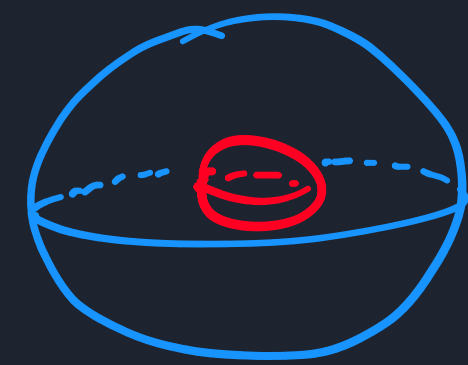
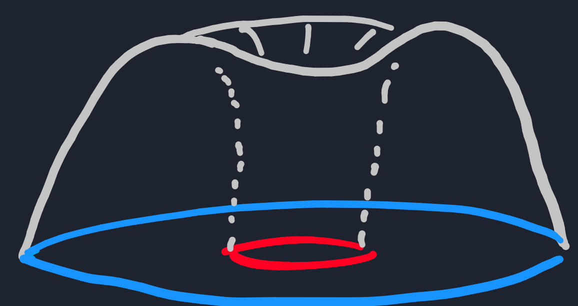
SERRIN-TYPE THEOREMS FOR DOMAINS WITH DISCONNECTED BOUNDARIES AND RELATED RESULTS



Lorenzo Mazzieri – Università degli Studi di Trento



BIRS-IASM – CONFERENCE

"RECENT ADVANCES IN COMPARISON GEOMETRY"

Hangzhou – 26th February – 1st March 2024

PRELIMINARIES / MOTIVATIONS

$\Omega \subseteq \mathbb{R}^2$ bounded domain with $\partial\Omega$ smooth (i.e. of class $\mathcal{C}^2$)

$$\begin{cases} \Delta u = -2 , & \Omega \\ u = 0 , & \partial\Omega \end{cases} \qquad (*)$$

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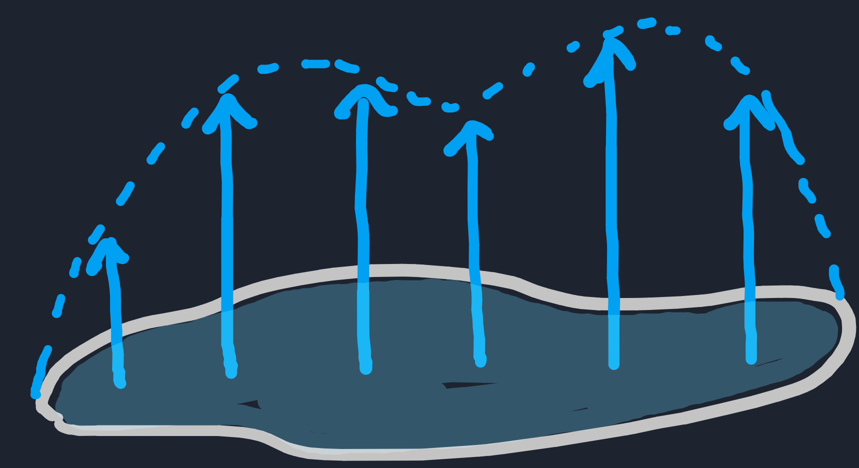

FLUID DYNAMICS

u represents the velocity of a steady laminar flow of a homogeneous incompressible fluid through a cylindrical pipe with cross-section Ω.

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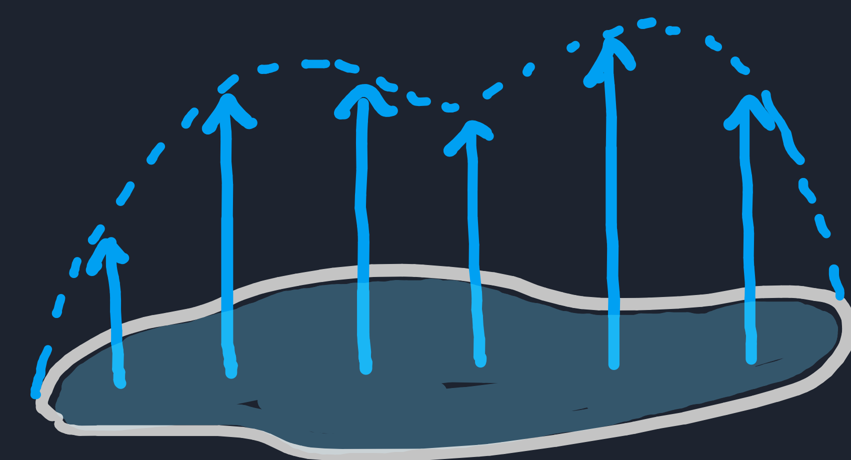

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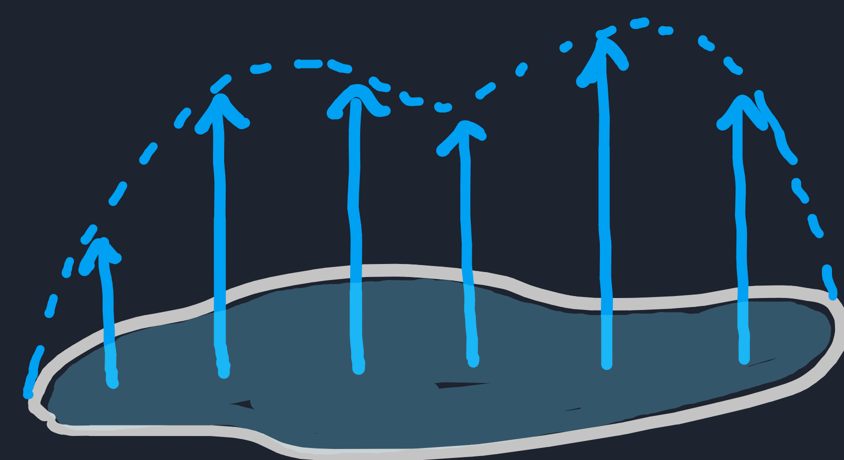

FLUID DYNAMICS

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- $u = 0$ at $\partial\Omega$: NO SLIP CONDITION.
- $|Du|$ at $\partial\Omega$ $\simeq$ WALL SHEAR STRESS.

$\text{WSS} = \mu |Du|$ at $\partial\Omega$ (μ ← DYN. VISC.)

"tangential force per unit area exerted by the flowing fluid on the surface of the conduit tube"

THEOREM (SERRIN, 1971)

Let u be a solution to (*), then

$|Du| \equiv c$ on $\partial\Omega$ $\Longrightarrow$ Ω = BALL & u ROT. SYMM

OVERDETERMINING CONDITION.

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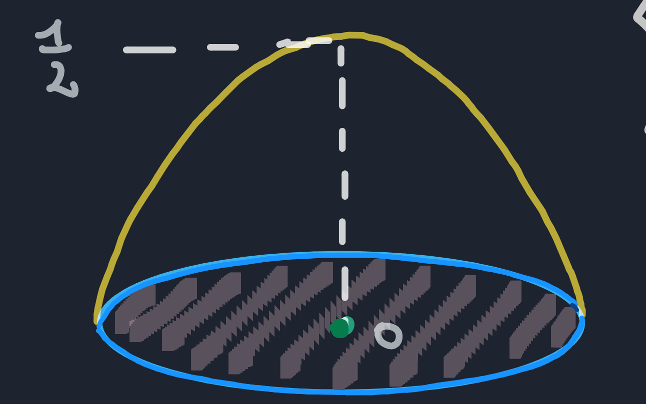
i.e. up to translations and rescalings (Ω, u) is a SERRIN SOLUTION

$$\left(\Omega_0 = \mathbb{B}^2,\ u_0(x) = \frac{1-|x|^2}{2}\right)$$



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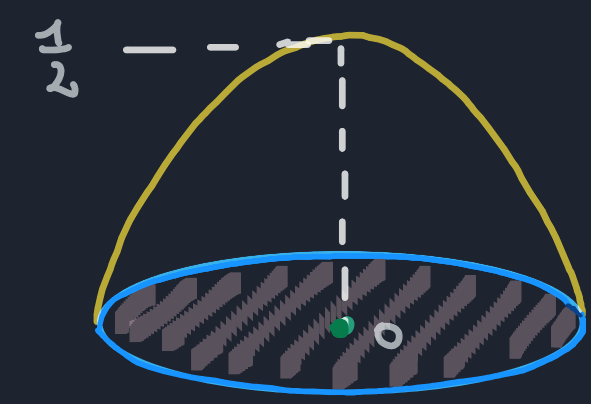

FLUID DYNAMICS

"The WSS is the same at all points of the wall if and only if the pipe has circular cross section"

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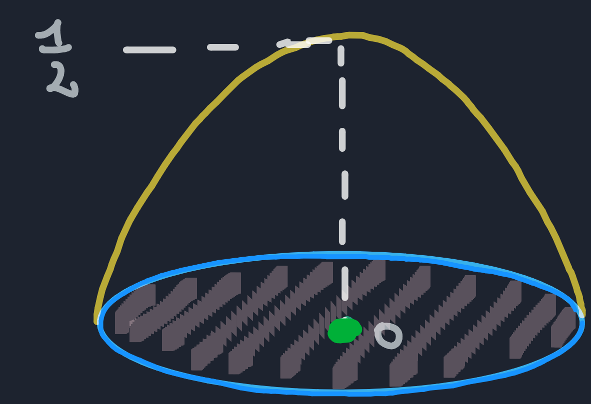


FLUID DYNAMICS

"The WSS is the same at all points of the wall if and only if the pipe has circular cross section"

Rmk "$\partial\Omega$ CONNECTED" is a consequence of "$|Du| \equiv$ const. on the WHOLE BDARY"

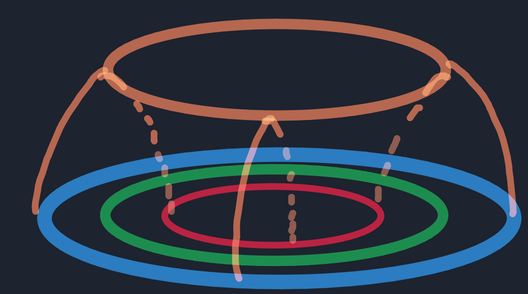
Rmk $\exists$ ROTATIONALLY SYMMETRIC SLN'S to (*) with DISCONNECTED BOUNDARY

Rmk ∃ ROTATIONALLY SYMMETRIC SLN'S to (*) with DISCONNECTED BOUNDARY

It is sufficient to add a NEGATIVE GREEN'S function to a SERRIN SLN. :

free parameter

$$\begin{cases} u_R(x) = \dfrac{1-|x|^2}{2} + R^2 \log|x| , & 0 < R < 1 \\ \Omega_R = \mathbb{B}^2 \setminus B(0, r_i(R)) \end{cases}$$



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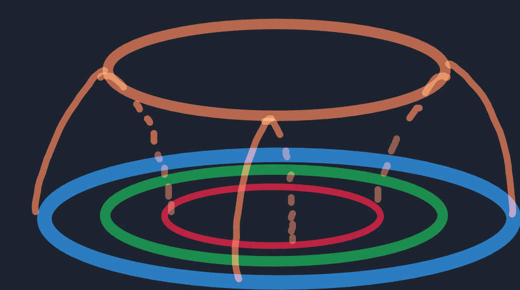
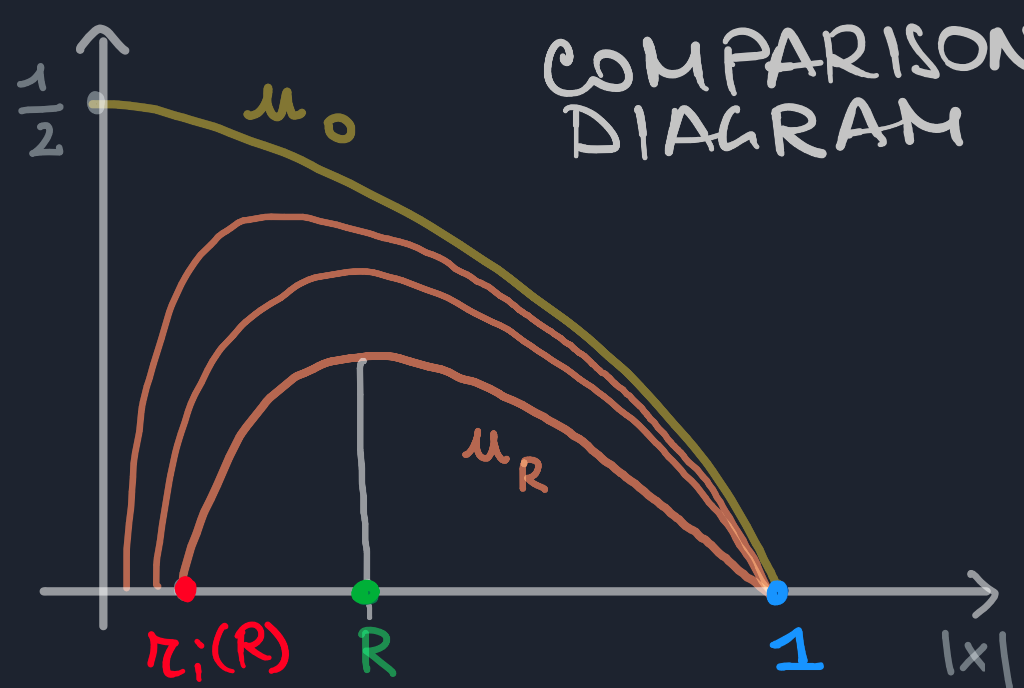


COMPARISON DIAGRAM



Rmk ∃ ROTATIONALLY SYMMETRIC SLN'S to (⁎) with DISCONNECTED BOUNDARY

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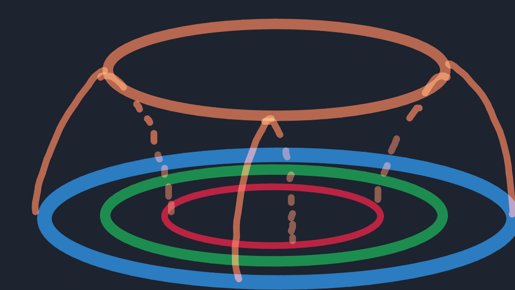
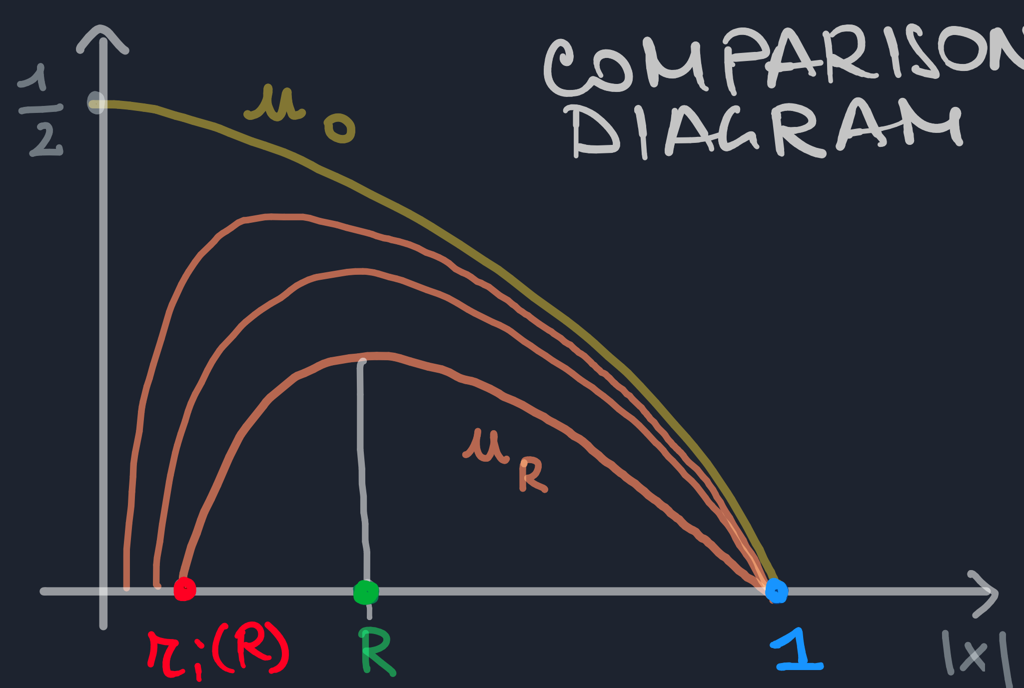




COMPARISON DIAGRAM

Notation:

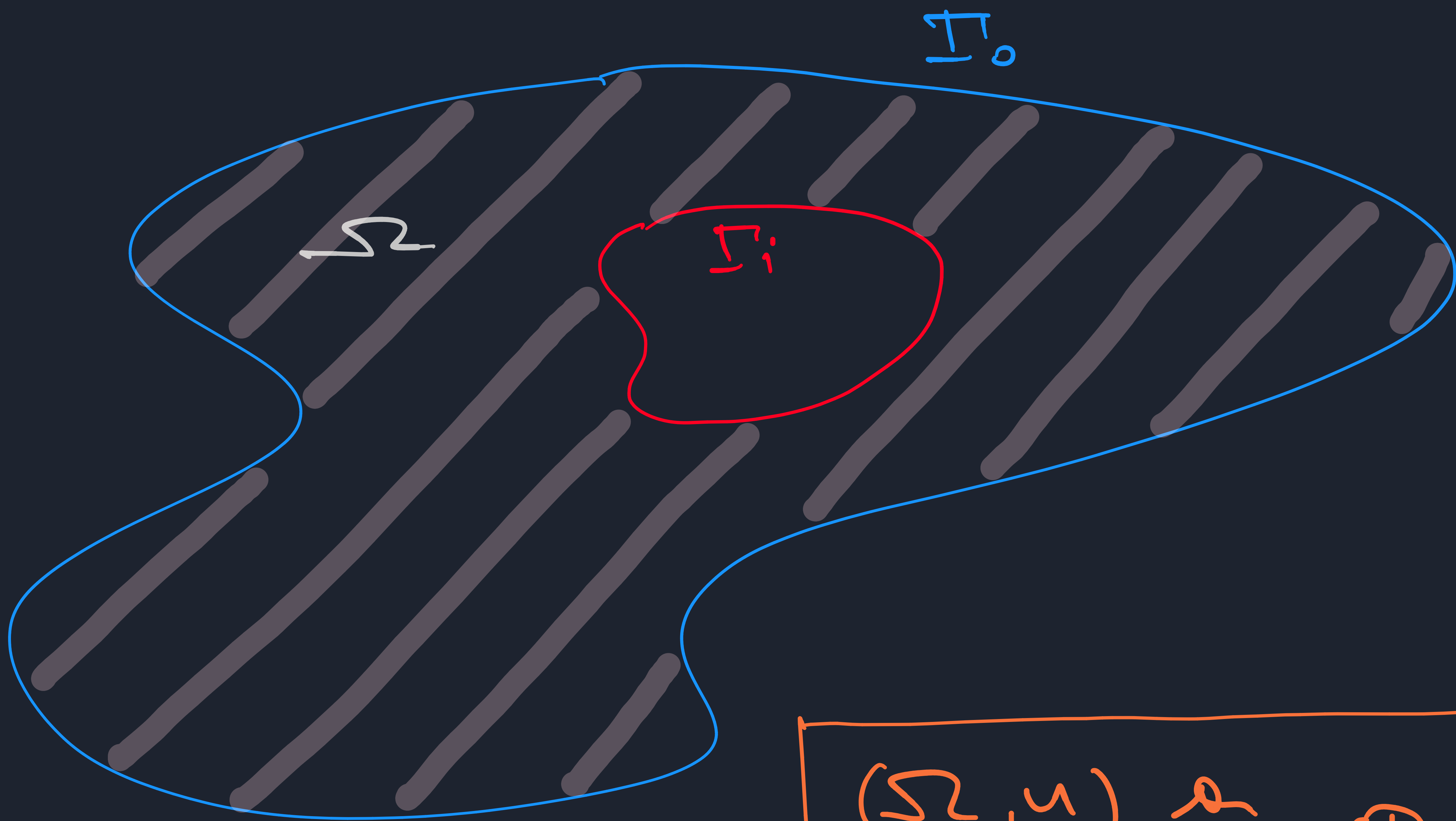
$$\mathrm{MAX}(u_R) = \{ |x| = R \}$$

$$\max_{\Omega_R} (u_R) = \frac{1-R^2}{2} + R^2 \log R$$

QUESTION: Are there SERRIN'S TYPE THEOREMS for domains with DISCONNECTED BDARY ?

i.e. characterizations of the rotationally symmetric solutions to pb. (*) defined on RING SHAPED DOMAINS

SETTING of THE PROBLEM



$\Omega \subseteq \mathbb{R}^2$

$\partial\Omega$ smooth

$\partial\Omega = \Gamma_o \sqcup \Gamma_i$

(Γ_o: outer boundary; Γ_i: inner boundary)

(Ω, u) a solution to $(*)$

$$(*) \quad \begin{cases} \Delta u = -2, & \Omega \\ u = 0, & \partial\Omega \end{cases}$$

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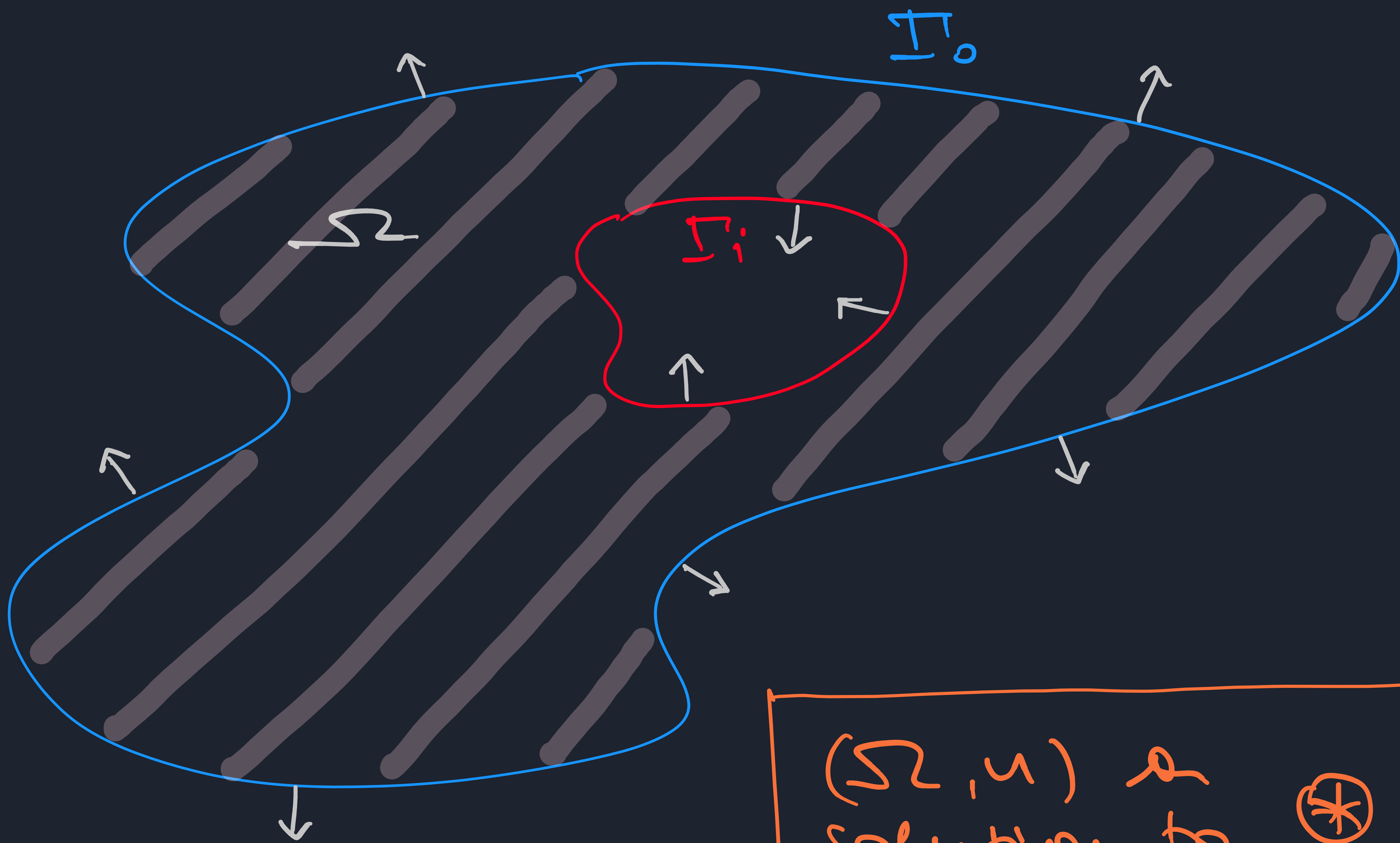

ν = outward pointing unit normal

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(Ω, u) a solution to $\circledast$ $\begin{cases} \Delta u = -2, & \Omega \\ u = 0, & \partial\Omega \end{cases}$

Hopf lemma $\Longrightarrow$ $\dfrac{\partial u}{\partial \nu} < 0$ on $\partial\Omega$.

Look for statements like:

$$\circledast \; + \; \boxed{\begin{array}{l}\text{OVERDET}\\ \text{CONDITIONS}\\ \text{on } \Gamma_0 \text{ and}\\ \text{on } \Gamma_i\end{array}} \; \Longrightarrow \; (\Omega, u) \underset{\text{ISOM}}{\cong} (\Omega_R, u_R)$$

Look for statements like:

$\circledast$ + [OVERDET CONDITIONS on Γ_o and on Γ_i] $\Rightarrow$ $(\Omega, u) \underset{\text{ISOM}}{\cong} (\Omega_R, u_R)$

e.g. "DREAM STATEMENT"

$\circledast$ + $\left[\begin{array}{ll} \dfrac{\partial u}{\partial \nu} \equiv c_o < 0 & \text{on } \Gamma_o \\ \dfrac{\partial u}{\partial \nu} \equiv c_i < 0 & \text{on } \Gamma_i \end{array}\right]$ $\Rightarrow$ $(\Omega, u) \underset{\text{ISOM}}{\cong} (\Omega_R, u_R)$

STATE of THE ART:

THEOREM (REICHEL 1995, SYRAKOV 2001)

$$\begin{cases} \Delta u = -2 & , \ \Omega \\ u = 0 & , \ \Gamma_0 \\ u = a > 0 & , \ \Gamma_a \end{cases}$$

with $0 < u < a$ in Ω

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OVERDET. CONDITIONS:

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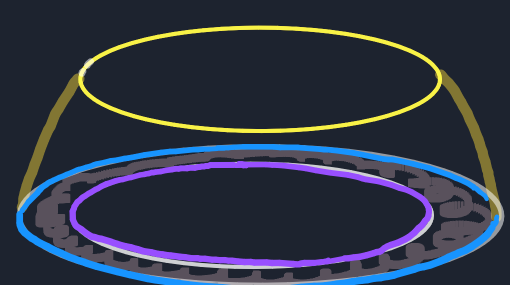
HOPF: $c_0 \leq 0$

ASSUME: $c_a \geq 0$

$\Longrightarrow$ Ω is an ANNULUS



u is ROT. SYMMETRIC & MONOTONICALLY DECREASING in r

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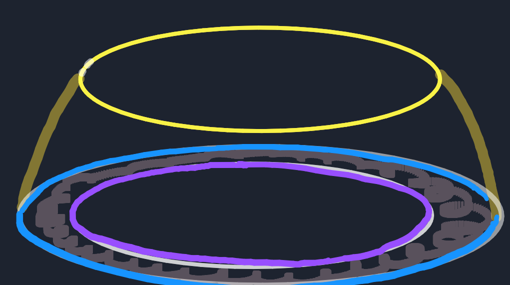

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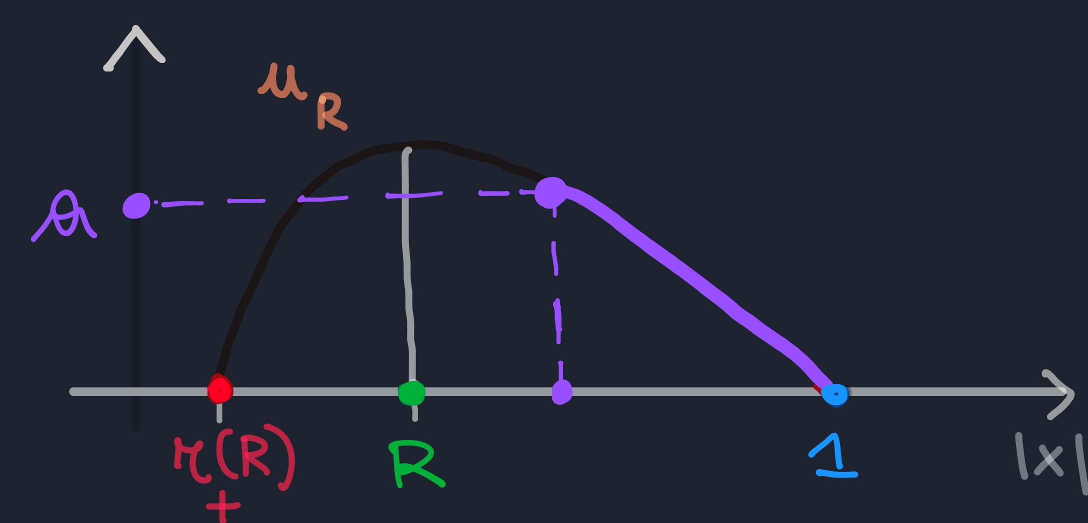

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SYRAKOV 2001

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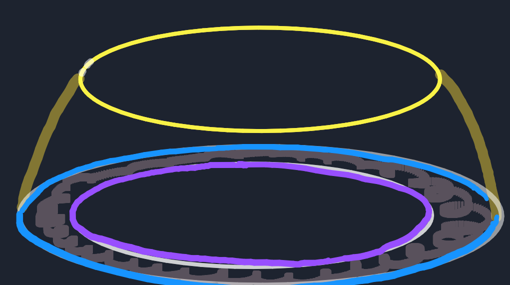

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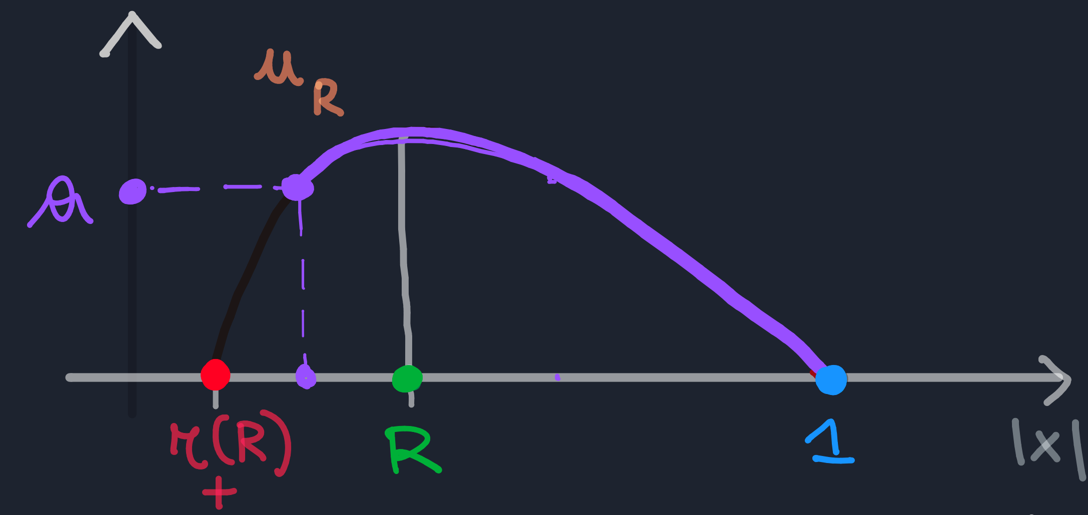
Fundamental features of the MOVING PLANES METHOD

Q What if $c_\ell < 0$?

Q What if $c_a < 0$?



A_1 Moving Planes Method DOES NOT APPLY! Model Solutions are no longer expected to be monotonically decreasing.

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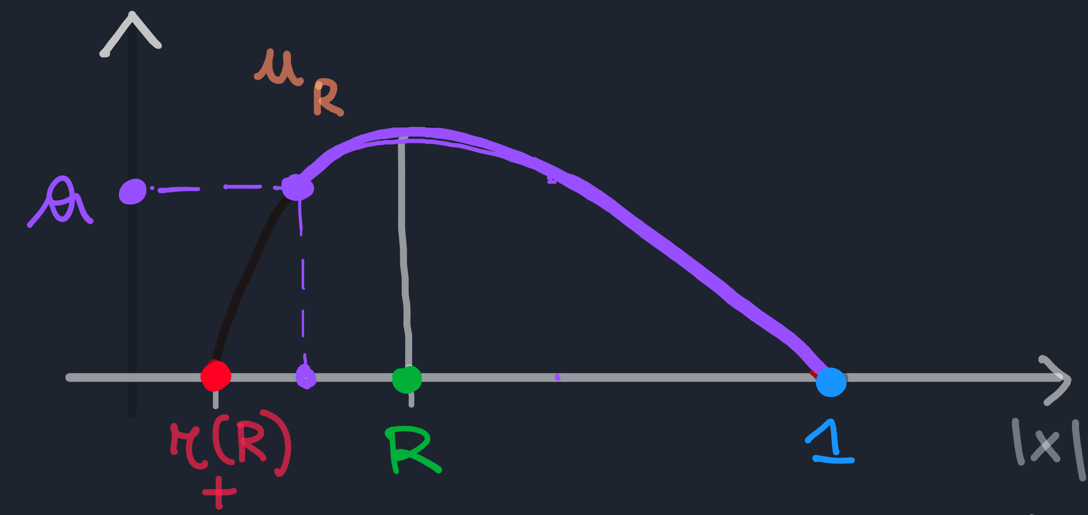

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A_2  + $\frac{\partial u}{\partial \nu}$ locally constant on $\partial\Omega = \Gamma_0 \sqcup \Gamma_a$ $\not\Longrightarrow$ (Ω, u) is rotationally symmetric

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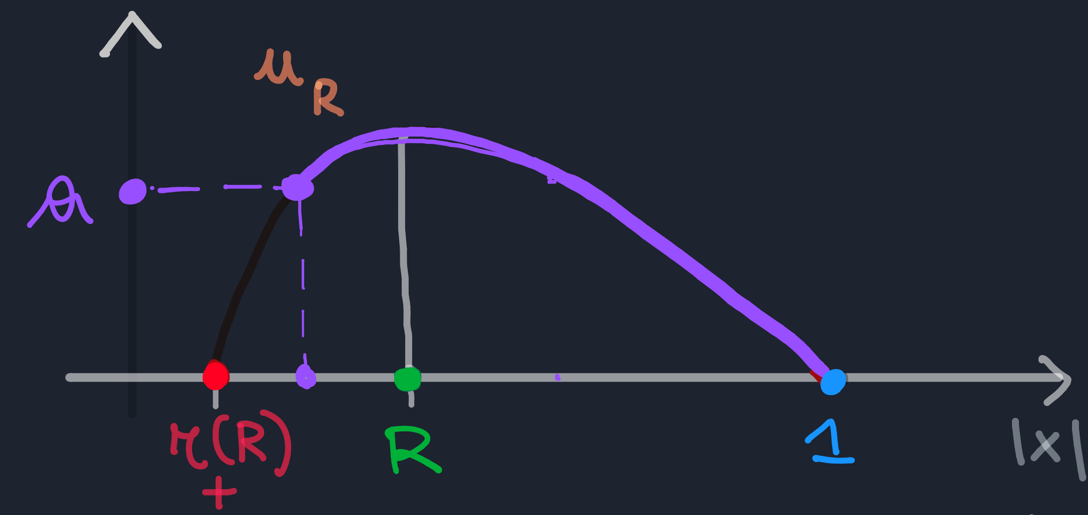

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CTREX / THEOREM (Kamburov - Sciaraffia 2019)

$\exists$ domains bifurcating from the annular ones admitting NON RADIALLY SYMMETRIC SOLUTIONS to  with

$$\boxed{\left.\frac{\partial u}{\partial \nu}\right|_{\Gamma_0} \equiv -K \equiv \left.\frac{\partial u}{\partial \nu}\right|_{\Gamma_a}, \quad K > 0.}$$

Q What about the "DREAM STATEMENT"?

$$\circledast + \boxed{\begin{array}{l} \dfrac{\partial u}{\partial \nu} \equiv c_0 < 0 \quad \text{on } \Gamma_0 \\[2ex] \dfrac{\partial u}{\partial \nu} \equiv c_i < 0 \quad \text{on } \Gamma_i \end{array}} \Longrightarrow (\Omega, u) \underset{\text{isom}}{\cong} (\Omega_R, u_R)$$

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~~(crossed out)~~

FALSE!

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FALSE!

THEOREM (A) (Agostiniani, Borghini, ——)

$\exists$ ∞-many solutions to pb. $\circledast$ with $\frac{\partial u}{\partial \nu}$ locally constant on $\partial\Omega$ that are not rotationally symmetric.

THEOREM (B) (Agostiniani, Borghini, ——)

Let u be a solution to $\circledast$. Then

$$|\mathrm{MAX}(u)| = +\infty$$

&

$$\frac{\partial u}{\partial \nu} \equiv c_0 \quad \text{on } \Gamma_0$$

$$\frac{\partial u}{\partial \nu} \equiv c_i \quad \text{on } \Gamma_i$$

$\Longleftrightarrow$ $(\Omega, u) \simeq (\Omega_R, u_R)$ up to rescaling and translation.

THEOREM Ⓑ (Agostiniani, Borghini, ——)

Let u be a solution to $\circledast$. Then

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&

EITHER

$$\frac{\partial u}{\partial \nu} \equiv C_0 \quad \text{on } \Gamma_0$$

OR

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Fluid-dynamical interpretation of THEOREM (A) and THEOREM (B)

"The velocity (of a steady laminar flow of a homogeneous incompressible fluid through a hollow cylindrical pipe with locally constant wall shear stress) attains its maximal value only at finitely many points, unless the cross section Ω of the tube is a rotationally symmetric annular domain."

EXTREMAL LEVEL SETS OF ANALYTIC FUNCTIONS

$u : (M, g) \longrightarrow \mathbb{R}$ REAL ANALYTIC function.

↑ complete REAL ANALYTIC Riem. mfld.

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LOJASIEWICZ STRUCTURE THEOREM: $\mathrm{Crit}(u) \overset{\text{loc.}}{=} \Sigma^{(0)} \sqcup \Sigma^{(1)} \sqcup \dots \sqcup \Sigma^{(n-1)}$

TOP STRATUM ($\Sigma^{(n-1)}$)

$\Sigma^{(i)}$: "the i-th stratum" finite union of i-dimensional real analytic submflds.

EXTREMAL LEVEL SETS OF ANALYTIC FUNCTIONS

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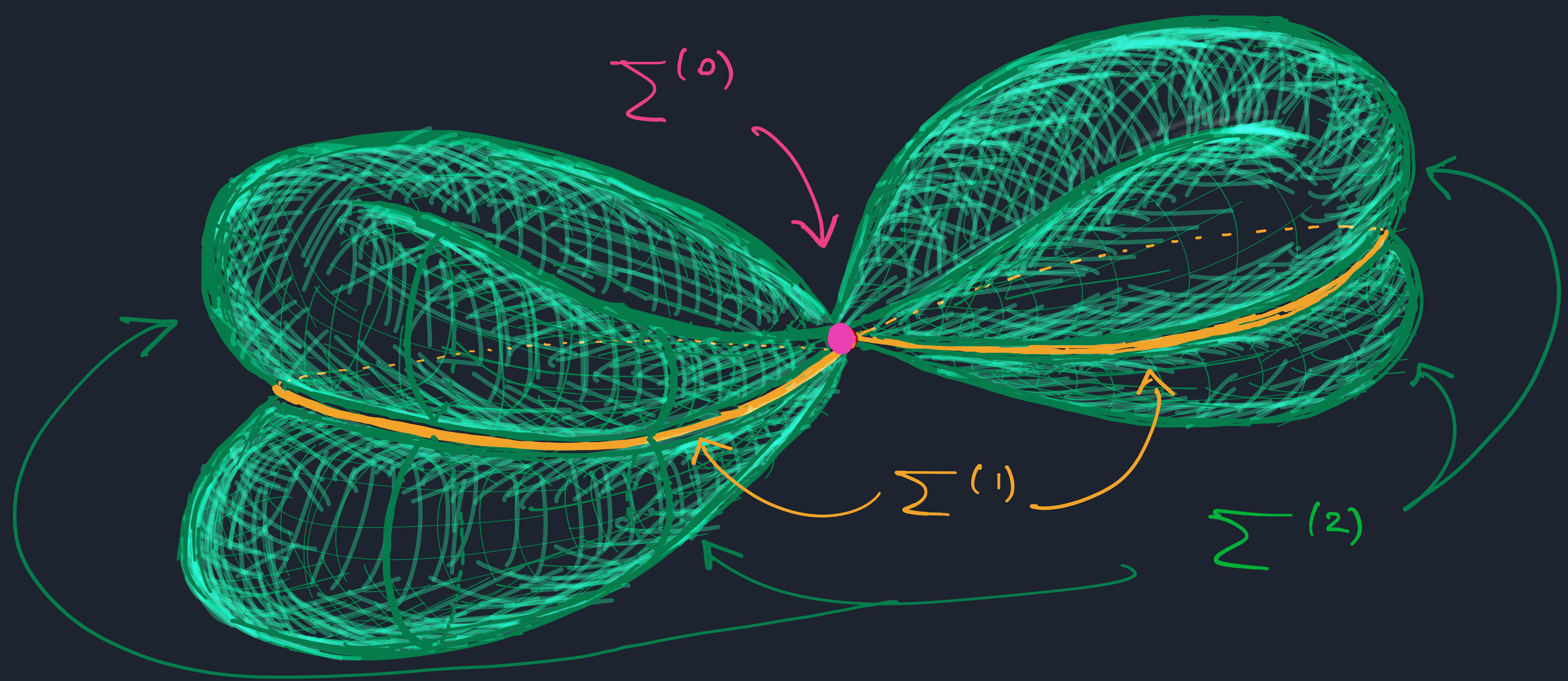

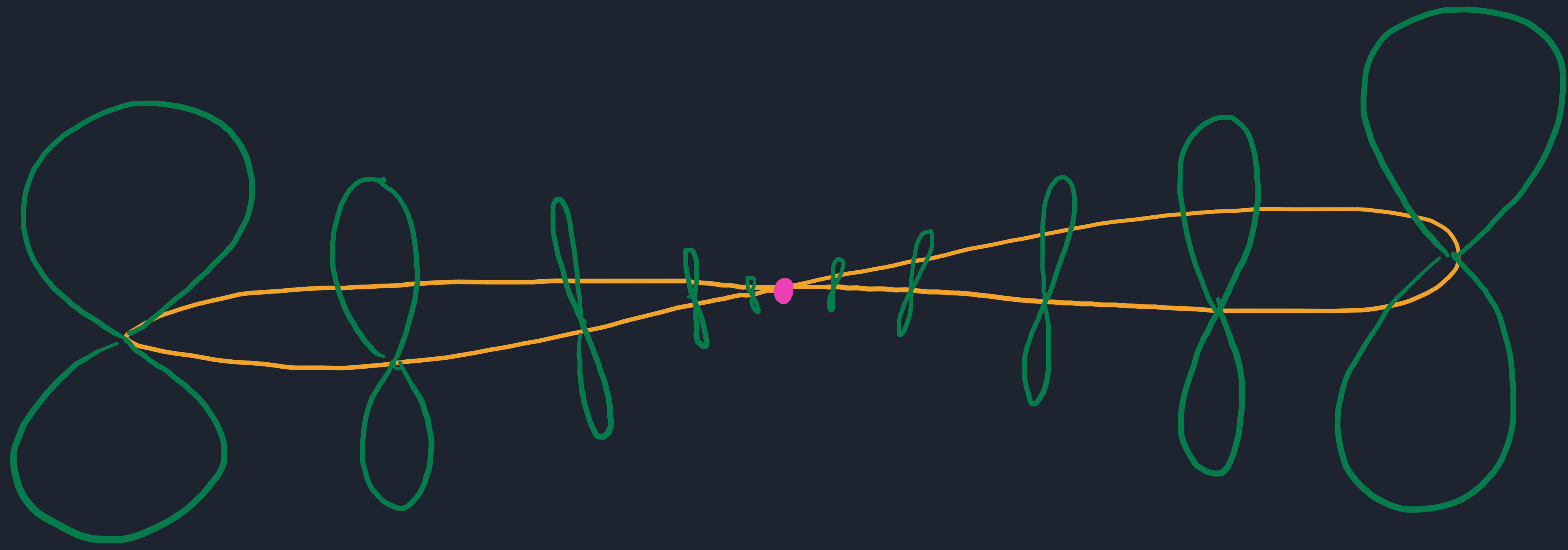

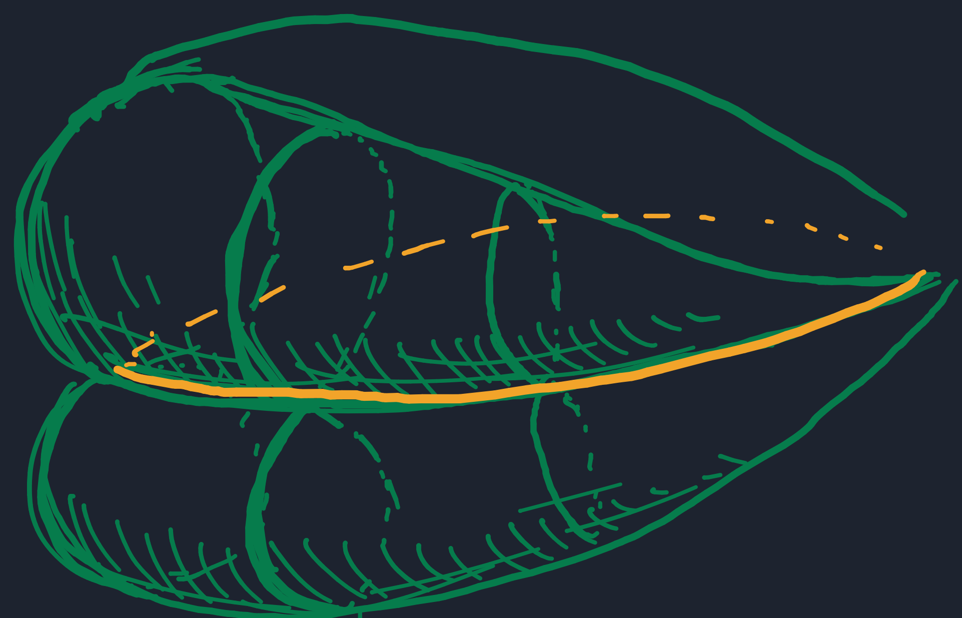

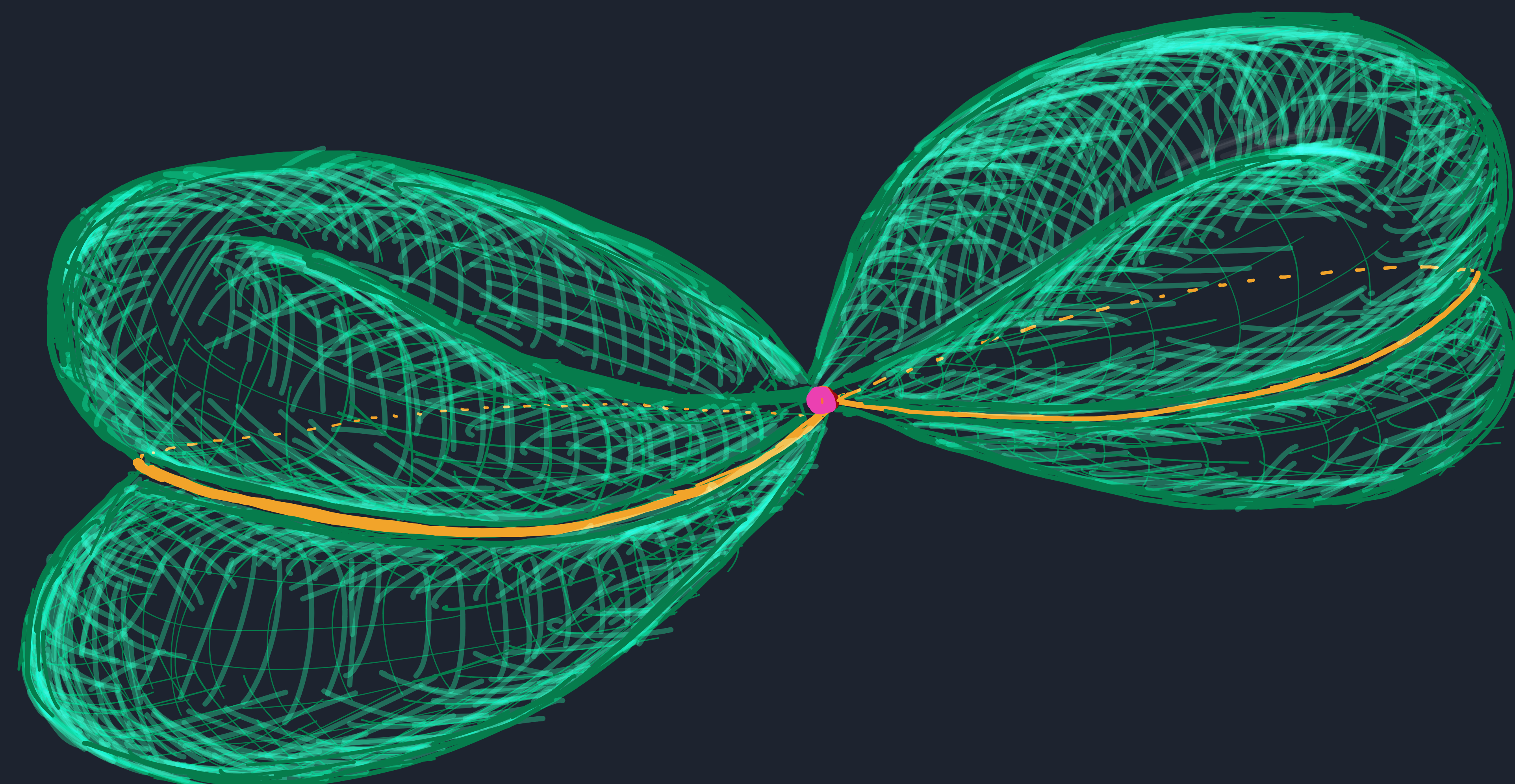

THEOREM (Borghini, Chruściel, ——)

Let Σ be the TOP STRATUM of $MAX(u)$. Then

$$\nabla u \neq 0 \text{ on } \overline{\Sigma} \implies \Sigma = \overline{\Sigma}$$

so that Σ is a REAL ANALYTIC HYPERSURFACE with $\partial\Sigma = \emptyset$

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Back to our problem:

$$MAX(u) = \Sigma_1 \sqcup \dots \sqcup \Sigma_\ell \sqcup \{p_1, \dots, p_k\}$$

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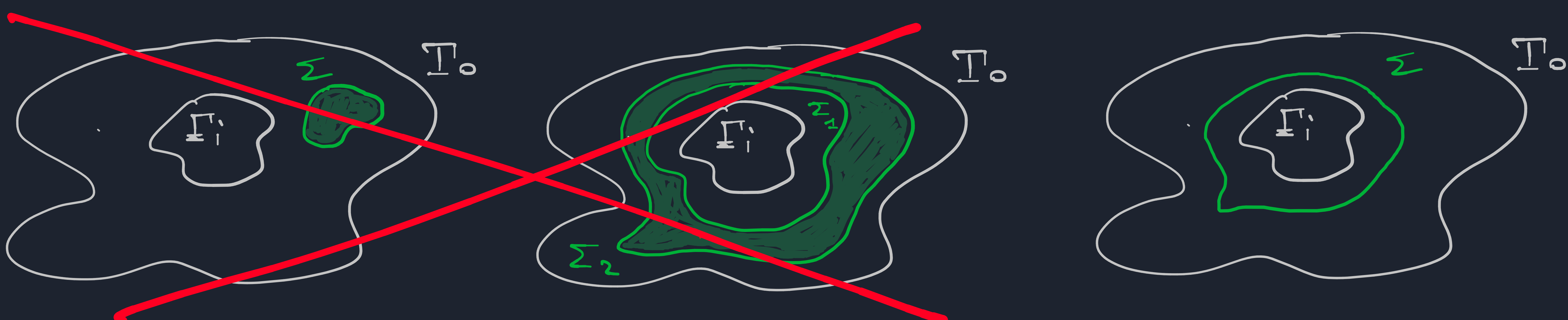

NO ISLAND LEMMA

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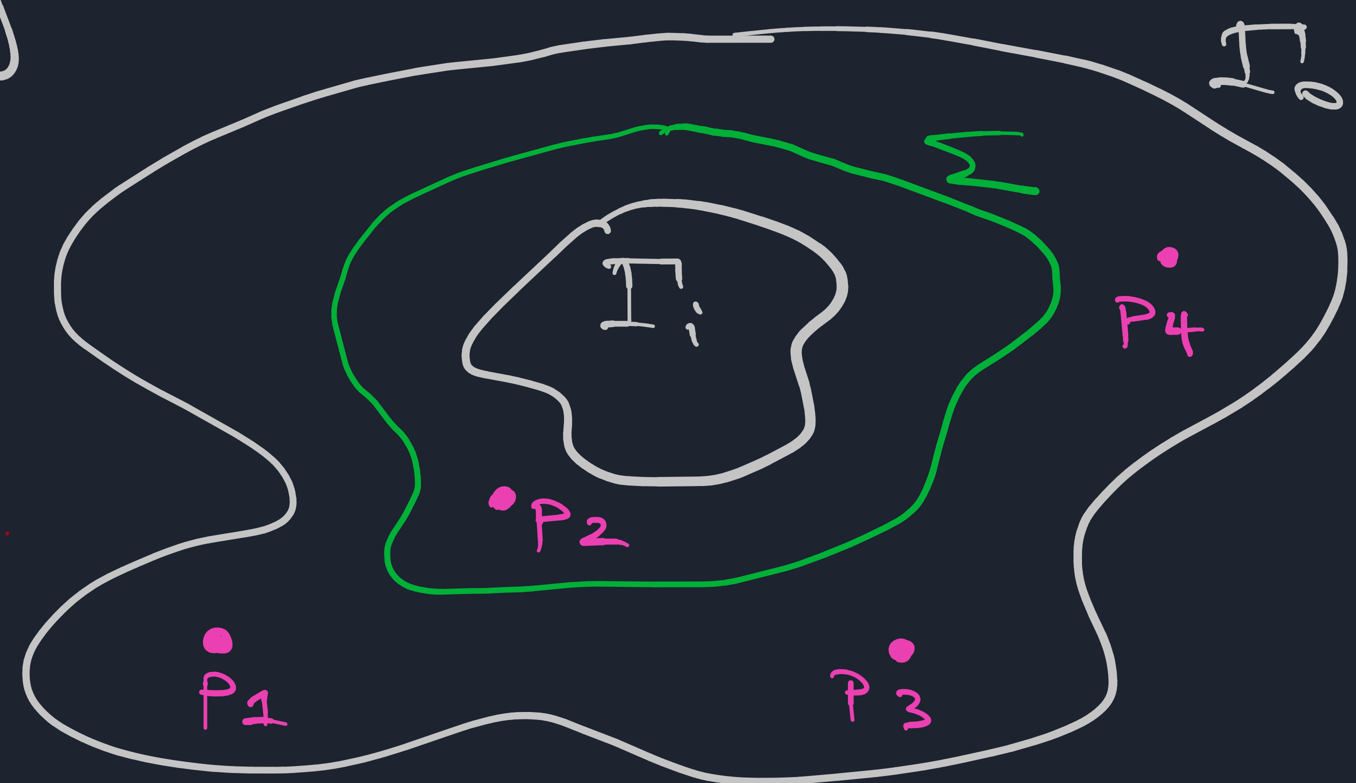
<u>Only admissible scenario:</u>

$$MAX(u) = \Sigma \sqcup \{p_1, \dots, p_k\}$$

A simple smooth closed curve separating Γ_0 from Γ_i

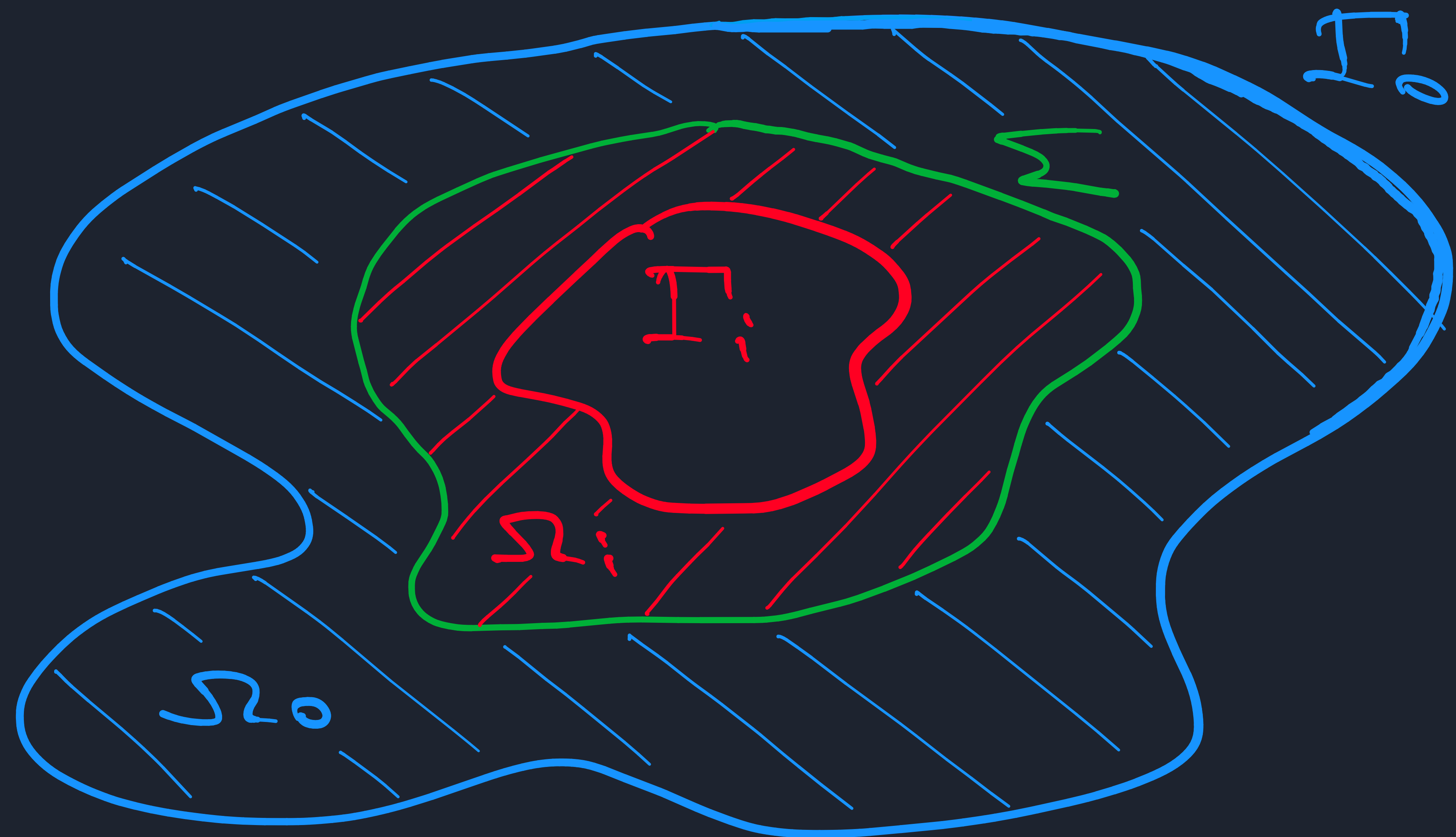


$|\mathrm{MAX}(u)| = +\infty \implies$ The TOP STRATUM $\Sigma \neq \emptyset$ is a REAL ANALYTIC CLOSED CURVE

and Σ divides Ω into 2 regions

$$\Omega = \Omega_0 \sqcup \Sigma \sqcup \Omega_i$$

outer region (for Ω_0); inner region (for Ω_i)



1/2 PROOF OF THEOREM (B)

Assume:

$$\boxed{\frac{\partial u}{\partial \nu} \equiv c_0 < 0 \text{ on } \Gamma_0}$$

1/2 PROOF OF THEOREM (B)

ASSUME: $$\frac{\partial u}{\partial \nu} \equiv c_0 < 0 \quad \text{on } \Gamma_0$$

$$(\circledast)\quad \begin{cases} \Delta u = -2 & \text{in } \Omega_0 \\ u = 0 & \text{on } \Gamma_0 \\ u = u_{max} & \text{on } \Sigma \end{cases} \quad \& \quad \boxed{\begin{array}{ll} \frac{\partial u}{\partial \nu} \equiv c_0 < 0 & \text{on } \Gamma_0 \\ \frac{\partial u}{\partial \nu} \equiv 0 \leq 0 & \text{on } \Sigma \end{array}}$$

Conclude using REICHEL and SYRAKOV THM

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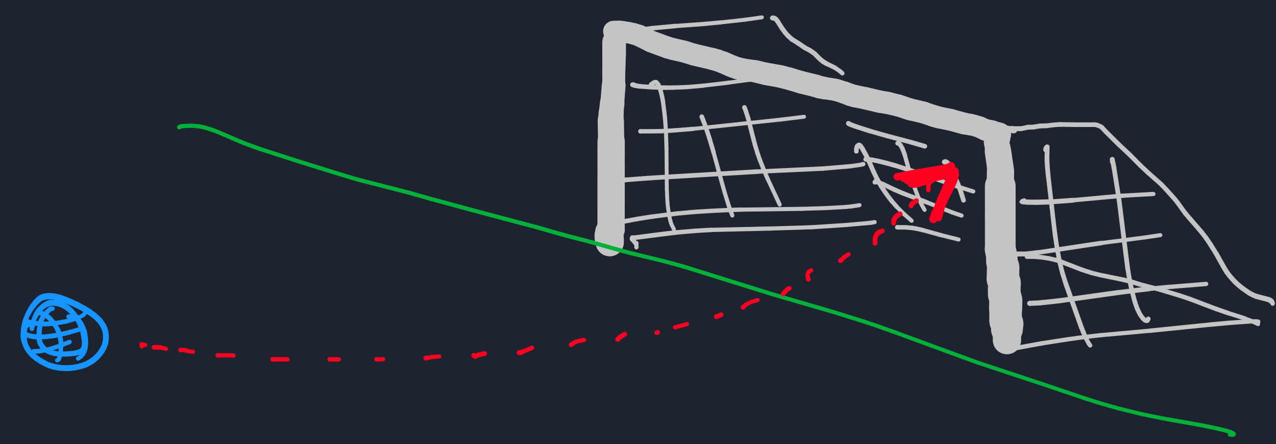
RmK "EXTREMAL" CASE $a = u_{max}$ $\quad C_a = 0 \quad \Gamma_a = \Sigma$

Conclude using REICHEL and SYRAKOV THM

□

COMPARISON GEOMETRY

GOAL:



Introduce a "COMPARISON ALGORITHM" in order to compare the generic solution with the model ones, establishing SHARP AND RIGID A PRIORI BOUNDS for relevant geometric/analytic quantities.

Rmk A NORMALISATION is required in order to get rid of the INVARIANCES of the equation and run the COMPARISON ALGORITHM SUCCESSFULLY.

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INVARIANCE by (TRANSLATIONS and) RESCALINGS:

(Ω, u) SOLUTION to $\circledast$ $\implies$ $\forall \lambda \in \mathbb{R}^+, \forall a \in \mathbb{R}^n$:

$$\begin{cases} \Omega_{\lambda,a} := \lambda\Omega + a \\ u_{\lambda,a} := \lambda^2 u(\lambda x + a) \end{cases}$$

is still a SOLUTION to $\circledast$

<u>RECALL:</u> Model solutions are parametrized by the value of the "CORE RADIUS"

$$\boxed{0 < R < 1}$$

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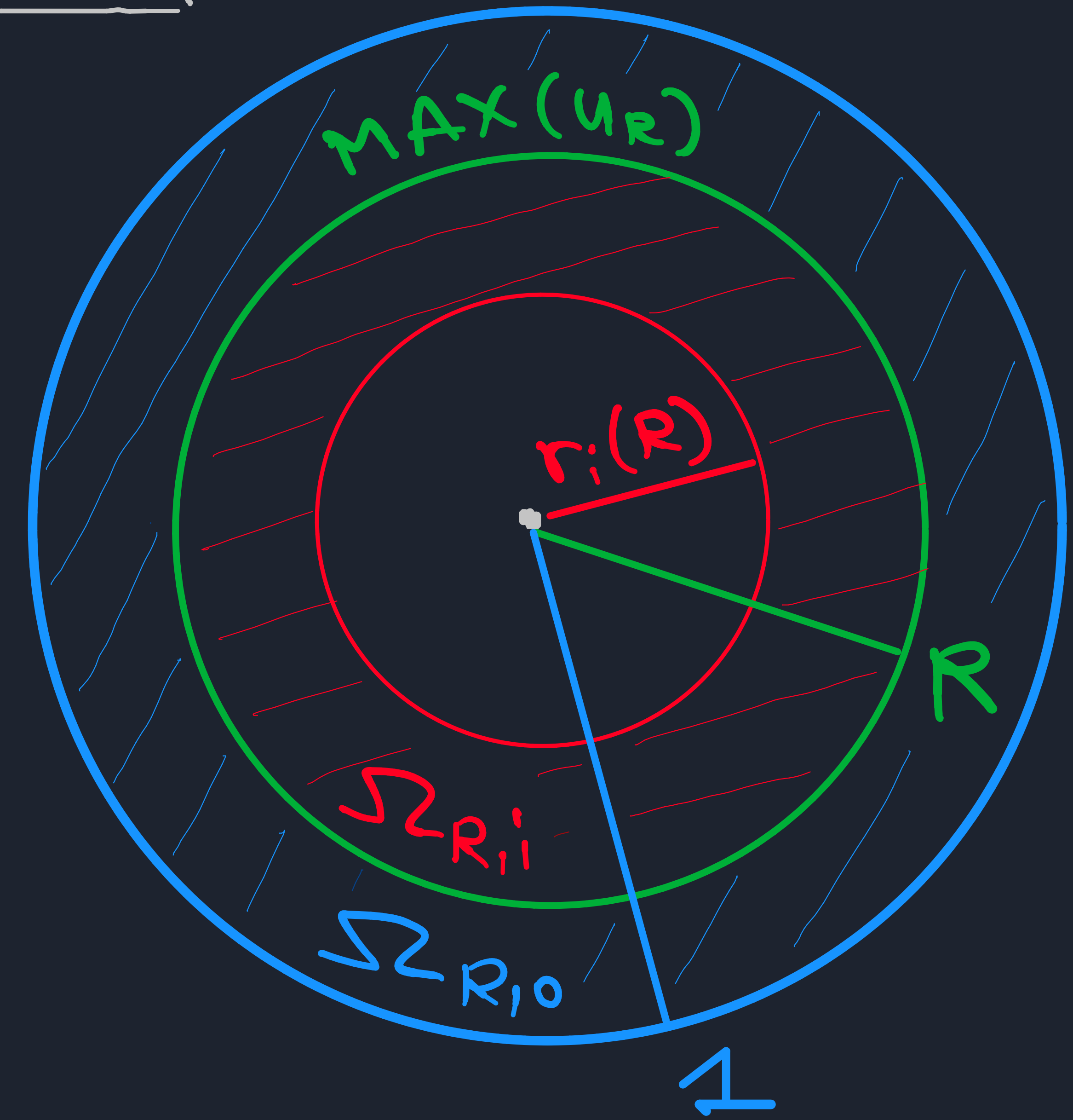
MODEL SOLUTIONS:

$$\begin{cases} u_R(x) = \dfrac{1-|x|^2}{2} + R^2 \log|x| \\ \Omega_R = \mathbb{B}^2 \setminus B(0, r_i(R)) \end{cases}$$



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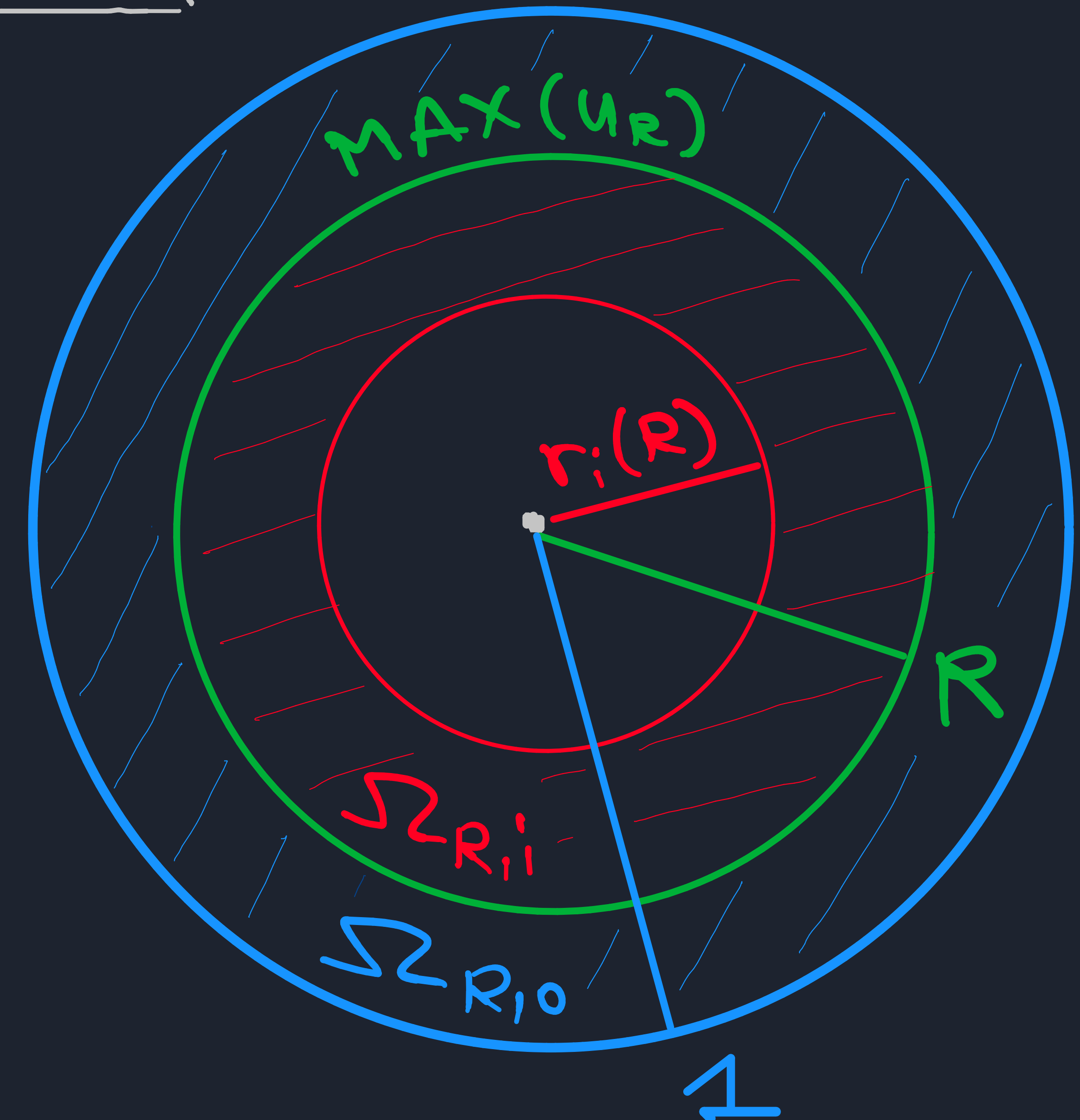

PROBLEM: Find the good R!

STRATEGY:

$(\Omega_i, u) \rightsquigarrow R = R(\Omega_i, u)$ **GUESS**

↓

RESCALE (Ω_i, u) so that $\max_{\Omega} u = \max_{\Omega_R} u_R$

↓

COMPARE (Ω_i, u) with $(\Omega_{R,i}, u_R)$

GRADIENT ESTIMATES

$$\forall x \in \Omega_i : \quad |Du|\Big|_x \leq |Du_R|\Big|_{\psi_R(x)}$$

ψ_R "pseudo radial function" s.t. $u(x) = u_R(\psi_R(x))$

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Rmk 1 The above estimates are SHARP and RIGID.

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$\textcircled{=}$ at some point $\Rightarrow$ $\textcircled{=}$ everywhere $\Rightarrow$

$$\Rightarrow \left| DDu + \frac{2R^2}{(R^2-\psi_R^2)^2}\, du \otimes du + \left(1 - \frac{R^2}{(R^2-\psi_R^2)^2}|Du|^2\right) g_{\mathbb{R}^2} \right|^2 \equiv 0$$

GRADIENT ESTIMATES

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Rmk 1 The above estimates are SHARP and RIGID.

Rmk 2 The gradient estimates holds in a fairly great generality:

The proof DOES NOT make use of the overdet. cond. $\frac{\partial u}{\partial \nu} \equiv c_i$ on Γ_i

Sketch of the proof:

$$F_R := \frac{\psi_R^2}{|R^2 - \psi_R^2|} \left(|Du|^2\Big|_{(\cdot)} - |Du_R|^2\Big|_{\psi_R(\cdot)} \right) \quad \text{satisfies:}$$

$$\Delta F_R - \left(\frac{8R^2 |Du|^2}{|R^2 - \psi_R^2|^4} \right) F_R \geq 0 \quad \text{in } \Omega;$$

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$\Downarrow$

$F_R \leq 0$ on Ω; BY MAXIMUM PRINC.

REVERSE ŁOJASIEWICZ INEQ.

$$\forall p \in MAX(u) \quad \forall \theta < 1 \quad \exists \Omega \ni p \quad \exists C > 0 :$$

$$|Du|^2(x) \leq C \left[u_{max} - u(x) \right]^{\theta}, \quad x \in \Omega$$

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Application:

$$F_R(x) = \ldots = \psi_R \frac{|Du|^2}{|Du_R|} - \psi_R |Du_R|$$

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$$\frac{|Du|^2}{|Du_R|} \sim \frac{|Du|^2}{2(u_{max} - u)} \longrightarrow 0 , \quad \text{as } x \to MAX(u)$$

REVERSE ŁOJASIEWICZ

$$\Longrightarrow \quad F_R(x) \longrightarrow 0 \quad \text{as } x \longrightarrow p \in MAX(u)$$

SHARP AND RIGID AREA BOUNDS

INTERIOR REGION Ω_i AND $R = R(\Omega_i)$

(1)

(2)

(3)

SHARP AND RIGID AREA BOUNDS

INTERIOR REGION Ω_i AND $R = R(\Omega_i)$

① $2\pi R \leq |\Sigma|$ ← Follows from GRADIENT ESTIMATES

②

③

SHARP AND RIGID AREA BOUNDS

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(2)

(3) $|\Gamma_i| \leq 2\pi\, r_i(R)$ || ← Follows from OVERDET. COND.

$$\frac{\partial u}{\partial \nu} \equiv c_i \text{ on } \Gamma_i$$

SHARP AND RIGID AREA BOUNDS

INTERIOR REGION Ω_i AND $R = R(\Omega_i)$

① $2\pi R \leq |\Sigma|$ ← Follows from GRADIENT ESTIMATES

② $\dfrac{|\Sigma|}{R} \leq \dfrac{|\Gamma_i|}{r_i(R)}$ ←

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$\dfrac{\partial u}{\partial \nu} \equiv c_i$ on Γ_i

1/2 PROOF OF THEOREM (B)

Assume: $\frac{\partial u}{\partial \nu} \equiv C_1$ on Γ_1;

Use the area bounds:

1/2 PROOF OF THEOREM (B)

Assume: $\frac{\partial u}{\partial \nu} \equiv C_1$ on Γ_i

Use the area bounds:

$$2\pi \leq \frac{|\Sigma|}{R}$$

↑ (3)

1/2 PROOF OF THEOREM (B)

Assume: $\boxed{\dfrac{\partial u}{\partial \nu} \equiv C_1 \text{ on } \Gamma_i}$

Use the area bounds:

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1/2 PROOF OF THEOREM (B)

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Use the area bounds:

$$2\pi \leq \frac{|\Sigma|}{R} \leq \frac{|\Gamma_i|}{r_i(R)} \leq 2\pi$$

(3) (2) (1)

1/2 PROOF OF THEOREM (B)

Assume: $\boxed{\frac{\partial u}{\partial \nu} \equiv c_1 \text{ on } \Gamma_i}$

Use the area bounds:

$$2\pi = \frac{|\Sigma|}{R} = \frac{|\Gamma_i|}{r_i(R)} = 2\pi$$

Use the RIGIDITY in the GRADIENT ESTIMATES.

□

EXTERIOR REGION Ω_0 and $R = R(\Omega_0)$

① $|\Sigma| \leq 2\pi R$ ‖ ← Follows from GRADIENT ESTIMATES

② $\dfrac{|\Gamma_0|}{1} \geq \dfrac{|\Sigma|}{R}$ ‖ ←

③ $2\pi \leq |\Gamma_0|$ ‖ ← Follows from OVERDET. COND.

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EXTERIOR REGION Ω_0 and $R = R(\Omega_0)$

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③ $2\pi \leq |\Gamma_0|$ $\quad \|$ ← Follows from OVERDET. COND.

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WRONG WAY !

Rmk The following AREA BOUNDS do not depend on the OVERDET. COND'S!

$$2\pi R(\Omega_i) \leq |\Sigma| \leq 2\pi R(\Omega_o)$$

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THEOREM (G) (Agostiniani, Borghini, ———)

Let (Ω, u) be a solution to $\circledast$ s.t.

$|\mathrm{MAX}(u)| = +\infty$ & $\tau(\Omega_o) < 2$

$\Longrightarrow$ $\begin{cases} R(\Omega_i) \leq R(\Omega_o) \\ \text{with } (=) \iff \text{ROT. SYMM.} \end{cases}$

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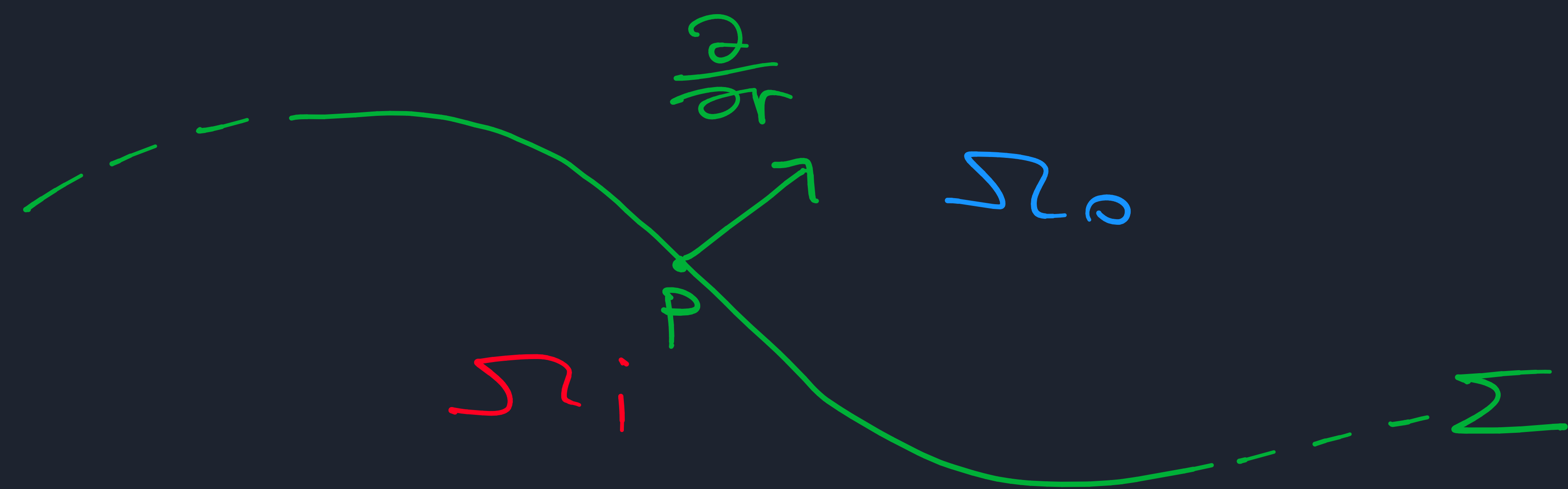
Sketch of the proof:

Taylor expansion of $|Du|^2$ & $|Du_R|^2\big|_{\psi_R(\cdot)}$ about Σ:

$$|Du|^2 = 4r^2\left[1 + K(p)\, r\right] + \mathcal{O}(r^4)$$

$$|Du_R|^2 = 4r^2\left[1 + \left(\frac{K(p)}{3} \mp \frac{2}{3R}\right) r\right] + \mathcal{O}(r^5)$$



K(p) = CURVATURE of Σ (w.r.t. $\frac{\partial}{\partial r}$)

GRADIENT ESTIMATES w.r.t. $R(\Omega_i)$, $R(\Omega_o)$ $\Longrightarrow$

$$\frac{1}{R(\Omega_o)} \leq K(p) \leq \frac{1}{R(\Omega_i)} \quad \forall p \in \Sigma$$

Integrating along Σ gives:

$$\int_{\Sigma} \frac{ds}{R(\Omega_o)} \leq \int_{\Sigma} K \, ds \leq \int_{\Sigma} \frac{ds}{R(\Omega_i)}$$

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$$\int_\Sigma \frac{ds}{R(\Omega_o)} = \frac{|\Sigma|}{R(\Omega_o)}, \qquad \int_\Sigma K \, ds = 2\pi, \qquad \int_\Sigma \frac{ds}{R(\Omega_i)} = \frac{|\Sigma|}{R(\Omega_i)}$$

Integrating along Σ gives:

$$\frac{|\Sigma|}{R(\Omega_o)} \leq 2\pi \leq \frac{|\Sigma|}{R(\Omega_i)}$$

(first $\leq$): AREA BOUND (1)

(second $\leq$): AREA BOUND (1)

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Rigidity: Assume $R(\Omega_0) = R = R(\Omega_i)$:

Integrating along Σ gives:

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Rigidity: Assume $R(\Omega_0) = R = R(\Omega_i)$:

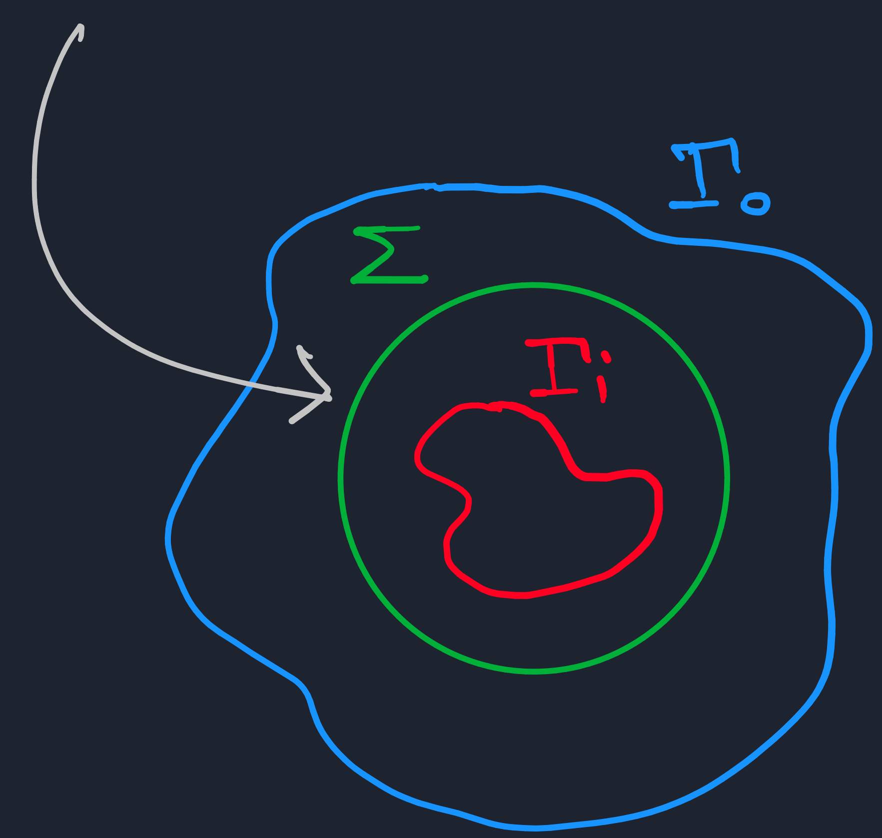
$$\Rightarrow K \equiv \frac{1}{R}$$

Integrating along Σ gives:

$$\frac{|\Sigma|}{R(\Omega_o)} \leq 2\pi \leq \frac{|\Sigma|}{R(\Omega_i)}$$

Rigidity: Assume $R(\Omega_o) = R = R(\Omega_i)$:

$\Rightarrow K \equiv \frac{1}{R} \Rightarrow \Sigma$ is a ROUND CIRCLE of RADIUS R !!!



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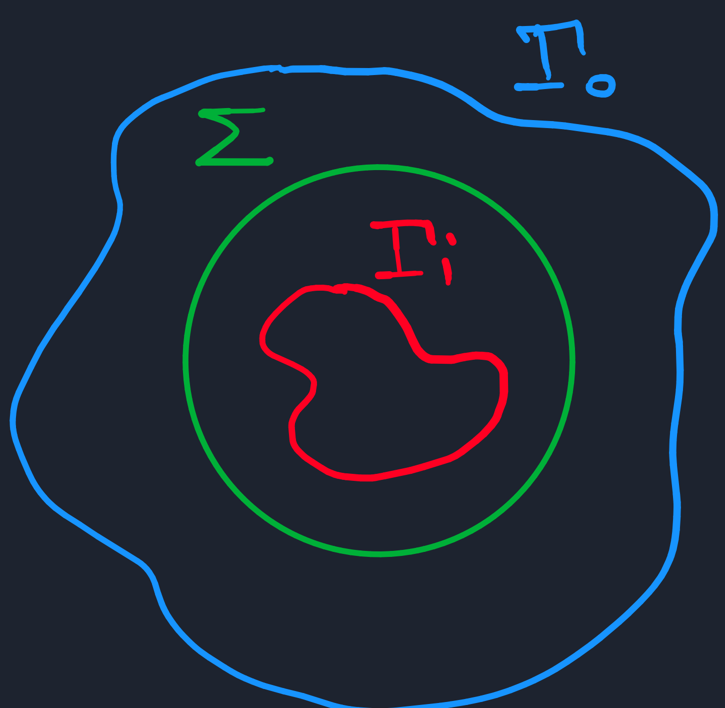

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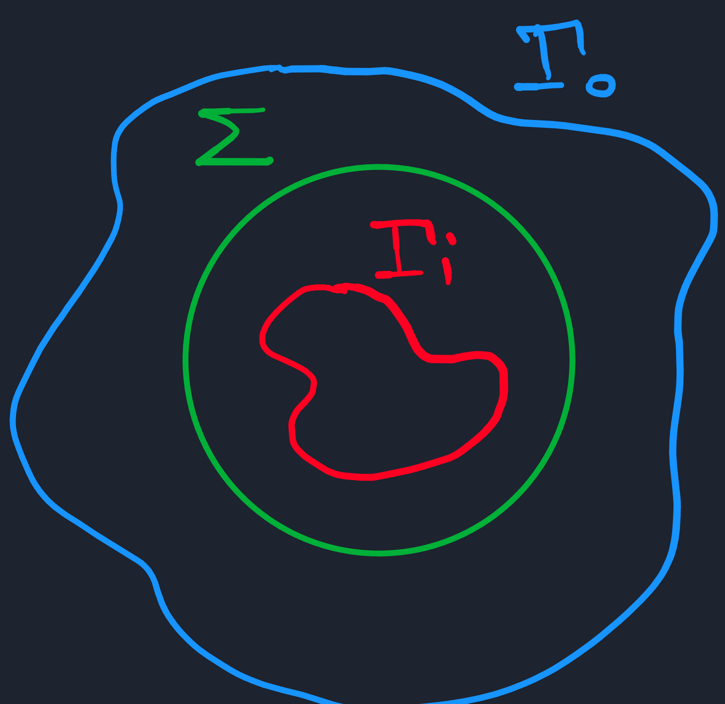
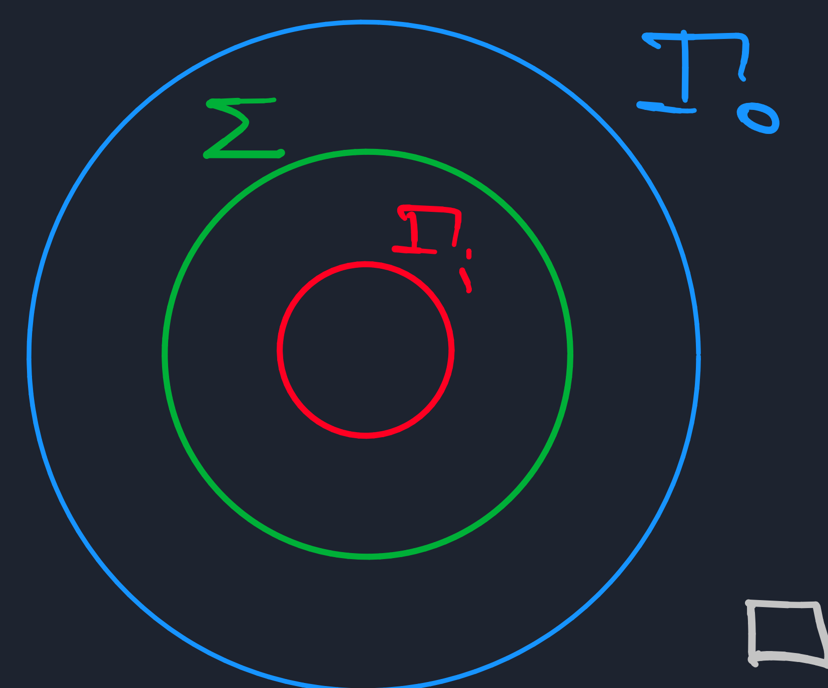


I.V.P.

$$\begin{cases} \Delta u = -2 & , \Omega \\ u = (u_R)_{max} & , \Sigma \\ \frac{\partial u}{\partial \nu} = 0 & , \Sigma \end{cases}$$

UNIQUENESS in CAUCHY-KOVALEVSKAYA'S THEOREM: $\Longrightarrow$



$\square$

Q How to find the "GOOD R" ?

A Use the following scaling invariant quantities:

- $\tau(\Omega_i) = \dfrac{\max_{I_i} |Du|^2}{2\, u_{max}}$

- $\tau(\Omega_0) = \dfrac{\max_{I_0} |Du|^2}{2\, u_{max}}$

Q How to find the "GOOD R"?

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- $$\tau(\Omega_i) = \frac{\max_{I_i} |Du|^2}{2\,u_{max}} = \frac{c_i^2}{2\,u_{max}}$$
- $$\tau(\Omega_0) = \frac{\max_{I_0} |Du|^2}{2\,u_{max}} = \frac{c_0^2}{2\,u_{max}}$$

(the last equalities hold) IF OVERDET. BOUNDARY CONDITIONS ARE IN FORCE

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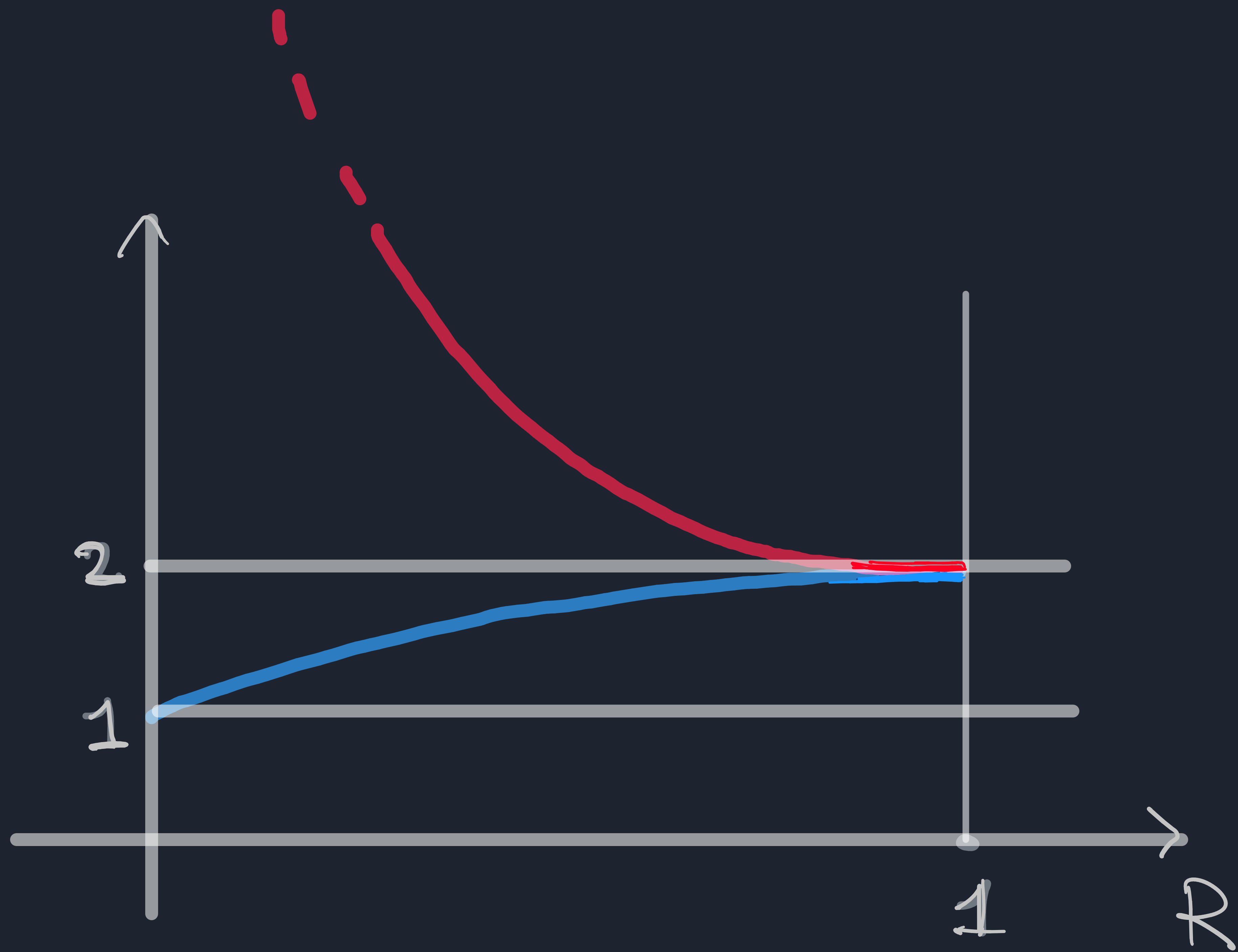
- $\tau(\Omega_o) = \dfrac{\max_{I_o} |Du|^2}{2\,u_{max}} = \dfrac{C_o^2}{2\,u_{max}}$

IF OVERDET. BOUNDARY CONDITIONS ARE IN FORCE

<u>Fluid dynamical interpretation</u>:

think of $\tau(\Omega_{\dots})$ as a "NORMALIZED WSS" due to the fluid moving along the portion of the pipe with section $\Omega_{\dots}$

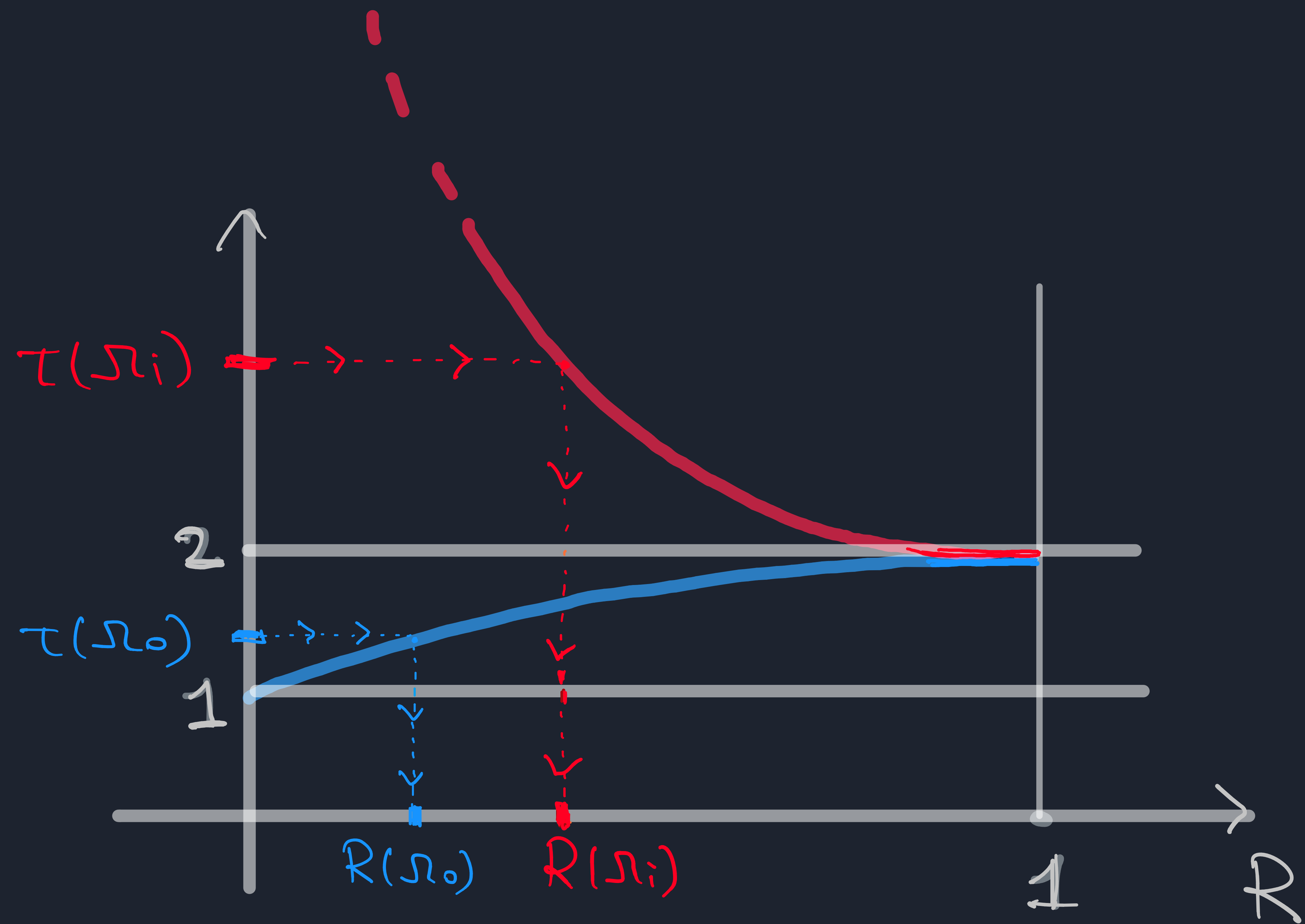
NORMALIZED WSS's as functions of the CORE RADIUS R



- graph of
  $R \mapsto \tau_i(\Omega_{R,i})$
  NORMALIZED WSS for MODEL SOLUTIONS at the INNER BOUNDARY

- graph of
  $R \mapsto \tau_o(\Omega_{R,o})$
  NORMALIZED WSS for MODEL SOLUTIONS at the OUTER BOUNDARY

NORMALIZED WSS's as functions of the CORE RADIUS R



- graph of $R \mapsto \tau_i(\Omega_{R,i})$

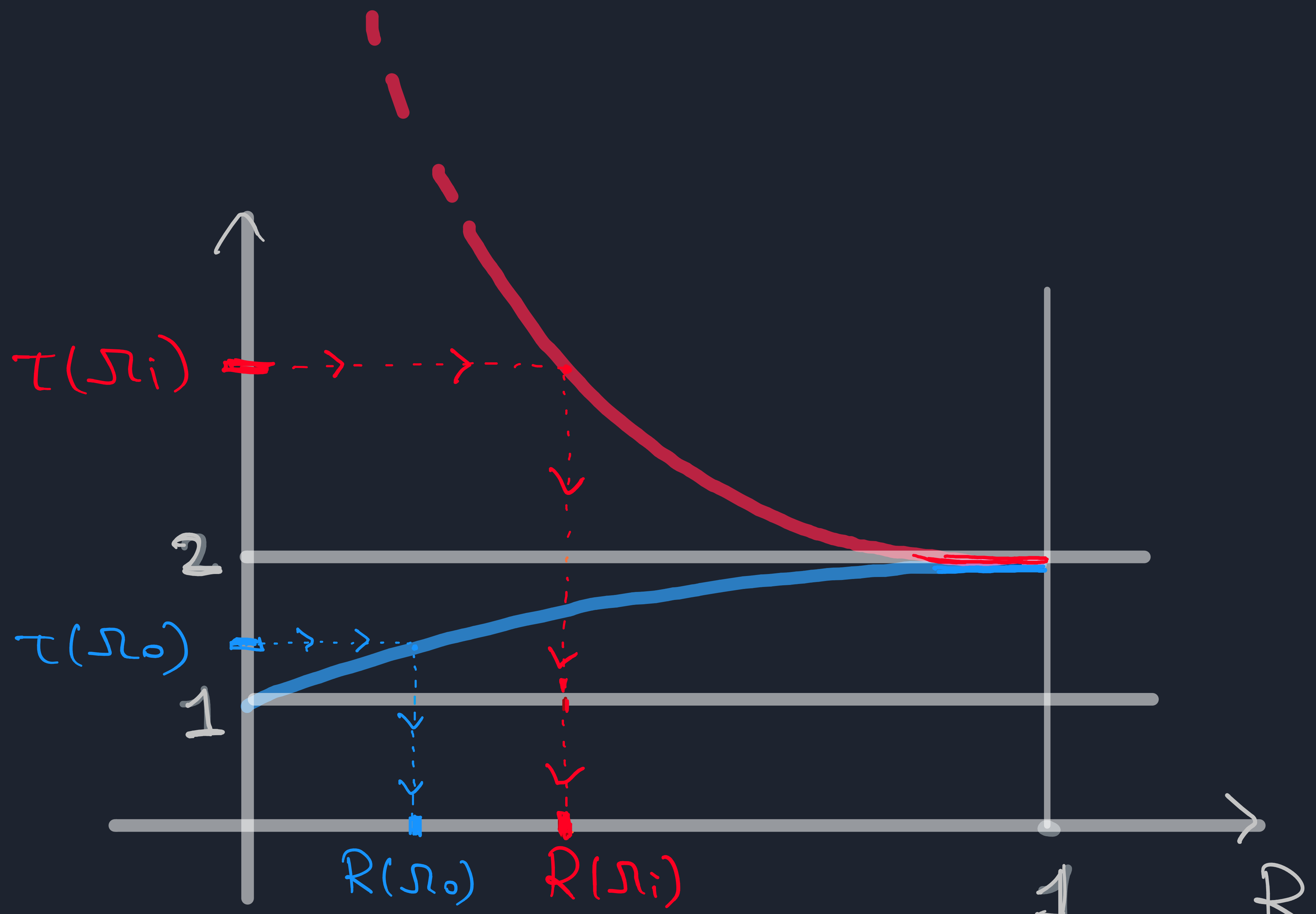
  NORMALIZED WSS for MODEL SOLUTIONS at the INNER BOUNDARY

- graph of $R \mapsto \tau_o(\Omega_{R,o})$

  NORMALIZED WSS for MODEL SOLUTIONS at the OUTER BOUNDARY

PROBLEMS: (1) Who guarantees that $\tau(\Omega_i) > 2$?

NORMALIZED WSS'S as functions of the CORE RADIUS R



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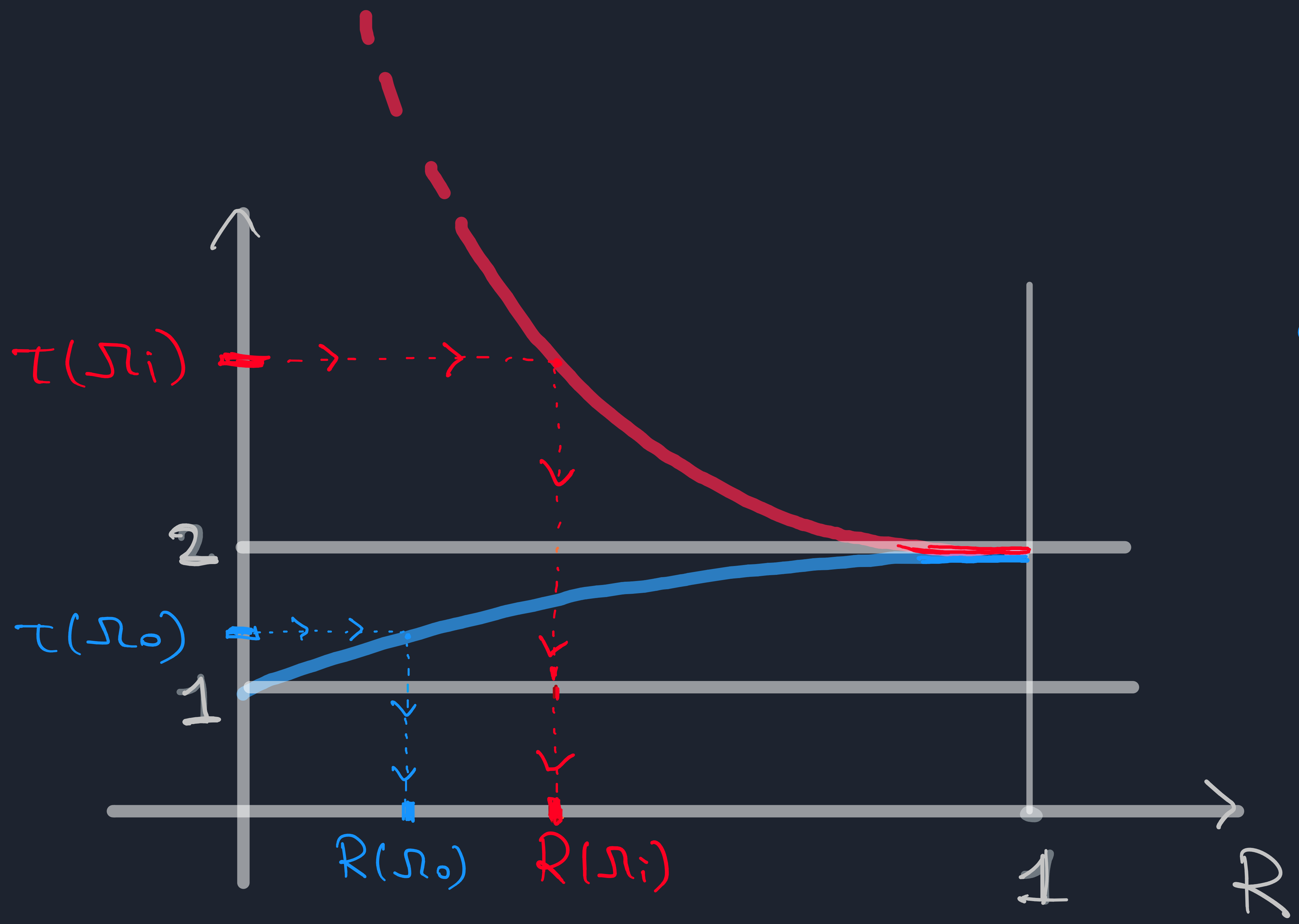
  NORMALIZED WSS for MODEL SOLUTIONS at the INNER BOUNDARY

- graph of
  $R \mapsto \tau_o(\Omega_{R,o})$
  NORMALIZED WSS for MODEL SOLUTIONS at the OUTER BOUNDARY

PROBLEMS:

1. Who guarantees that $\tau(\Omega_i) > 2$?
2. Who guarantees that $\tau(\Omega_o) < 2$?

NORMALIZED WSS's as functions of the CORE RADIUS R



- graph of $R \mapsto \tau_i(\Omega_{R,i})$

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- graph of $R \mapsto \tau_o(\Omega_{R,o})$

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PROBLEMS:

① Who guarantees that $\tau(\Omega_i) > 2$?

② Who guarantees that $\tau(\Omega_o) < 2$?

③ Who guarantees that $\tau(\Omega_i), \tau(\Omega_o) > 1$?

Using POHOZAEV'S IDENTITIES one gets:

LEMMA

Let N be a connected comp. of $\Omega \setminus MAX(u)$ and denote by Γ_N its (connected) boundary portion. If $|Du| \equiv const$ on Γ_N, then:

1. $\tau(N) > 2$ iff N is INNER,
2. $\tau(N) < 2$ iff N is OUTER.

Moreover one always has $\tau(N) \neq 2$.

③ Who guarantees that: $\tau(\Omega_0), \tau(\Omega_i) > 1$?

③ Who guarantees that: $\tau(\Omega_o), \tau(\Omega_i) > 1$?

THEOREM (Agostiniani, Borghini, ——)

Let N be a connected comp. of $\Omega \setminus MAX(u)$ and denote by Γ_N its boundary portion. Then:

$$\tau(N) \leq 1 \implies (\Omega, u) \sim \text{SERRIN SOLUTION}$$

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<u>Fluid-dynamics</u>: "SERRIN SLN'S have the least possible NORMALIZED WSS"

(3) Who guarantees that: $\tau(\Omega_o), \tau(\Omega_i) > 1$?

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Fluid-dynamics: "SERRIN SLN'S have the least possible NORMALIZED WSS"

COROLLARY The algorithm $(N, u) \rightsquigarrow R(N)$ is well defined !!!

G.R. STATIC METRICS with $\Lambda > 0$

(M^3, g, u) cpt. Riem mfld with ∂M^3 smooth

$u : M^3 \longrightarrow \mathbb{R}$ smooth funct.

$$\begin{cases} u \, Ric_g = DDu + 3u\, g \,, & \text{in } M^3 \\ \Delta_g u = -3u \,, & \text{in } M^3 \\ u = 0 \,, & \text{on } \partial M^3 \end{cases}$$

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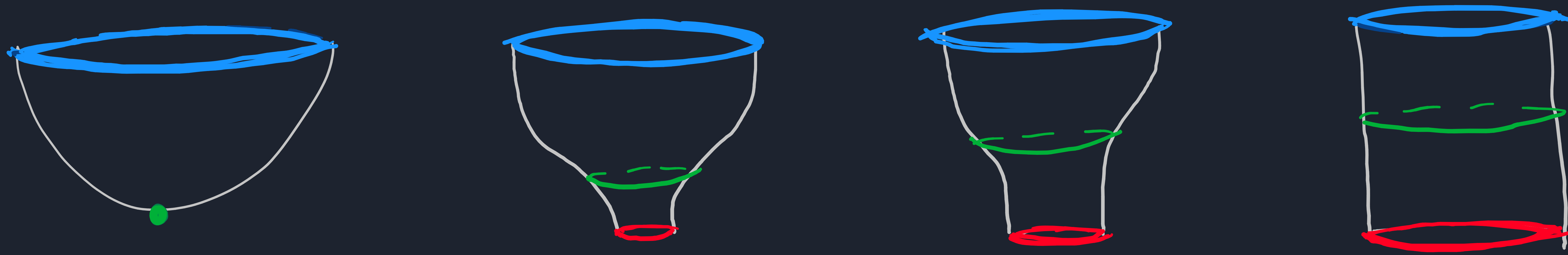
MODEL SOLUTIONS:

$$u_m = \sqrt{1 - r^2 - \frac{2m}{r}} \qquad M^3 = (r_-(m), r_+(m)) \times \mathbb{S}^2$$

$$g = \frac{dr \otimes dr}{u_m^2(r)} + r^2 g_{\mathbb{S}^2} \qquad \mathrm{MAX}(u_m) = \{ r = m^{1/3} \}$$



DE SITTER

$m = 0$

DESITTER SCHWARZSCHILD

$0 < m < 3\sqrt{3}$

NARIAI

$m = 3\sqrt{3}$

Γ_0 = COSMOLOGICAL HORIZON

Γ_i = BLACK HOLE HORIZON

Σ = MAX(u_m)

COMPARISON ALGORITHM

$$(N, u, g) \rightsquigarrow m(N) \geq 0$$

"VIRTUAL MASS"

COMPARISON ALGORITHM

$(N, u, g) \rightsquigarrow m(N) \geq 0$

"VIRTUAL MASS"

THEOREM (Borghini, Chruściel, ——)

Let $\Sigma = MAX(u)$ and assume

$$\Pi = \Pi_o \sqcup \Sigma \sqcup \Pi_i$$

$$\Rightarrow \quad m(\Pi_o) \leq m(\Pi_i)$$

with $=$ $\Leftrightarrow$ MODEL SOLUTION