boldsymbol{\eta}'(\boldsymbol{x})\,d\boldsymbol{x}$$
$$= 0.$$

Evaluation of the gradient

- Objective functional: $\Phi_T(\boldsymbol{\eta}) := \|\boldsymbol{u}(T;\boldsymbol{\eta})\|^2_{\dot{H}^3}$.

$$\begin{aligned}\Phi_T'(\boldsymbol{\eta};\boldsymbol{\eta}') &= \langle \nabla\Phi_T(\boldsymbol{\eta}), \boldsymbol{\eta}'\rangle_{G^\sigma} \\ &= 2\,\langle \boldsymbol{u}(\cdot,T), \boldsymbol{u}'(\cdot,T)\rangle_{\dot{H}^3} = 2\left\langle |D|^6\boldsymbol{u}(\cdot,T), \boldsymbol{u}'(\cdot,T)\right\rangle_{L^2} \\ &= \langle \boldsymbol{u}^*(0), \boldsymbol{\eta}'\rangle_{L^2},\end{aligned}$$

where $|D| = \sqrt{-\Delta}$.

- $\nabla\Phi_T(\boldsymbol{\eta}) = (1+|D|^2)^{-3}e^{-2\sigma|D|}\boldsymbol{u}^*(0)$.

- Adjoint system

$$\mathcal{L}^*\begin{bmatrix}\boldsymbol{u}^*\\ p^*\end{bmatrix} := \begin{bmatrix}-\partial_t\boldsymbol{u}^* - (\boldsymbol{\nabla}\boldsymbol{u}^* + (\boldsymbol{\nabla}\boldsymbol{u}^*)^T)\boldsymbol{u} - \boldsymbol{\nabla}p^* \\ -\boldsymbol{\nabla}\cdot\boldsymbol{u}^*\end{bmatrix} = \begin{bmatrix}\boldsymbol{0}\\ 0\end{bmatrix},$$

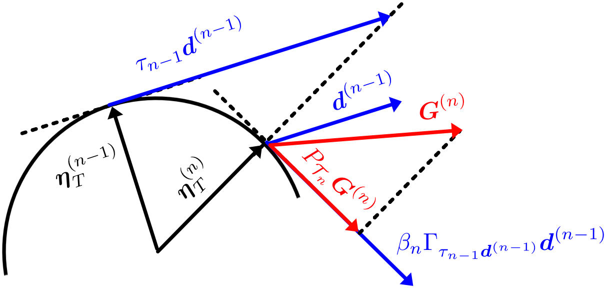
$$\boldsymbol{u}^*(T) = 2|D|^6\boldsymbol{u}(T).$$

Riemannian conjugate gradient method

$$\boldsymbol{\eta}^{(n+1)} = \mathrm{R}\left[\boldsymbol{\eta}^{(n)} + \tau_n \boldsymbol{d}^{(n)}\right], \quad n = 0, 1, 2, \cdots,$$

$$\boldsymbol{\eta}^{(0)} = \boldsymbol{\eta}_0. \quad \Longleftarrow \text{ Taylor-Green, Random, Hou 2022, Kerr 1993}$$

- $\mathrm{R}(\boldsymbol{v}) = \boldsymbol{v}/||\boldsymbol{v}||_{\dot{H}^3}, \quad \boldsymbol{v} \neq \boldsymbol{0}.$
- $\boldsymbol{d}^{(n)} = \mathrm{P}_{\mathcal{T}_n} \boldsymbol{G}^{(n)} + \beta_n \Gamma_{\tau_{n-1}\boldsymbol{d}^{n-1}}(\boldsymbol{d}^{n-1})$, $n \geq 1$, $G^{(n)} := \nabla\Phi_T\left(\boldsymbol{\eta}^{(n)}\right)$, where β_n is computed using the Polak-Ribière approach.



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- $\tau_n = \mathrm{argmax}_{\tau > 0} \left\{\Phi_T\left(\mathrm{R}\left[\boldsymbol{\eta}^{(n)} + \tau \boldsymbol{d}^{(n)}\right]\right)\right\}$.

Numerical results

We use uniform meshes with N^3 grid points and denote the solution we obtain at time T by $\widetilde{\boldsymbol{\eta}}_T^N$.

- Shorter times T: $\lim_{N\to\infty} \Phi_T\left(\widetilde{\boldsymbol{\eta}}_T^N\right) < \infty$

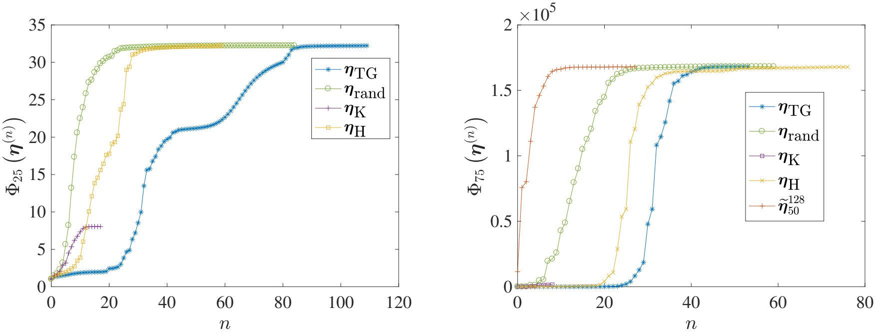
  $\Longrightarrow$ equation is well-posed on $[0, T]$;

- Longer times T: $\lim_{N\to\infty} \Phi_T\left(\widetilde{\boldsymbol{\eta}}_T^N\right) = \infty$

  $\Longrightarrow$ there is a possible singularity at $T^* \leq T$.

- In practice, we use $N^3 = 128^3, 256^3, 512^3, 1024^3$.

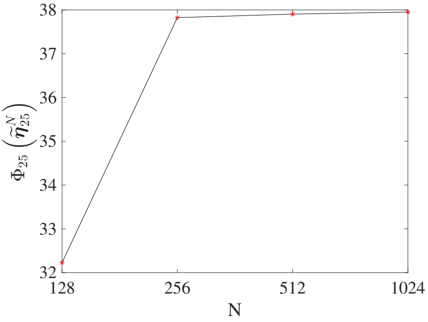
Different initial guesses



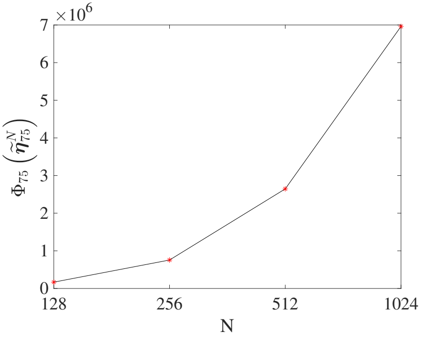
(a) $T = 25$ (b) $T = 75$

Figure: Dependence of the objective functional $\Phi_T\left(\boldsymbol{\eta}^{(n)}\right)$ on the iteration index n for different initial guesses.

Behavior of Φ_T for $T = 25$ and $T = 75$



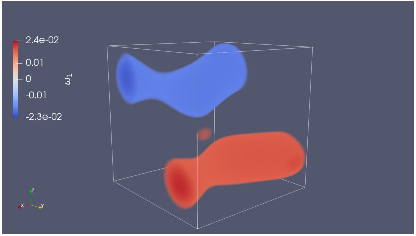
(a) $T = 25$



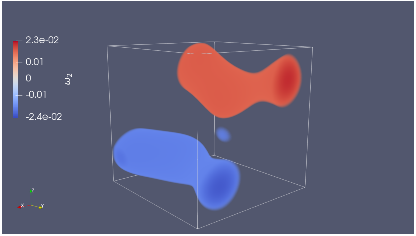
(b) $T = 75$

Figure: Comparison of the objective functional at different spatial resolutions for different time windows.

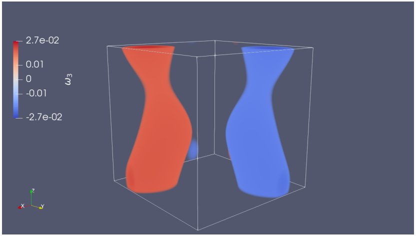
Vorticity components of $\widetilde{\eta}_{75}^{1024}$



(a)

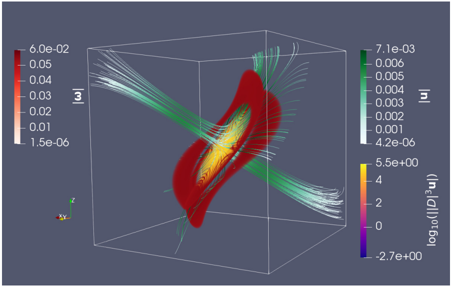


(b)

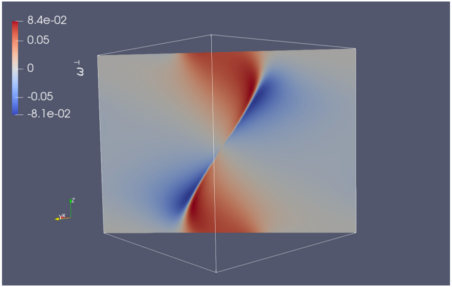


(c)

Vorticity field of $\boldsymbol{u}\left(75; \widetilde{\boldsymbol{\eta}}_{75}^{1024}\right)$



(a)



(b)

Figure: Visualization of the vorticity field.

Time evolution of $\left|\boldsymbol{\omega}\left(t; \widetilde{\boldsymbol{\eta}}_{75}^{1024}\right)\right|$

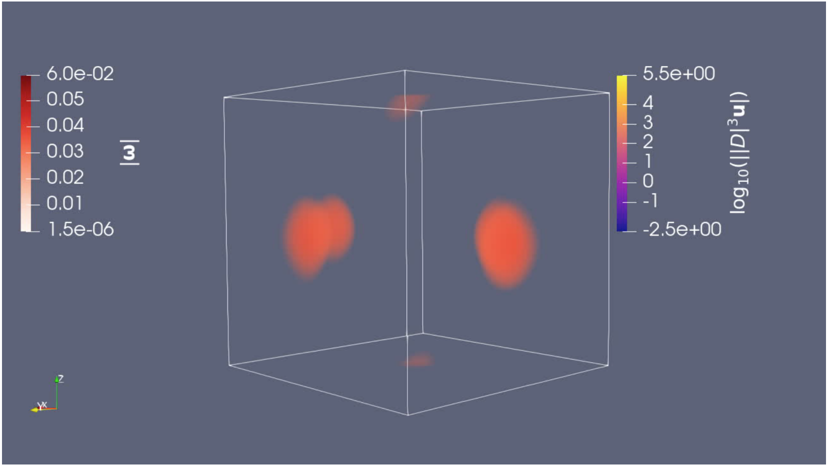


Figure: Time evolution of the vorticity field.

Time evolution of $\omega^{\perp}\left(t; \widetilde{\boldsymbol{\eta}}_{75}^{1024}\right)$

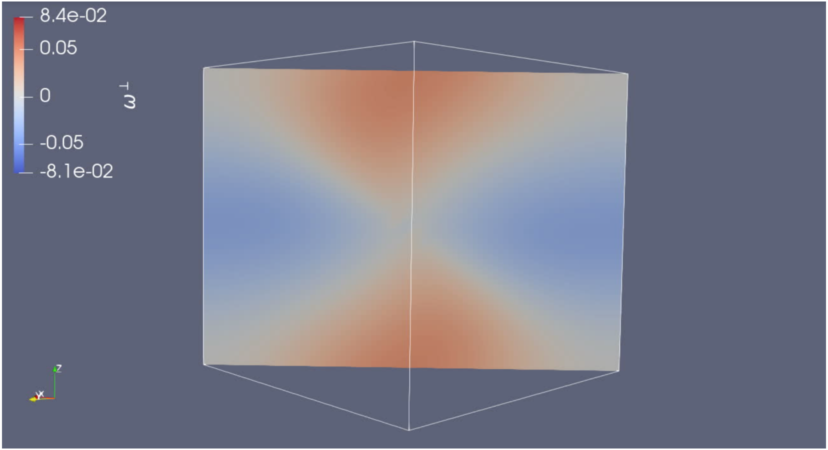


Figure: Time evolution of the vorticity component orthogonal to the symmetry plane.

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Thank you!

Time evolution of $\left|\omega\left(t;\widetilde{\boldsymbol{\eta}}_{75}^{1024}\right)\right|$

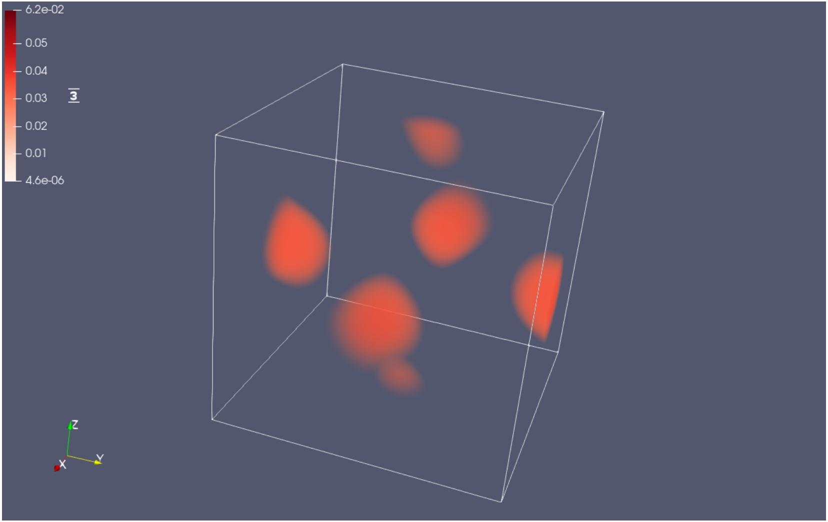
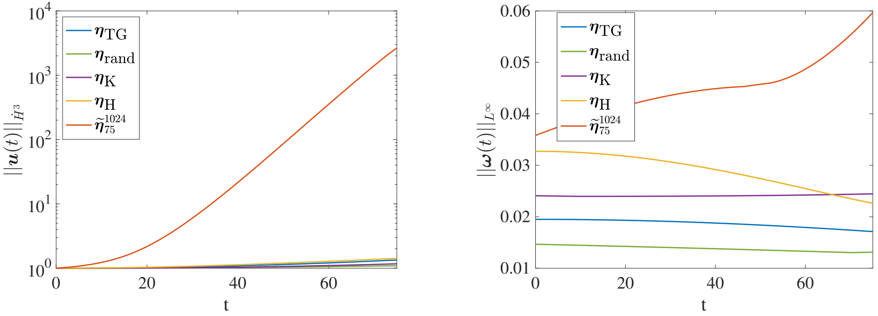


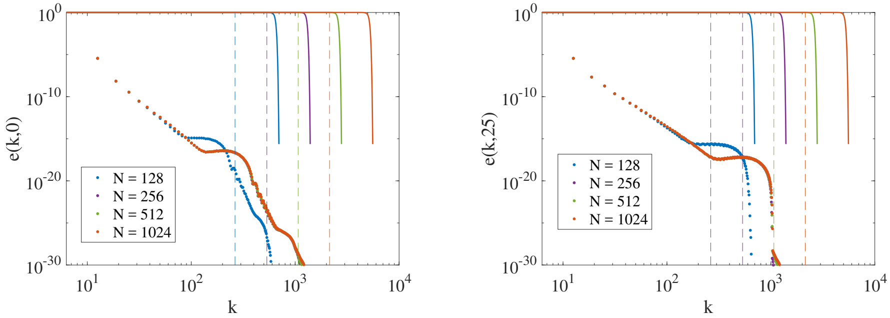
Figure: Time evolution of the vorticity field.



(a) (b)

Figure: Comparison of the time evolution of the $\dot{H}^3$-norm and the maximum vorticity of solutions with different initial conditions.

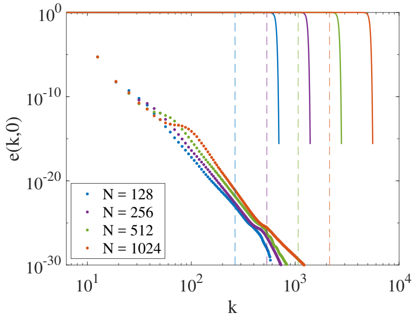
Energy spectrum of $\boldsymbol{u}^N\left(t, \widetilde{\boldsymbol{\eta}}_{25}^N\right)$ at $t = 0$ and $t = 25$



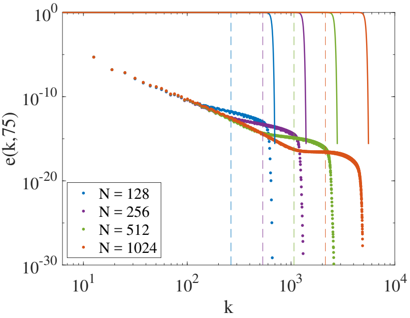
(a) (b)

$$e(k,t) := \frac{1}{2} \sum_{j \leq |\boldsymbol{j}| < j+1} |\hat{\boldsymbol{u}}_{\boldsymbol{j}}(t)|^2, \qquad k = 2\pi j, \quad j \in \mathbb{N}.$$

Energy spectrum of $\boldsymbol{u}^N\left(t, \widetilde{\boldsymbol{\eta}}_{75}^N\right)$ at $t = 0$ and $t = 75$



(a)

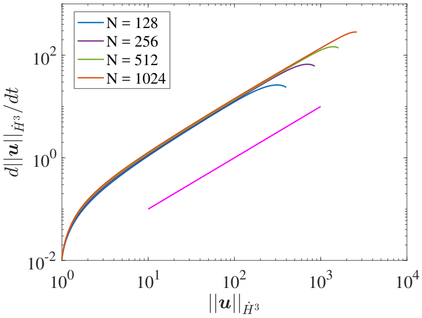


(b)

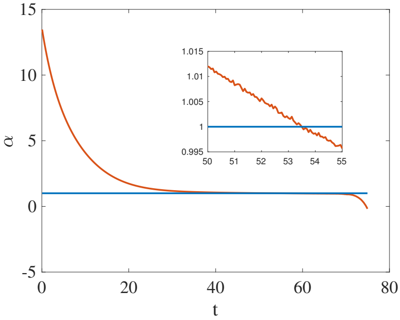
$$e(k,t) = \frac{1}{2} \sum_{j \le |\boldsymbol{j}| < j+1} |\hat{\boldsymbol{u}}_{\boldsymbol{j}}(t)|^2, \qquad k = 2\pi j, \quad j \in \mathbb{N}.$$

Numerical results for the long time window (T = 75)

$$\frac{d\|\boldsymbol{u}(t)\|_{\dot{H}^3}}{dt} = C\|\boldsymbol{u}(t)\|_{\dot{H}^3}^{\alpha} \Longrightarrow \ln\left(\frac{d\|\boldsymbol{u}(t)\|_{\dot{H}^3}}{dt}\right) = \ln(C) + \alpha \ln\left(\|\boldsymbol{u}(t)\|_{\dot{H}^3}\right).$$



(a)



(b)

Figure: Growth rate of $\|\boldsymbol{u}(t)\|_{\dot{H}^3}$.