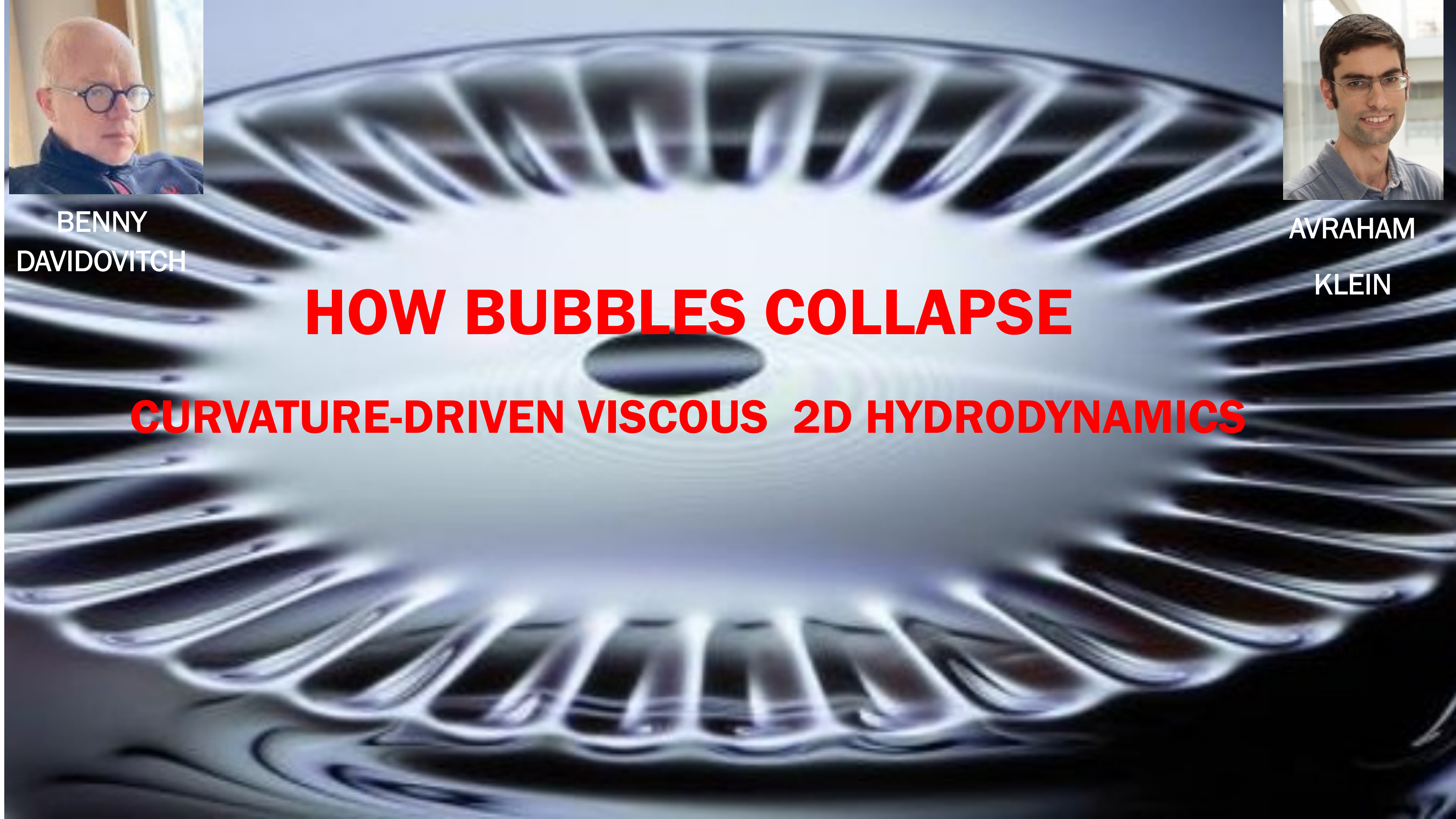


BENNY DAVIDOVITCH

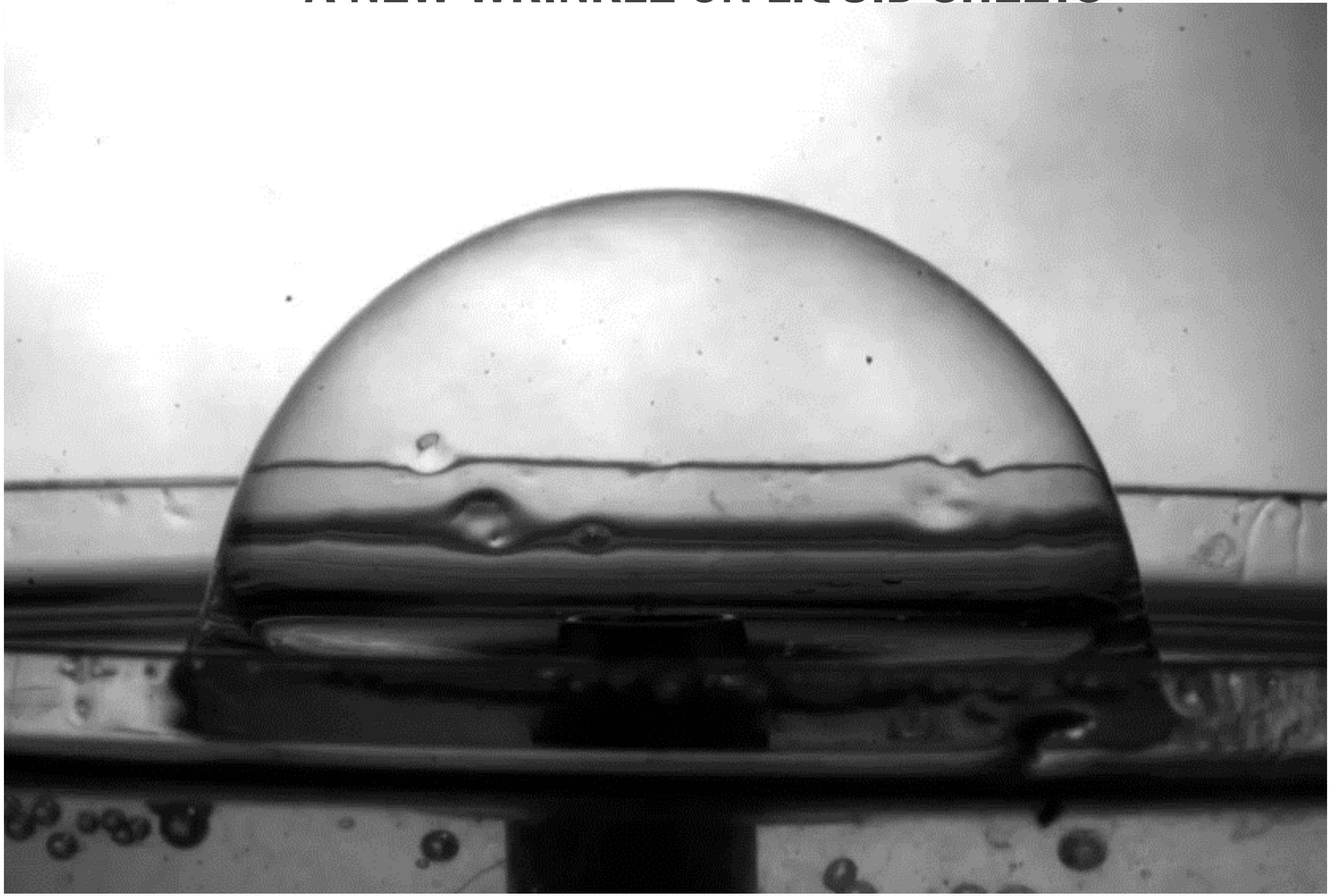
AVRAHAM KLEIN

HOW BUBBLES COLLAPSE

CURVATURE-DRIVEN VISCOUS 2D HYDRODYNAMICS

Debrégeas, … 1998; De Silviera 2000 …; Oratis, … 2020

BACKGROUND: A RECENT EXPERIMENT (ORATIS-BUSH-STONE-BIRD, SCIENCE 2020)
“A NEW WRINKLE ON LIQUID SHEETS”



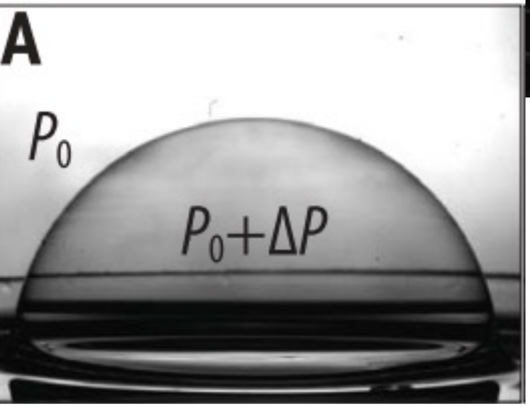
AT ORATIS

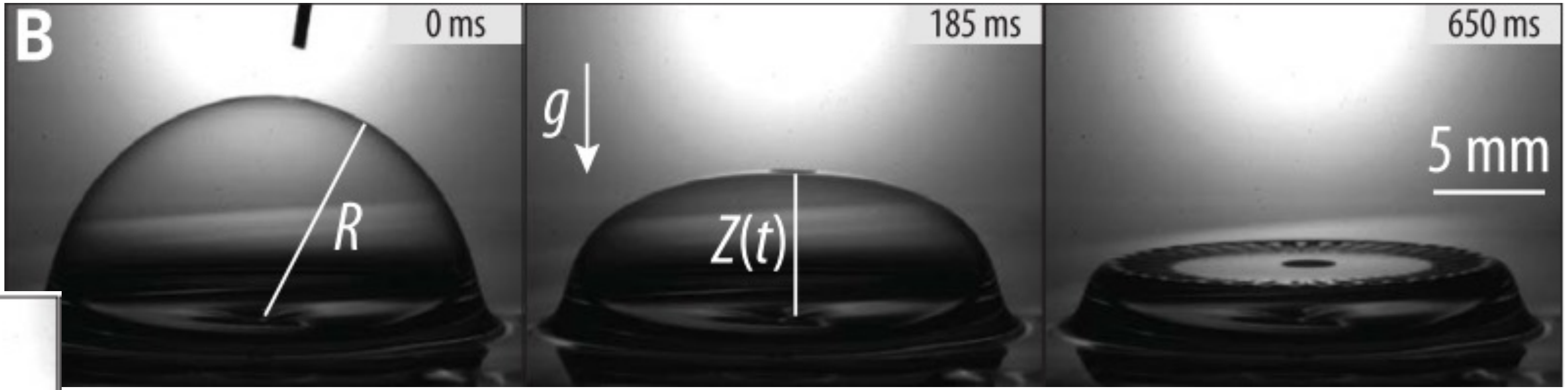
JC BIRD

Debrégeas, … 1998; Oratis, … 2020

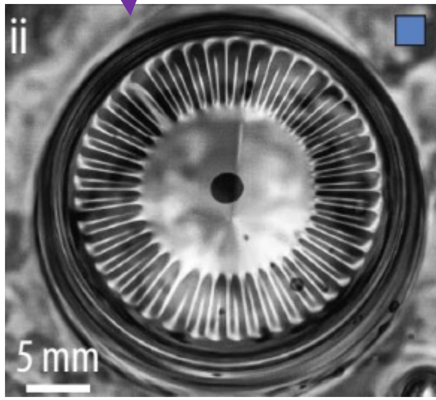
BACKGROUND: A RECENT EXPERIMENT (ORATIS-BUSH-STONE-BIRD, SCIENCE 2020)
"A NEW WRINKLE ON LIQUID SHEETS"

sudden decompression

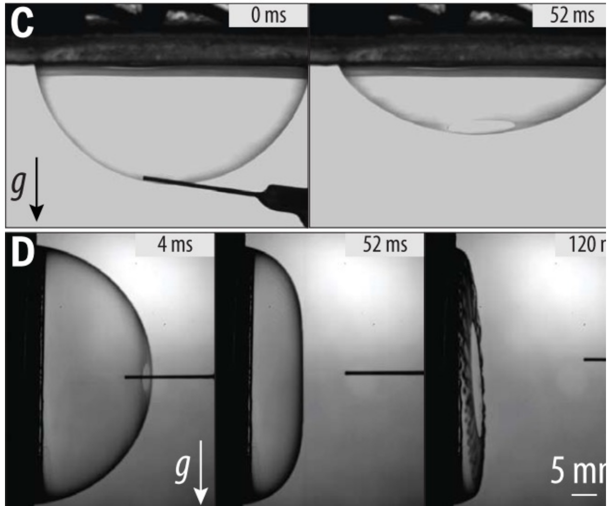


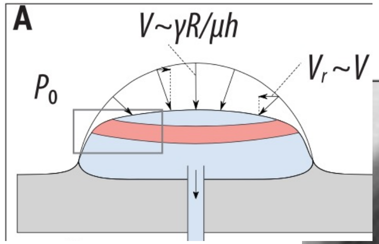


wrinkles

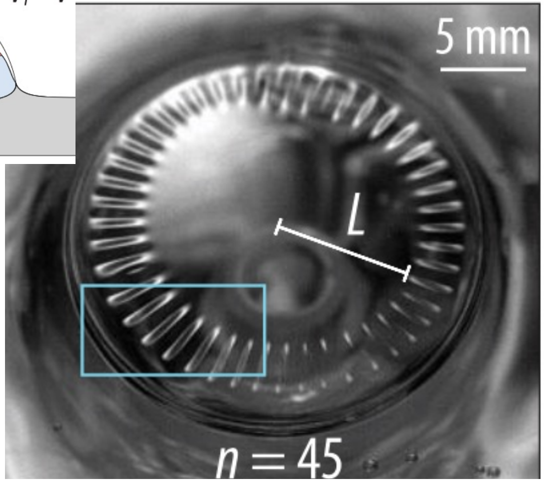


NO GRAVITY





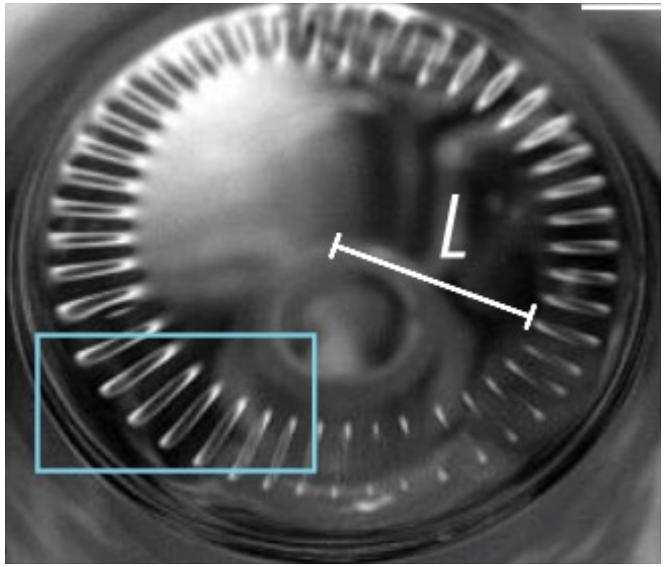
NO RUPTURE



CURVATURE DRIVEN HYDRODYNAMICS OF 2D LIQUIDS

Experiment suggests dynamics of bubble collapse is governed by:

viscous hydrodynamics ←→ interfacial thermodynamics



WHAT IS "HYDRODYNAMICS" ?

Macroscopic flow, characterized in the bulk by <u>local thermodynamics</u> & <u>conservation laws:</u>

"Ideal" (inviscid) fluid

Energy

Momentum

Mass

(Euler ~1750)

viscous fluid

~~Energy~~

Momentum
(diffusion)

Mass

(Navier-Stokes ~1820)

Trouton ~ 1900

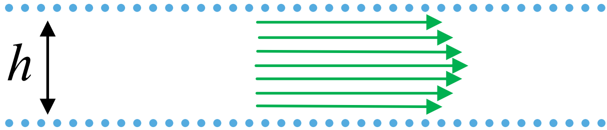
$$\rho \frac{d\mathbf{v}}{dt} \propto \nabla \sigma \qquad \sigma \propto \eta \nabla \mathbf{v}$$

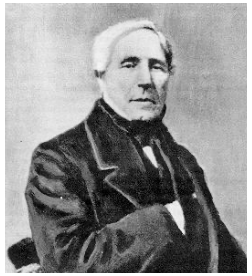
Films: **free surfaces**

~~Energy~~

Tangential Momentum

Mass





Poiseuille ~ 1840

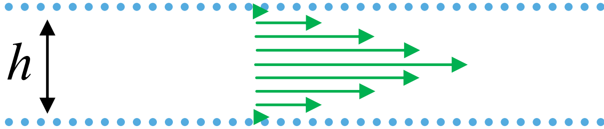
$$\rho \frac{d\mathbf{v}}{dt} \propto -\nabla p + \frac{\eta}{h^2}\mathbf{v}$$

Films (lubrication):

~~Energy~~

~~Tangential Momentum~~

Mass



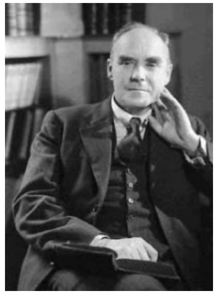
MOMENTUM-CONSERVING HYDRODYNAMICS OF CURVED VISCOUS FILMS

Trouton ~ 1900

~~Energy~~

Tangential Momentum

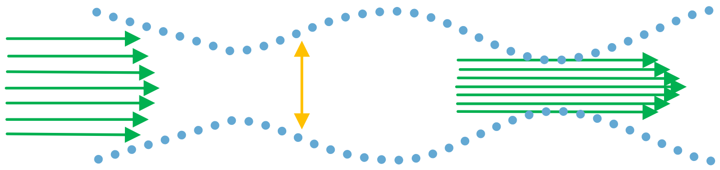
Mass



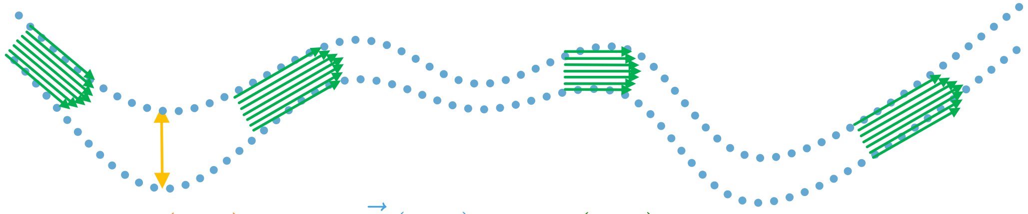
Taylor ~ 1960

Bucmaster-Nachman-Ting 1975

"viscida" ⟷ *elastica*



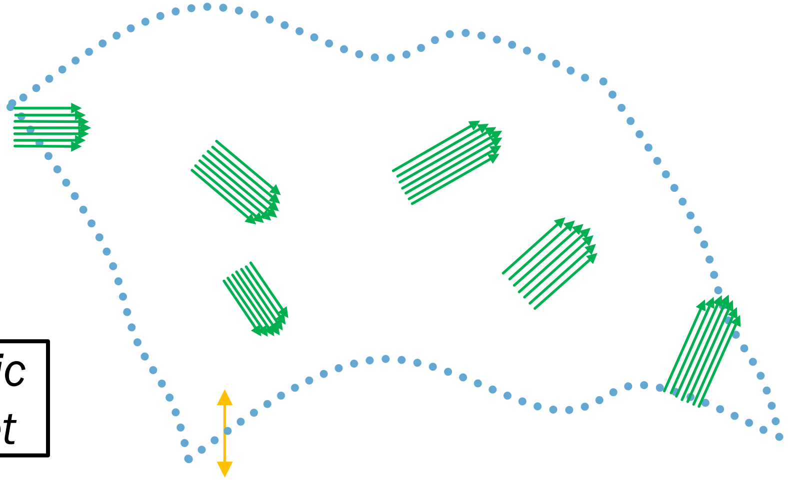
$h(x,t)$ $v(x,t)$



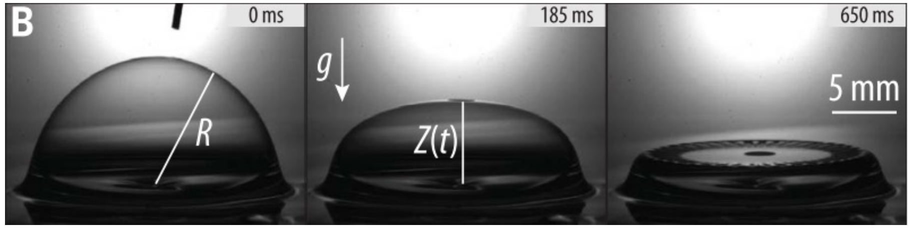
$h(s,t)$ $\vec{X}(s,t)$ $v(s,t)$

Howell 1996

viscous film ⟷ *elastic sheet*



$h(x,y,t)$ $\vec{X}(x,y,t)$ $\vec{v}_{\|}(x,y,t)$



compression ⟹ buckling/wrinkling

... but why compression ??

OUTLINE

- Introduction (+ bubble collapse in a nutshell)

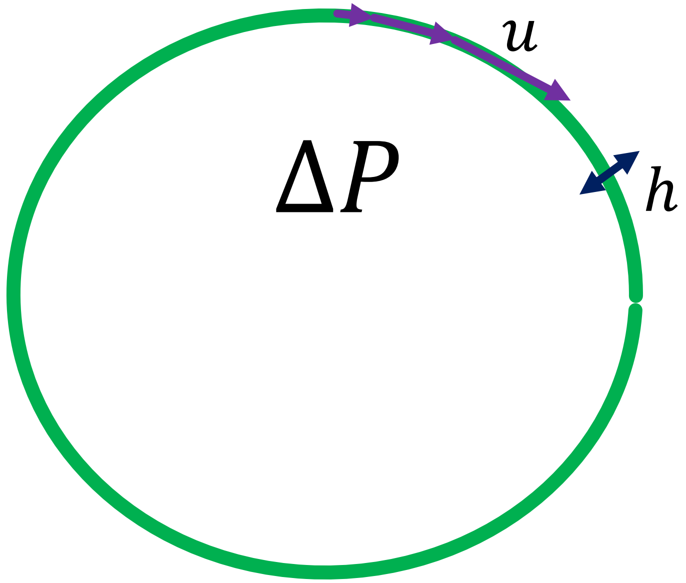
- *Hydrodynamics* of viscous films *vs.* *Electrostatics* in conducting media:

  momentum conservation ←→ *dynamo-geometric* ***charge*** & *curvature* ***current***

INTRODUCTION: ORIGIN OF STRESS

- Elastic systems:

strain $\nabla u \longrightarrow$ stress $\sigma \sim Eh\nabla u$

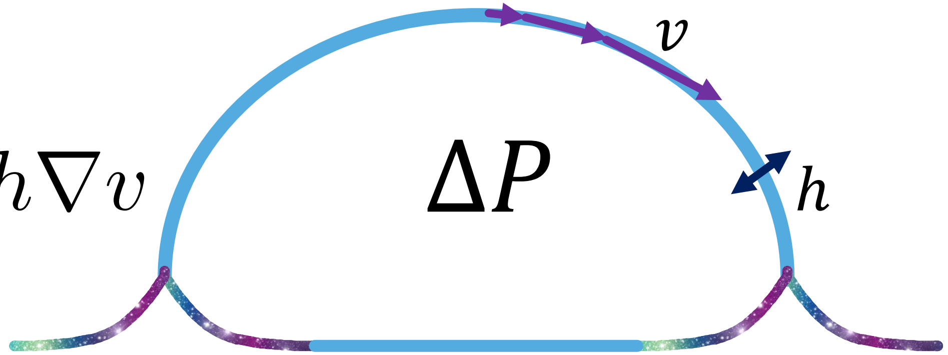


- Viscous systems:

strain rate ∇v

surface tension 2γ

$\longrightarrow$ stress $\sigma \sim 2\gamma + \eta h \nabla v$



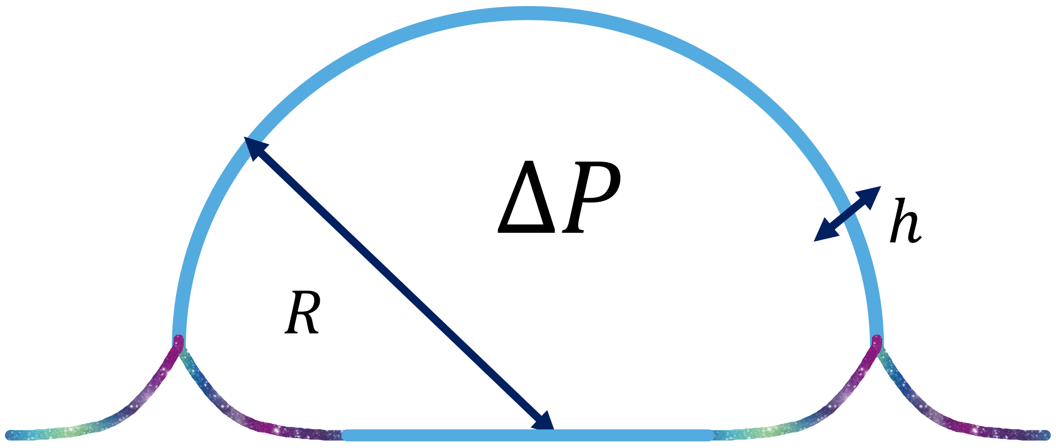
INTRODUCTION: LAPLACE LAW (NORMAL FORCE BALANCE)

equilibrium

= 0

$$\Delta P = \frac{2\sigma}{R} = \frac{2(2\gamma + \eta h \nabla v)}{R} \approx \frac{4\gamma}{R}$$

$$R \approx \frac{4\gamma}{\Delta P}$$



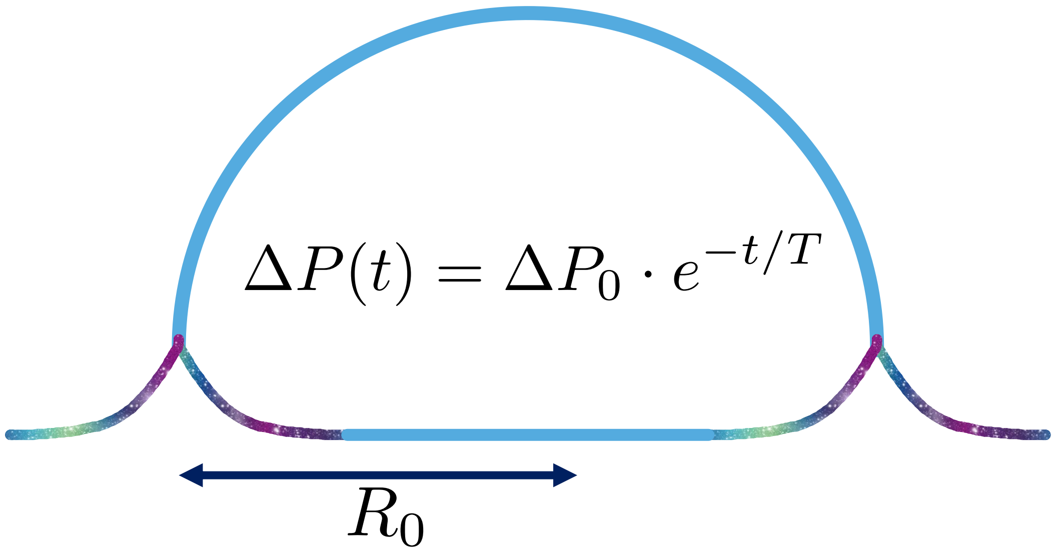
INTRODUCTION: ADIABATIC DEPRESSURIZATION

adiabatic:

$$|\eta h v / R_0| \ll 2\gamma$$

$$\Delta P(t) \approx \frac{2\sigma(t)}{R(t)} \approx \frac{2(2\gamma + \eta h \nabla v)}{R(t)} \approx \frac{4\gamma}{R(t)}$$

$$R(t) \approx R_0 e^{t/T}$$



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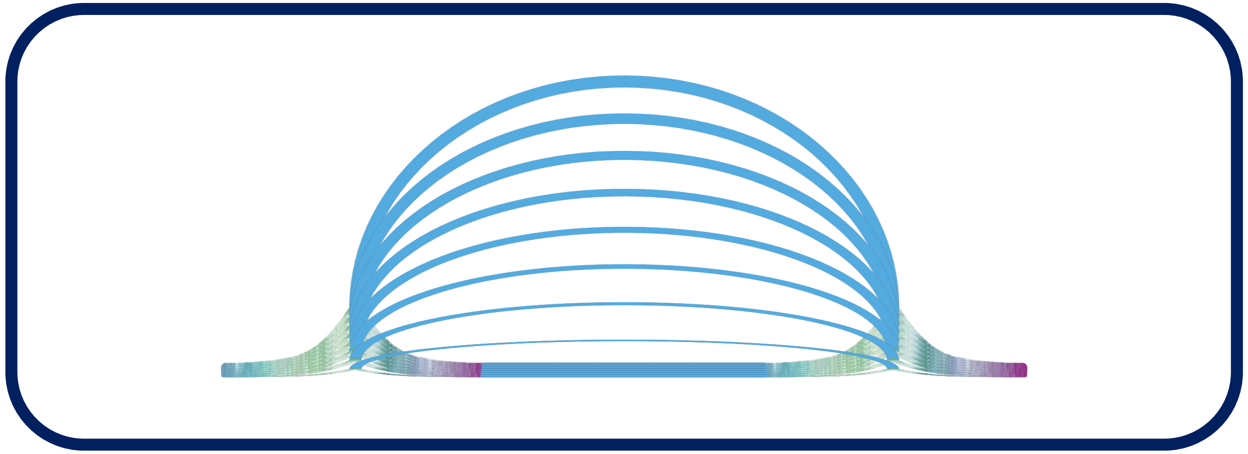

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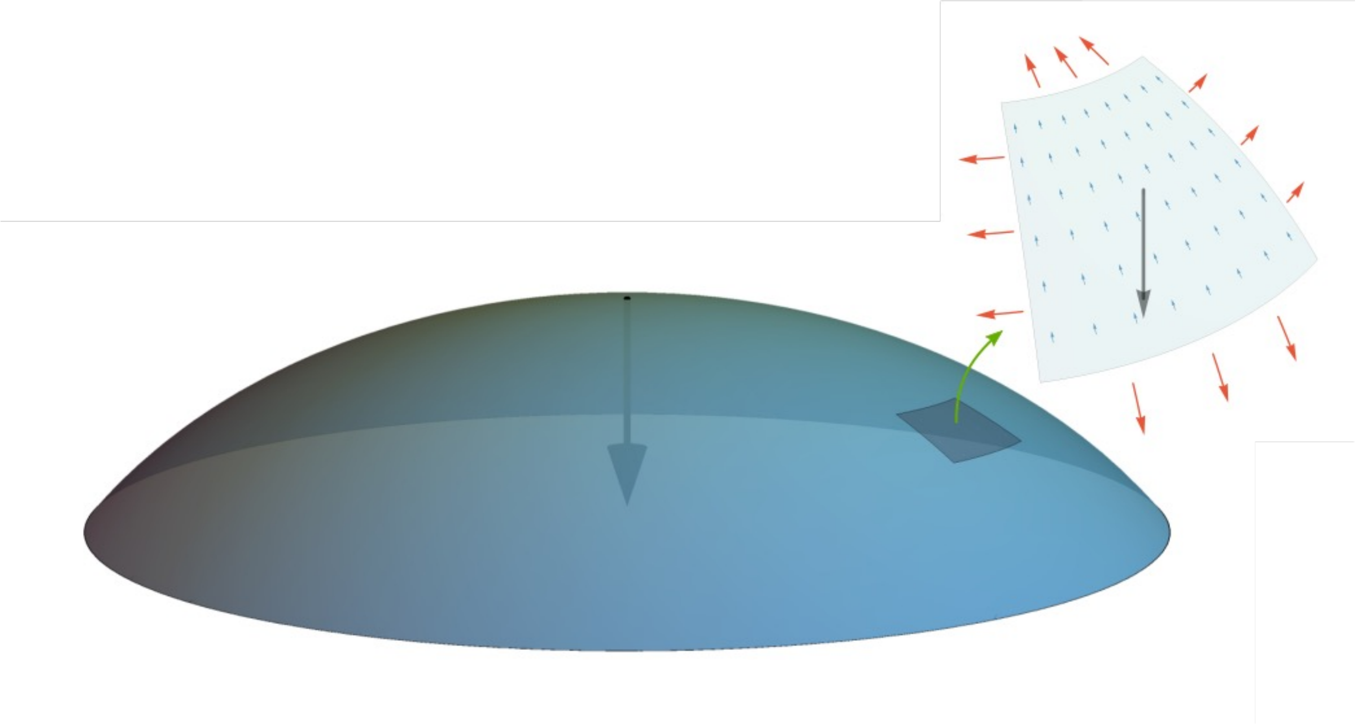
adiabatic condition:

$$\eta h v / R_0 \ll \gamma \implies \tau_{dep} \gg \tau_{vc}$$

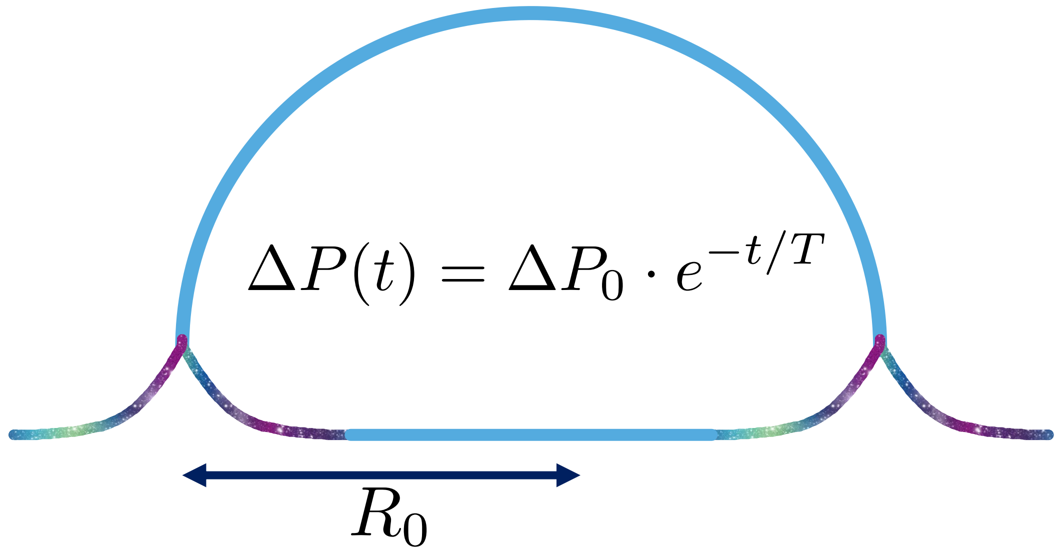
$$\boxed{\tau_{vc} \equiv \frac{\eta h}{\gamma}}$$



INTRODUCTION: RAPID DEPRESSURIZATION

$\tau_{dep} \ll \tau_{vc}$

$$\color{red}{0} \approx \Delta P(t) \approx \frac{2\sigma(t)}{R(t)} \approx \frac{2(2\gamma + \eta h \nabla v)}{R(t)}$$



INTRODUCTION: RAPID DEPRESSURIZATION

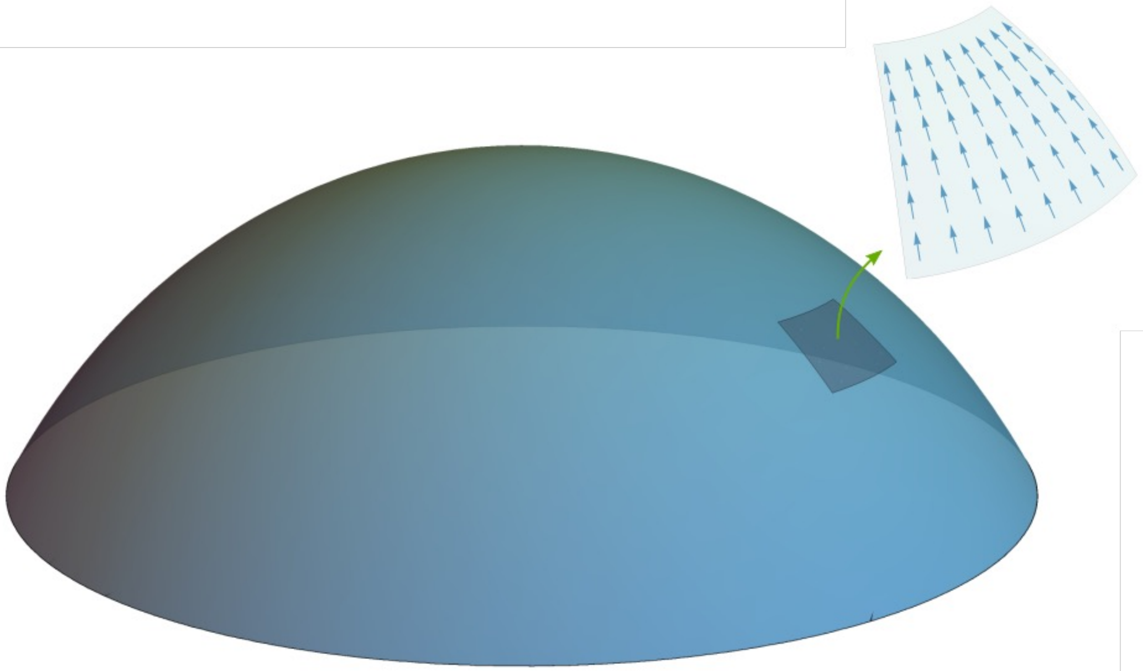
$\tau_{dep} \ll \tau_{vc}$

$$0 \approx \Delta P(t) \approx \frac{2\sigma(t)}{R(t)} \approx \frac{2(\sout{2\gamma + \eta h \nabla v})}{R(t)} \approx 0$$

$$\eta h \nabla v \approx -2\gamma$$

$\Longrightarrow$ purely tangential flow !

$$v \propto r/\tau_{vc}$$



INTRODUCTION: RAPID DEPRESSURIZATION $(T \ll \tau_{vc})$

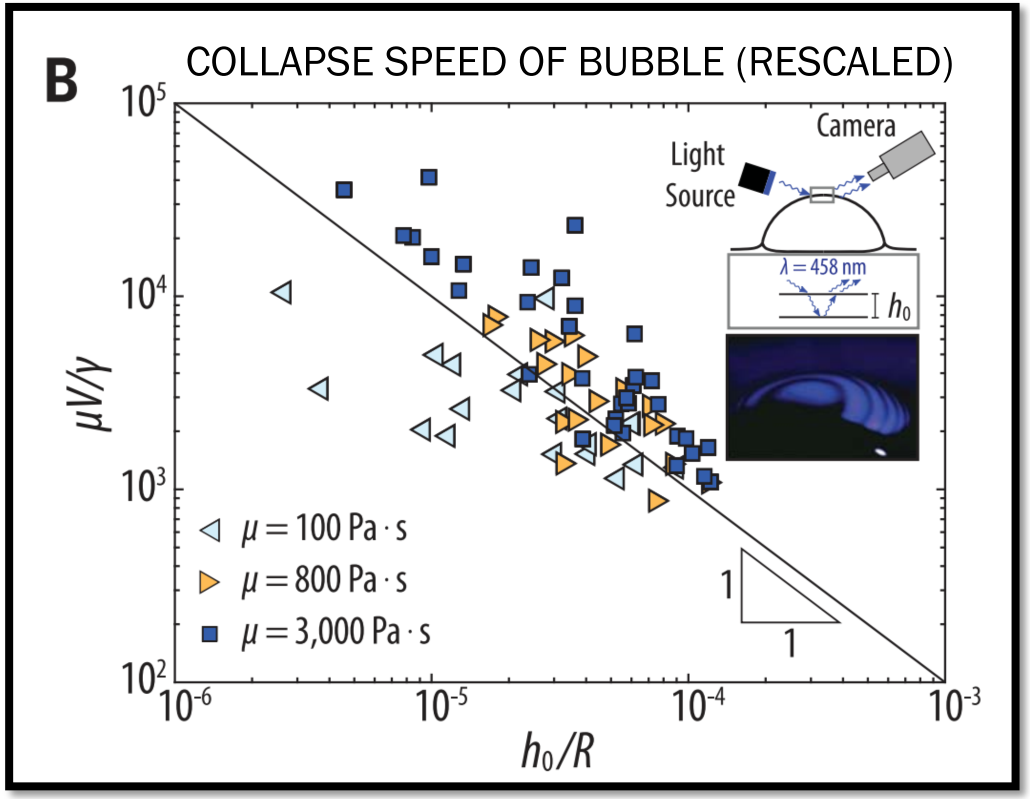
$$0 \approx \Delta P(t) \approx \frac{2\sigma(t)}{R(t)} \approx \frac{2(\cancel{2\gamma + \eta h \nabla v})}{R(t)} \approx 0$$

$$\eta h \nabla v \approx -2\gamma$$

$\Longrightarrow$ purely tangential flow !

$$v \propto r/\tau_{vc}$$

normal flow !? $\Longleftarrow$



INTRODUCTION: THE PHANTOM BUBBLE PARADOX

How does bubble's shape evolve after rapid depressurization ?

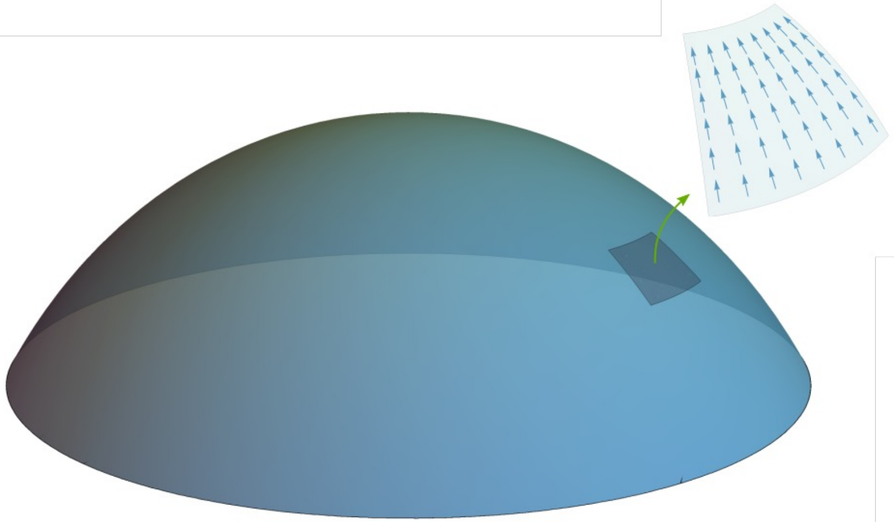
"phantom bubble"

mass & momentum conserved

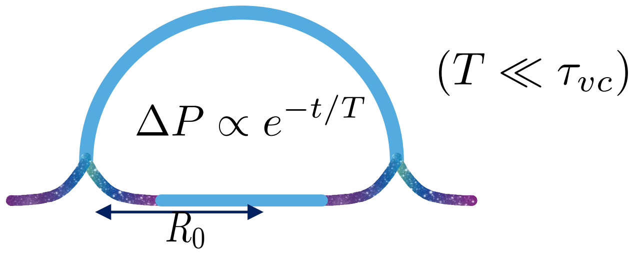
- nonequilibrium , stress-free state
  enabled by viscous flow
- steady spherical shape
  no net tangential or normal force

$\Delta U \neq Q - W$

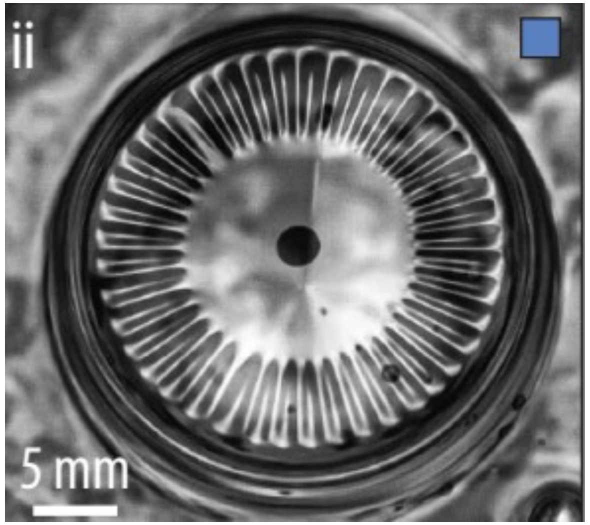
- violating the 1st law of thermodynamics
  no source of energy to fuel heat generated by viscous flow



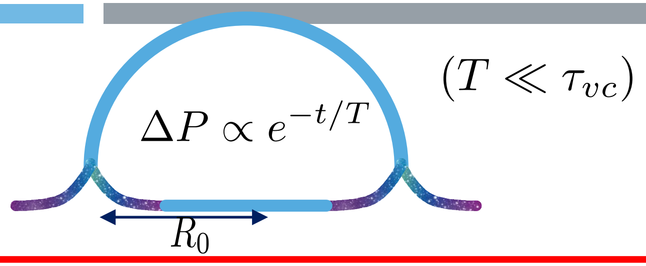
INTRODUCTION: THE PHANTOM BUBBLE PARADOX



- What drives flattening of rapidly-depressurized bubble ?
- What causes radial wrinkles ?
- How general is this behavior?

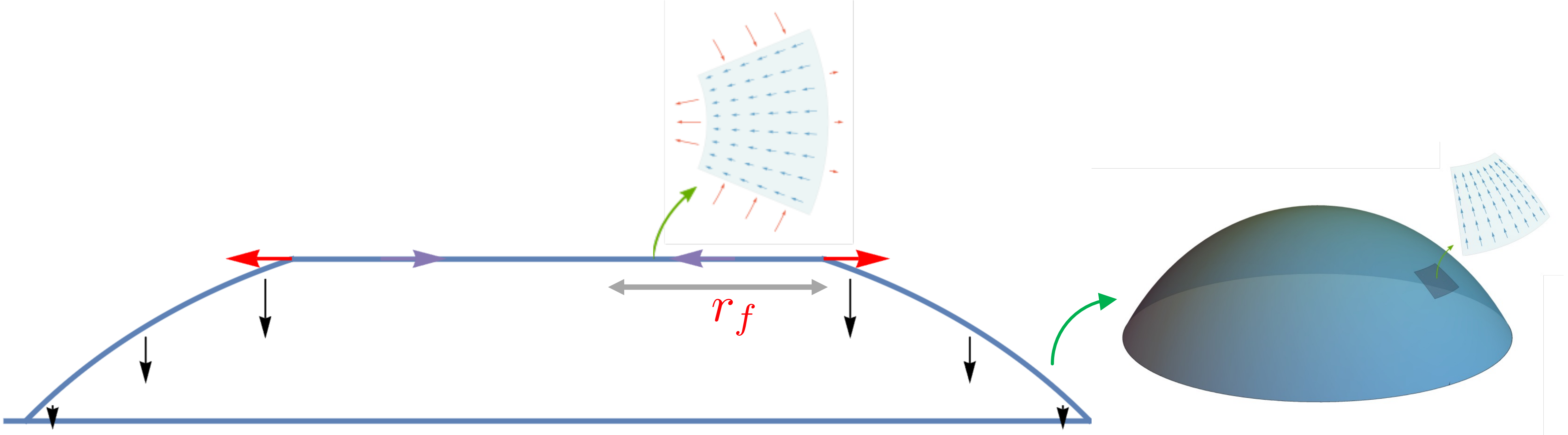


INTRODUCTION: PARADOX RESOLVED !

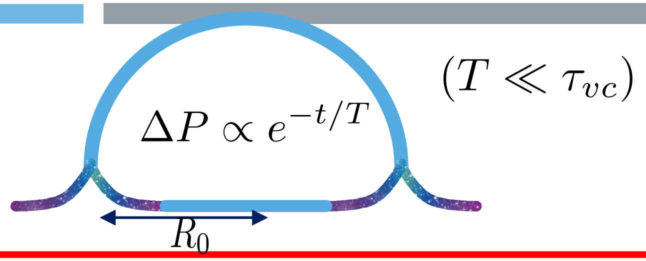


bubble flattens by <u>front propagation</u> (akin to phase transformation)

- flat "island" of radius $r_f \sim \sqrt{T}$ nucleates in a spherical stress-free "sea" during depressurization
- planar core invades spherical periphery at velocity $\dot{r}_f \sim R_0/\tau_{vc}$
- release of **surface energy** → **heat** generated by viscous flow in spherical (stress-free) portion

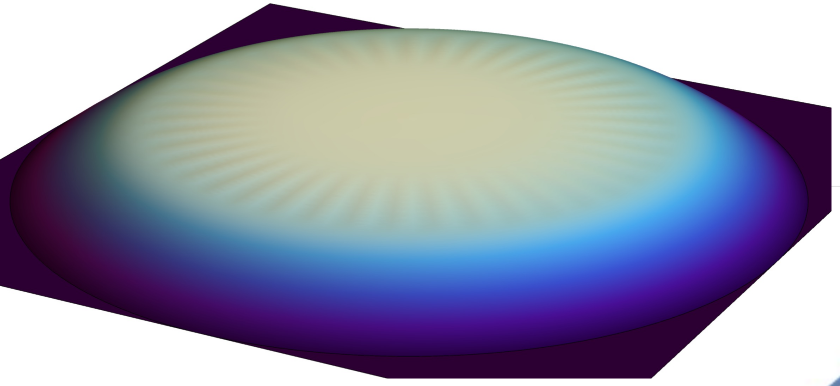


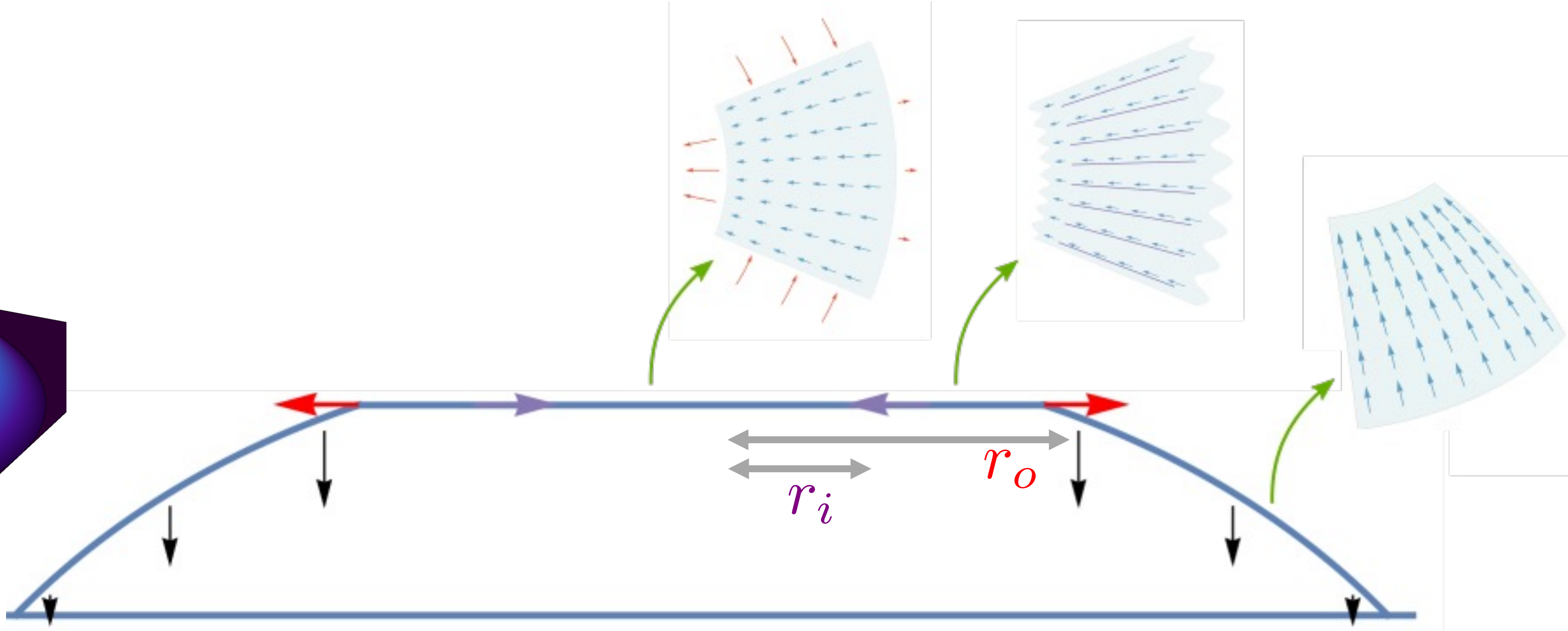
INTRODUCTION: WHY RADIAL WRINKLES ?



planar core is under hoop compression → wrinkling instability

- planar-wrinkled annulus $r_i < r < r_o$ expands between planar core and spherical periphery





OUTLINE

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- *Hydrodynamics* of viscous films *vs.* *Electrostatics* in conducting media:

momentum conservation ←→ *dynamo-geometric* ***charge*** & *curvature* ***current***

NON-INERTIAL DYNAMICS OF MOMENTUM CONSERVING FILM

$$\overset{\leftrightarrow}{\sigma} \propto 2\gamma \overset{\leftrightarrow}{I} + \eta h \cdot [\nabla \vec{v} + \partial_t \overset{\leftrightarrow}{g}]$$

in-plane stress (force/length) — tangential velocity — metric

$$\sigma_{ij} \equiv \varepsilon_{ijk}\varepsilon_{ijm}\partial_k\partial_m\Psi$$

$g_{ij} =$ metric

$\eta =$ viscosity

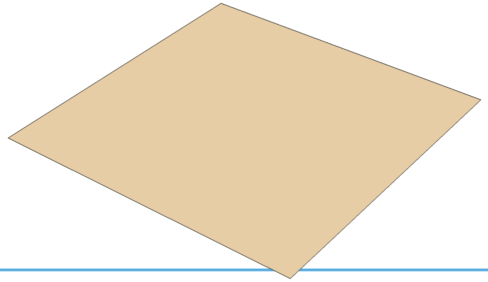
$\nu = 1/3$

$R^{-1}_{ij} =$ curvature

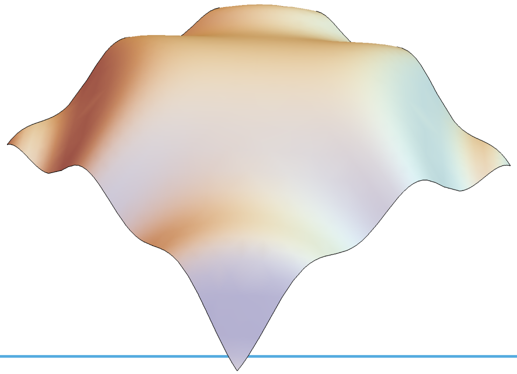
$$\sigma_{ij} = 2\gamma\delta_{ij} + \frac{\eta h}{1-\nu}[(1-\nu)(\partial_i v_j + \partial_j v_i + \partial_t g_{ij}) + 2\nu(\partial_k v_k + \partial_t g_{kk})g_{ij}]$$

$$R^{-1}_{ij} \approx \frac{\partial^2 z}{\partial x_i \partial x_j}$$

planar film



curved film



normal force balance

$$\Delta P = \overset{\leftrightarrow}{\sigma} \cdot \overset{\leftrightarrow}{R}^{-1} + \eta h^3 \partial_t \nabla^2 \mathrm{Tr}\, \overset{\leftrightarrow}{R}^{-1}$$

"viscous bending"

tangential force balance

$$\nabla \cdot \overset{\leftrightarrow}{\sigma} = 0$$

$$\nabla^4\Psi = 0$$

(2D Stokes flow)

$$\nabla^4\Psi = -\eta h[3\frac{\partial}{\partial t}\det \overset{\leftrightarrow}{R}^{-1} - \frac{1}{2}\nabla^2(v_n \,\mathrm{Tr}\, \overset{\leftrightarrow}{R}^{-1})]$$

Gaussian curvature — normal velocity — mean curvature

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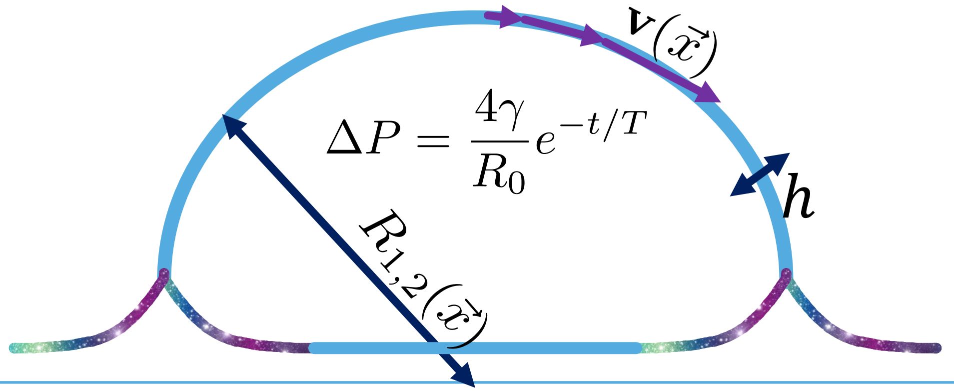

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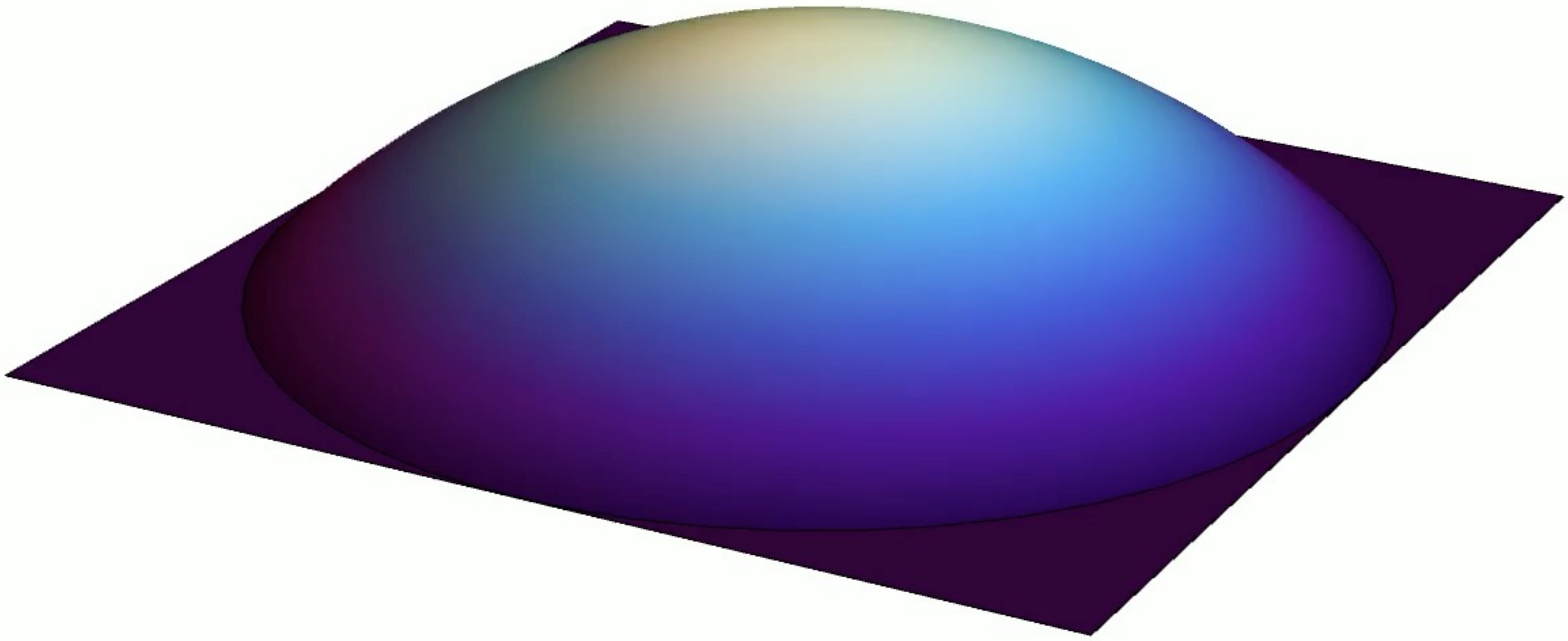

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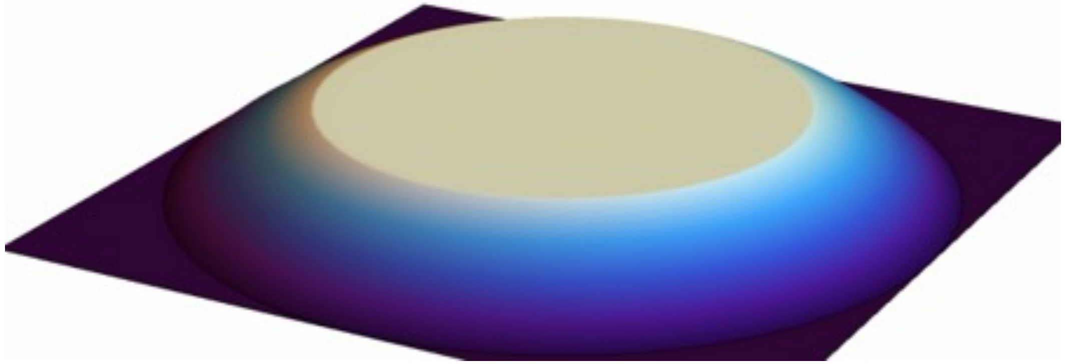

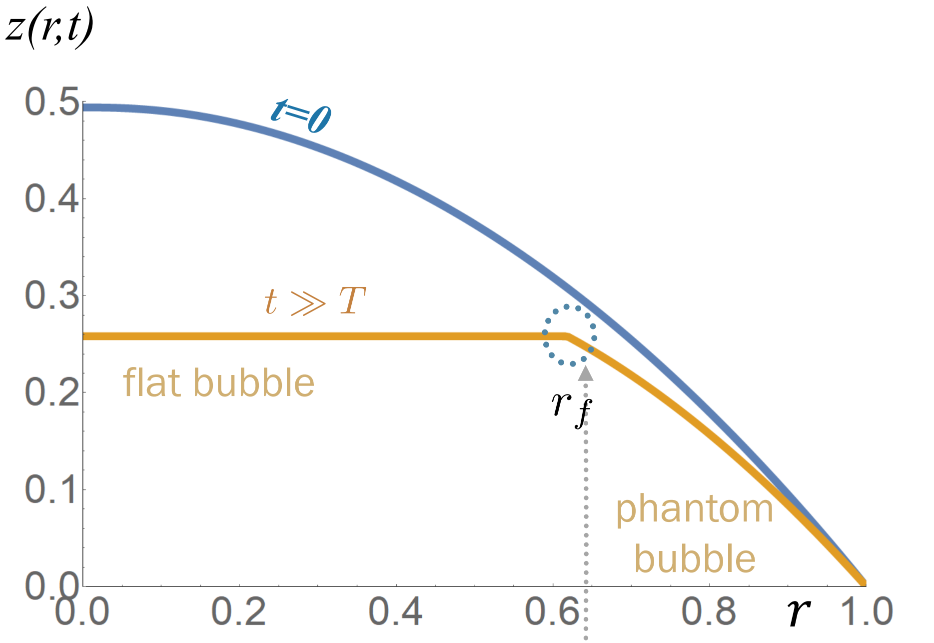

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$$\Delta P(t) = \overleftrightarrow{\sigma} \cdot \overleftrightarrow{R}^{-1} + \eta h^3 \partial_t \nabla^2 \mathrm{Tr}\, \overleftrightarrow{R}^{-1}$$

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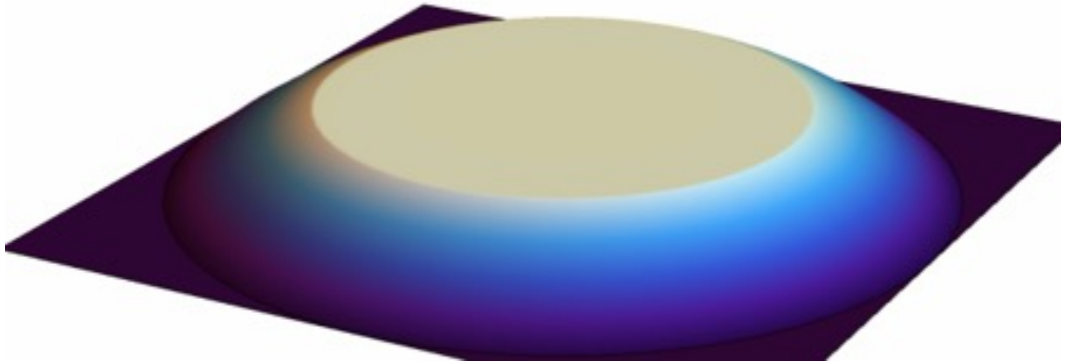

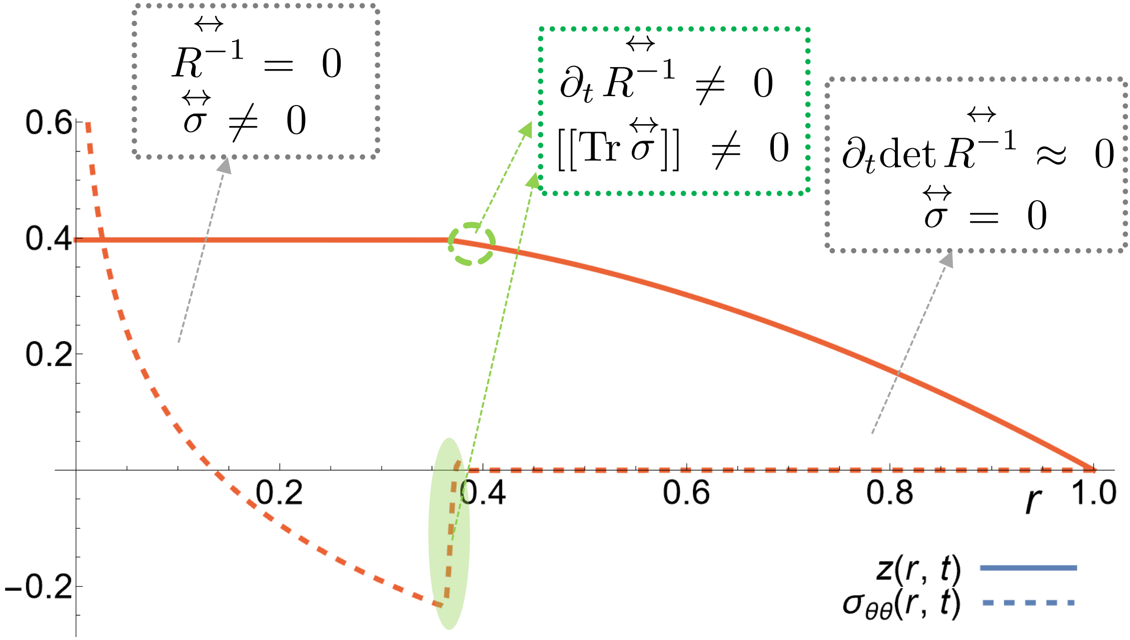

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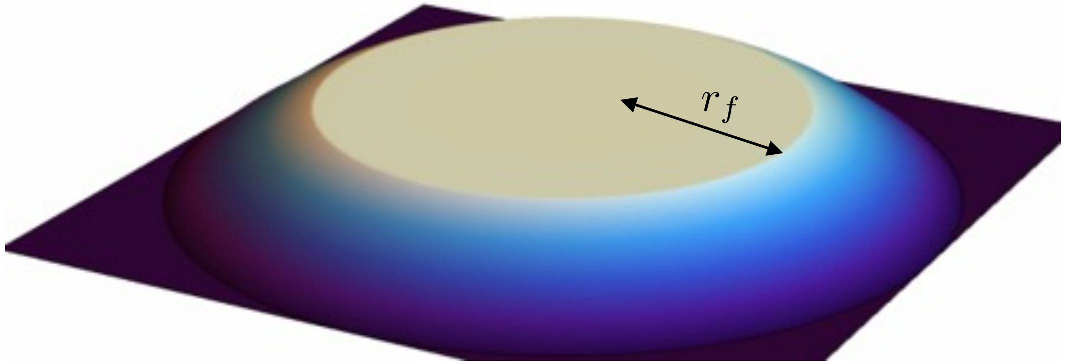
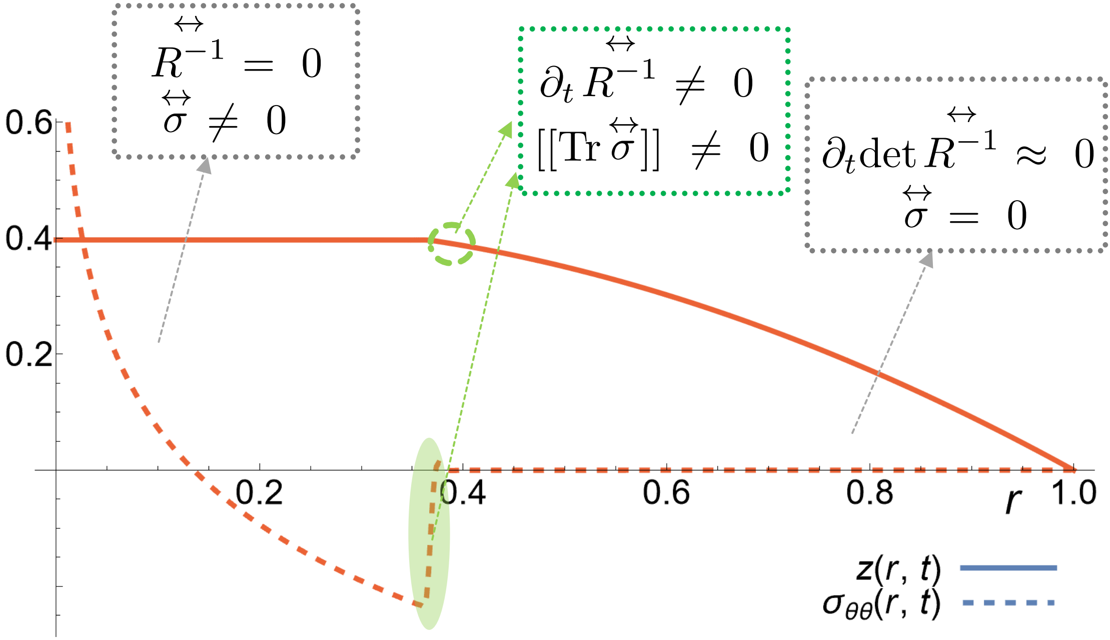




normal force balance

$\overleftrightarrow{\sigma} \neq 0$ $\quad \overleftrightarrow{R}^{-1} \approx 0 \quad r < r_f(t)$

(stressed , planar shape)

$\overleftrightarrow{\sigma} \approx 0$ $\quad \partial_t \det \overleftrightarrow{R}^{-1} \approx 0 \quad r > r_f(t)$

(stress-free , non-planar shape)

$$\Longleftarrow \quad 0 \approx \overleftrightarrow{\sigma} \cdot \overleftrightarrow{R}^{-1}$$

tangential force balance

$$\nabla^4 \Psi = -\eta h [3 \frac{\partial}{\partial t} \det \overleftrightarrow{R}^{-1} - \frac{1}{2} \nabla^2 (v_n \,\mathrm{Tr}\, \overleftrightarrow{R}^{-1})]$$

$$\nabla^2 [\mathrm{Tr}\, \overleftrightarrow{\sigma}] \propto \eta h \cdot \text{(rate of change of curvature)}$$

NON-INERTIAL DYNAMICS OF MOMENTUM CONSERVING FILM

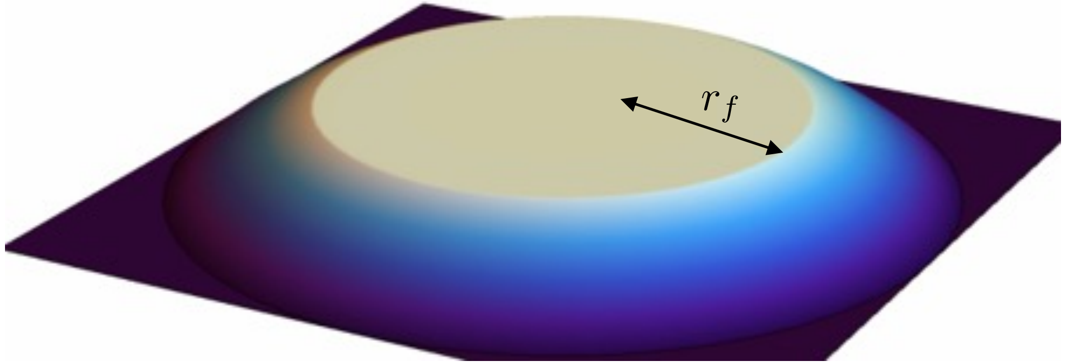
g_{ij} = metric

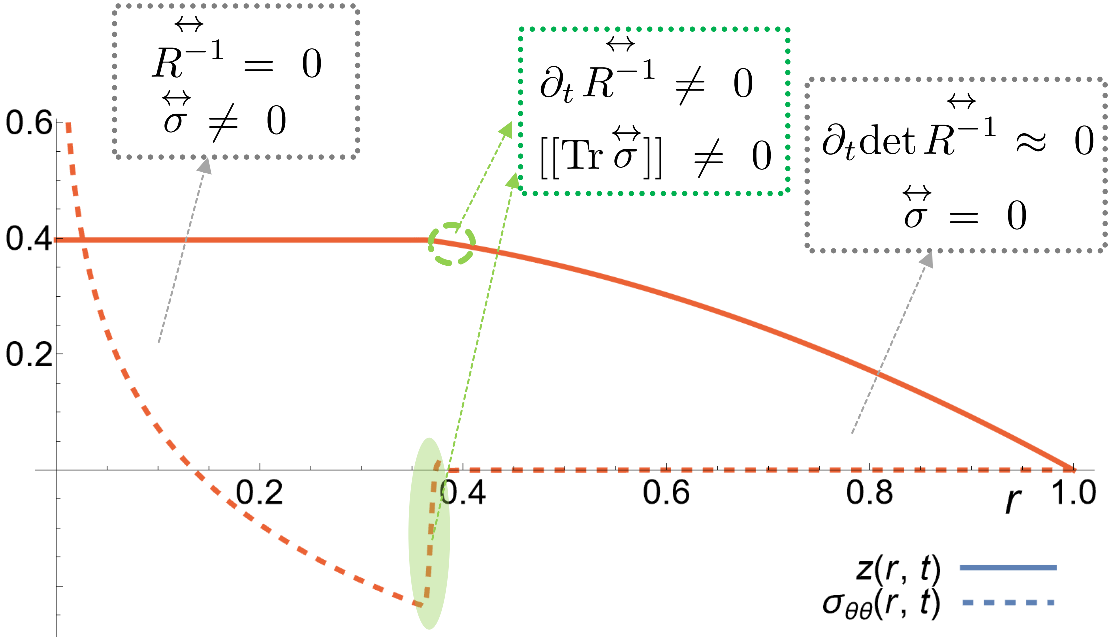
η = viscosity

R^{-1}_{ij} = curvature

$$\overset{\leftrightarrow}{\sigma} \propto 2\gamma \overset{\leftrightarrow}{I} + \eta h \cdot [\nabla \vec{v} + \partial_t \overset{\leftrightarrow}{g}]$$

$$\nabla^2 \Psi = \mathrm{Tr}\, \overset{\leftrightarrow}{\sigma}$$





(vacuum) ← $\overset{\leftrightarrow}{\sigma} \neq 0$ $\qquad \overset{\longleftrightarrow}{R^{-1}} \approx 0 \qquad r < r_f(t)$

(stressed , planar shape)

normal force balance

(conductor) ← $\overset{\leftrightarrow}{\sigma} \approx 0$ $\qquad \partial_t \det \overset{\leftrightarrow}{R}^{-1} \approx 0 \qquad r > r_f(t)$

(stress-free , non-planar shape)

$$\Longleftarrow \quad 0 \approx \overset{\leftrightarrow}{\sigma} \cdot \overset{\leftrightarrow}{R}^{-1}$$

tangential force balance

$$\rho_e = q \cdot \left(\frac{1}{r}\delta(r) - \frac{1}{r_f}\delta(r - r_f) \right) \longleftarrow \nabla^2 [\mathrm{Tr}\, \overset{\leftrightarrow}{\sigma}] \propto \eta h \cdot \text{(rate of change of curvature)}$$

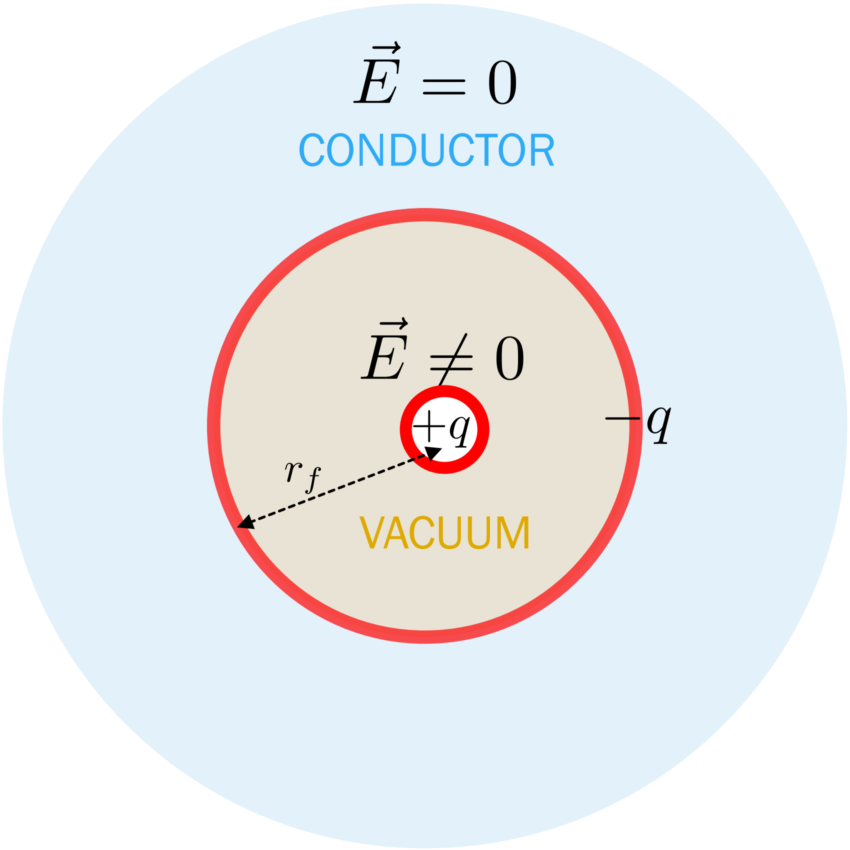
disclination (strain rate)

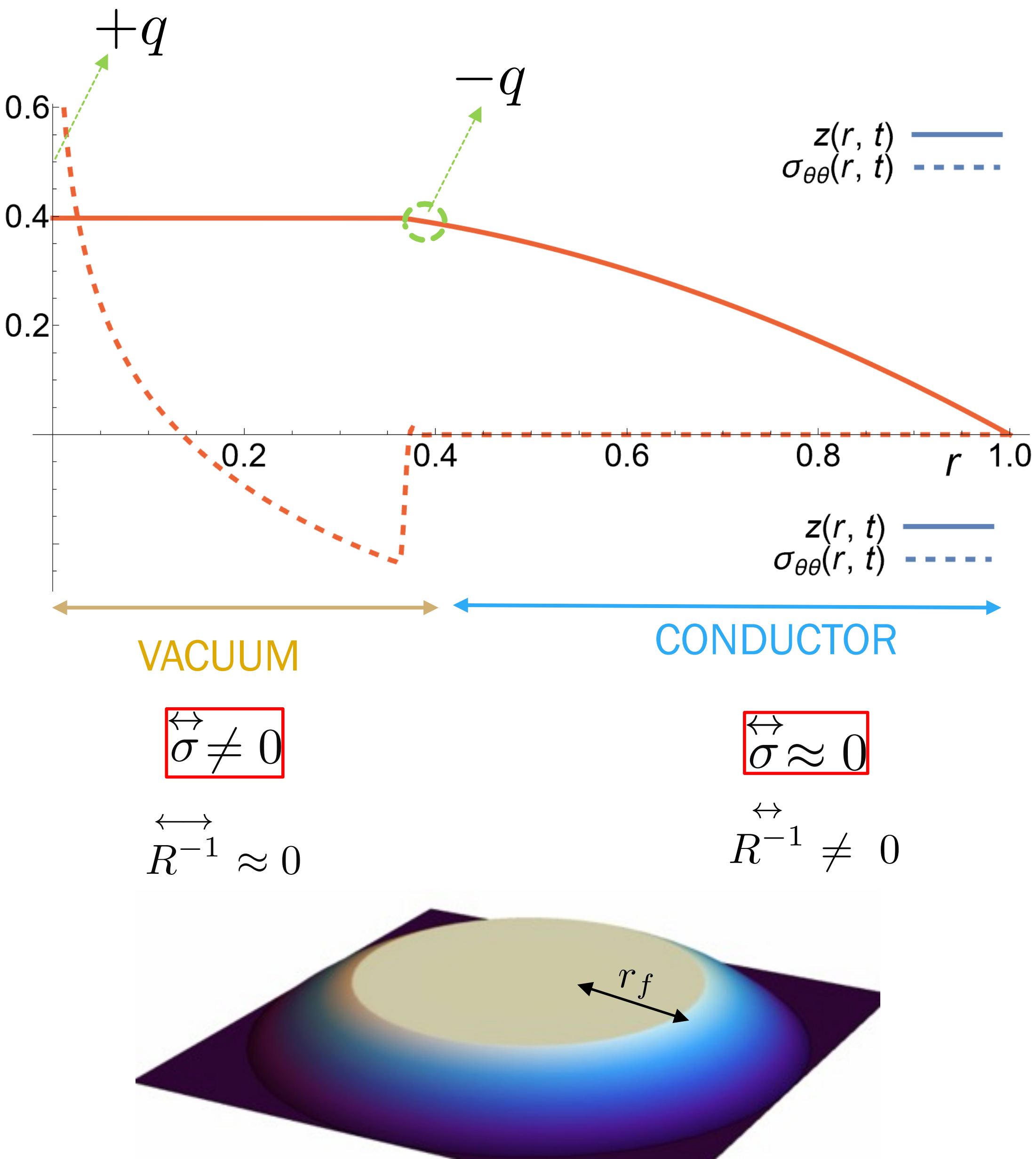
propagating front

"charge" density $\rho(\mathbf{x}, t)$

NON-INERTIAL DYNAMICS OF MOMENTUM CONSERVING FILM
AS FRONT PROPAGATION IMPOSED ON CHARGE-CONSERVING MEDIA

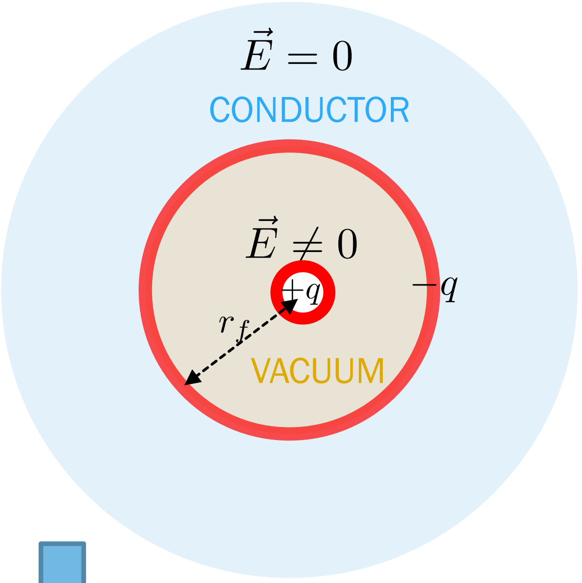
axisymmetric electrostatic quadrupole



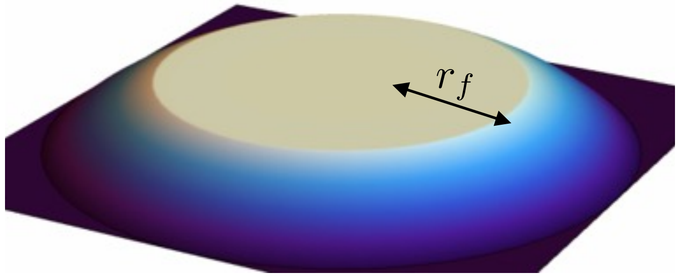


NON-INERTIAL DYNAMICS OF MOMENTUM CONSERVING FILM AS FRONT PROPAGATION IMPOSED ON CHARGE-CONSERVING MEDIA

Electrostatics of vacuum-conductor ←→ Momentum conservation in viscous film



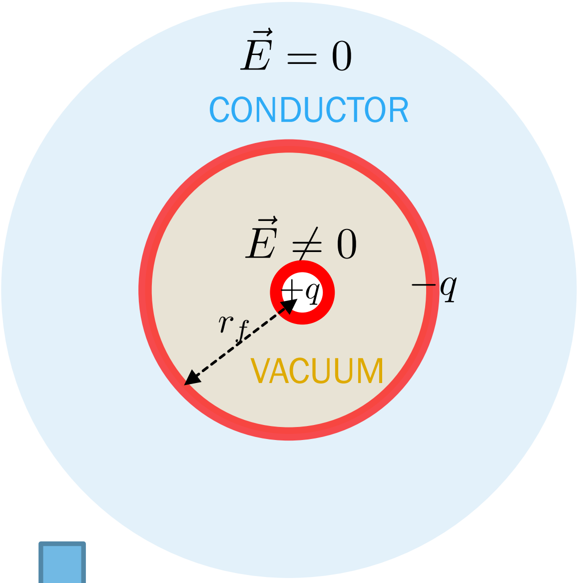
Electrostatics of vacuum-conductor	Momentum conservation in viscous film
V	$\mathrm{Tr}\,\overset{\leftrightarrow}{\sigma}$
$\vec{E} = \nabla V$	$\nabla[\mathrm{Tr}\,\overset{\leftrightarrow}{\sigma}]$
q	$\propto \eta h \partial_t \det \overset{\leftrightarrow}{R}^{-1}$
charge neutrality	absence of normal force



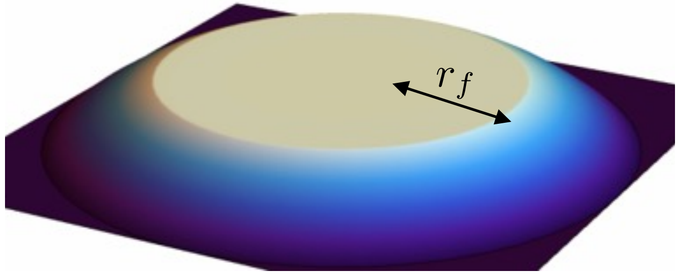
- Determine $\overset{\leftrightarrow}{\sigma}$ and $\overset{\leftrightarrow}{R}^{-1}$ for given $q(t)$, $r_f(t)$
- Dynamics (front propagation) $[[\mathrm{Tr}\,\overset{\leftrightarrow}{\sigma}]] \propto [[\det \overset{\leftrightarrow}{R}^{-1}]] \Longrightarrow$ $\dot{r}_f = q \dfrac{r_f}{r_f^2 - \int^t q(t')dt'}$
- Thermodynamics (1st law) change of surface energy = heat production by viscous flow $\Longrightarrow$ $q = \sqrt{r_f^{-2} - 4} - r_f^{-2}$

NON-INERTIAL DYNAMICS OF MOMENTUM CONSERVING FILM
AS FRONT PROPAGATION IMPOSED ON CHARGE-CONSERVING MEDIA

Electrostatics of vacuum-conductor $\leftrightarrow$ Momentum conservation in viscous film



V	$\mathrm{Tr}\,\overset{\leftrightarrow}{\sigma}$
$\vec{E} = \nabla V$	$\nabla[\mathrm{Tr}\,\overset{\leftrightarrow}{\sigma}]$
q	$\propto \eta h \partial_t \det \overset{\leftrightarrow}{R}^{-1}$
charge neutrality	absence of normal force



- Determine $\overset{\leftrightarrow}{\sigma}$ and $\overset{\leftrightarrow}{R}^{-1}$ for given $q(t)$, $r_f(t)$
- Dynamics (front propagation) $[[\mathrm{Tr}\,\overset{\leftrightarrow}{\sigma}]] \propto [[\det \overset{\leftrightarrow}{R}^{-1}]]$
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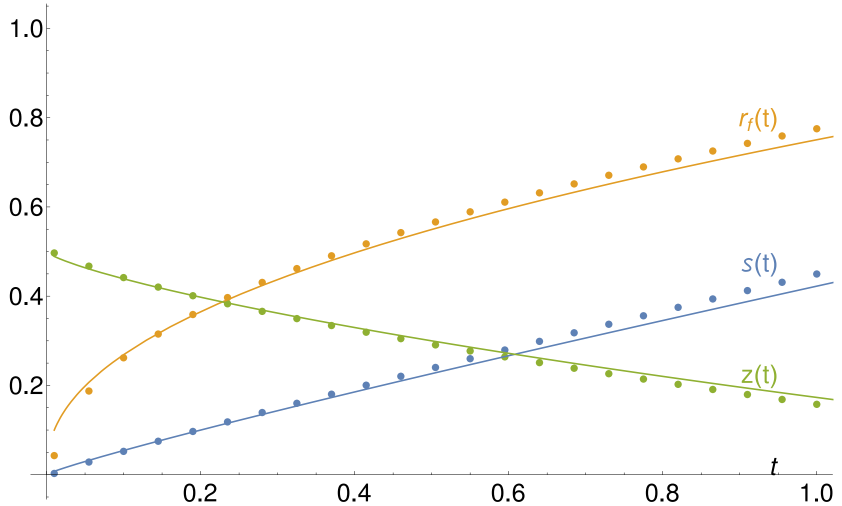


INTERIM SUMMARY

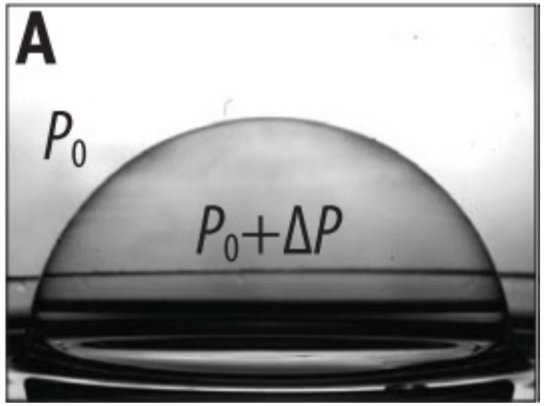
- Rapid depressurization → “Phantom bubble” – steady, spherical, stress-free state

- Topological instability: dynamical nucleation of a “disclination-front” pair

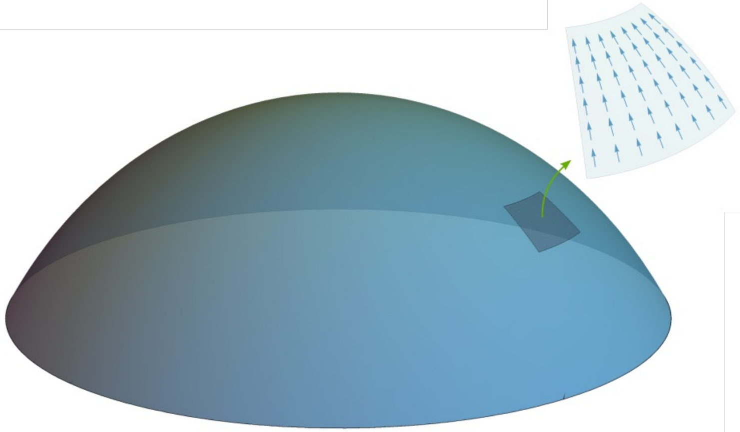
  (akin to electrostatic quadruple)

- Axisymmetric flattening process: curvature flows out, damping only on front

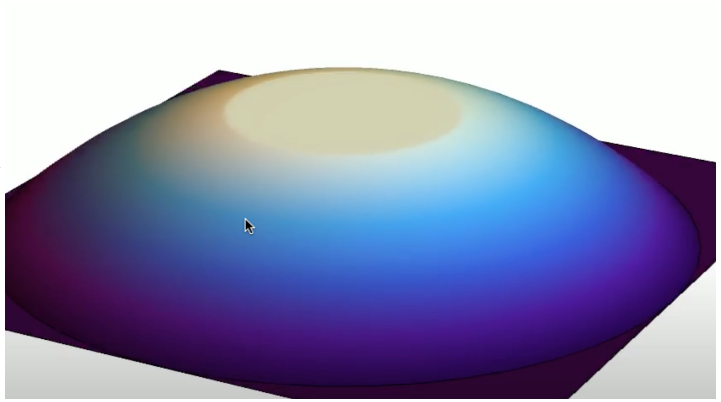
  (quadruple is expanding)



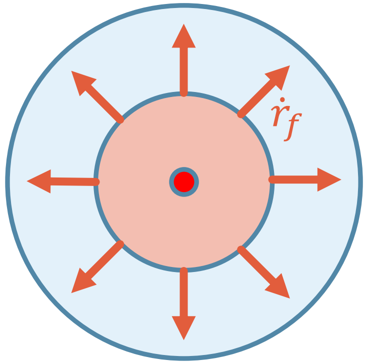
$t \leq 0$



$0 < t \lesssim T$



$t \sim T$



$T \ll t \lesssim \tau_{vc}$

WRINKLING INSTABILITY

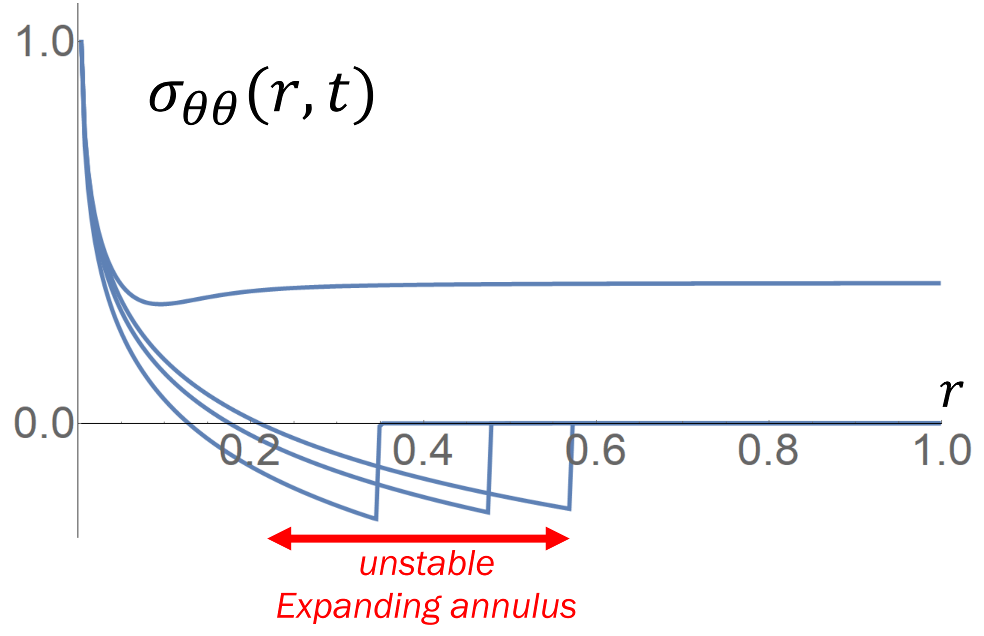
hoop compression → Instability
(radial wrinkles suppress compression)

↓

dynamics captured by single-mode ansatz

$$z \approx z_0(r,t) + z_m(r,t)\cos m\theta$$

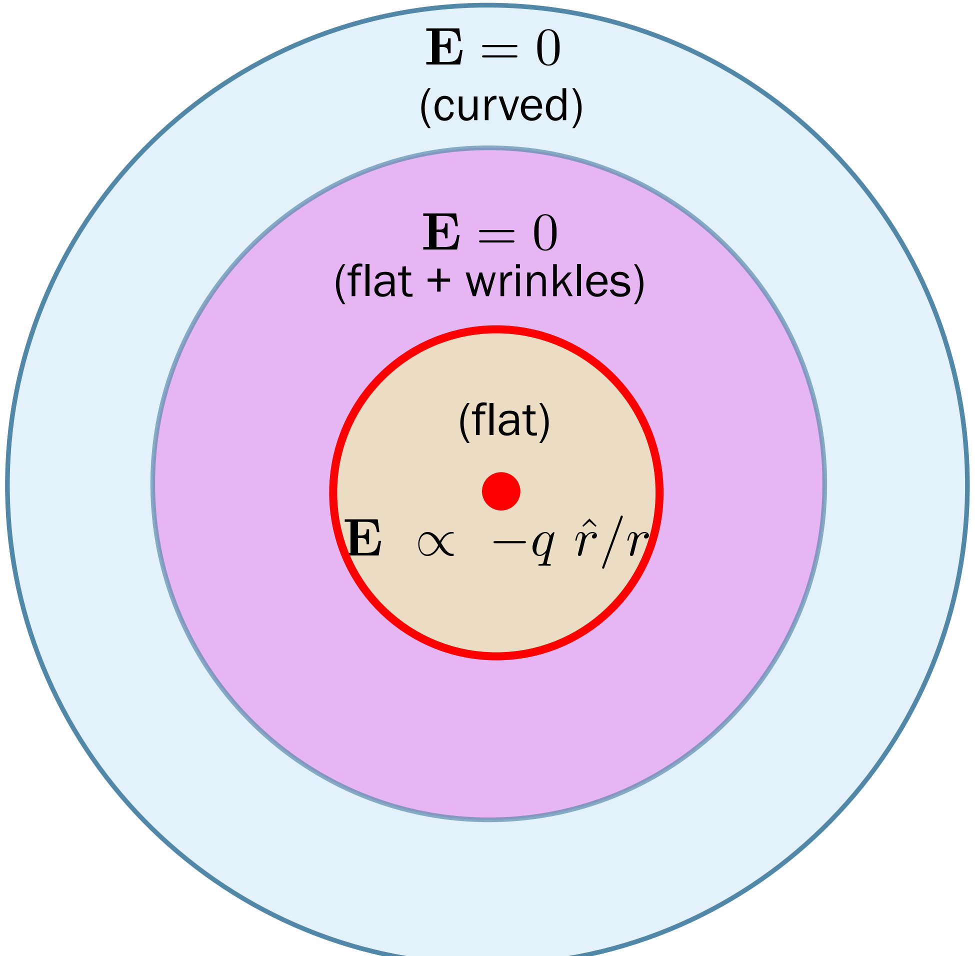
$$\sigma_{ij} \approx \sigma_{ij,0}(r,t) + \sigma_{ij,m}(r,t)\cos m\theta$$

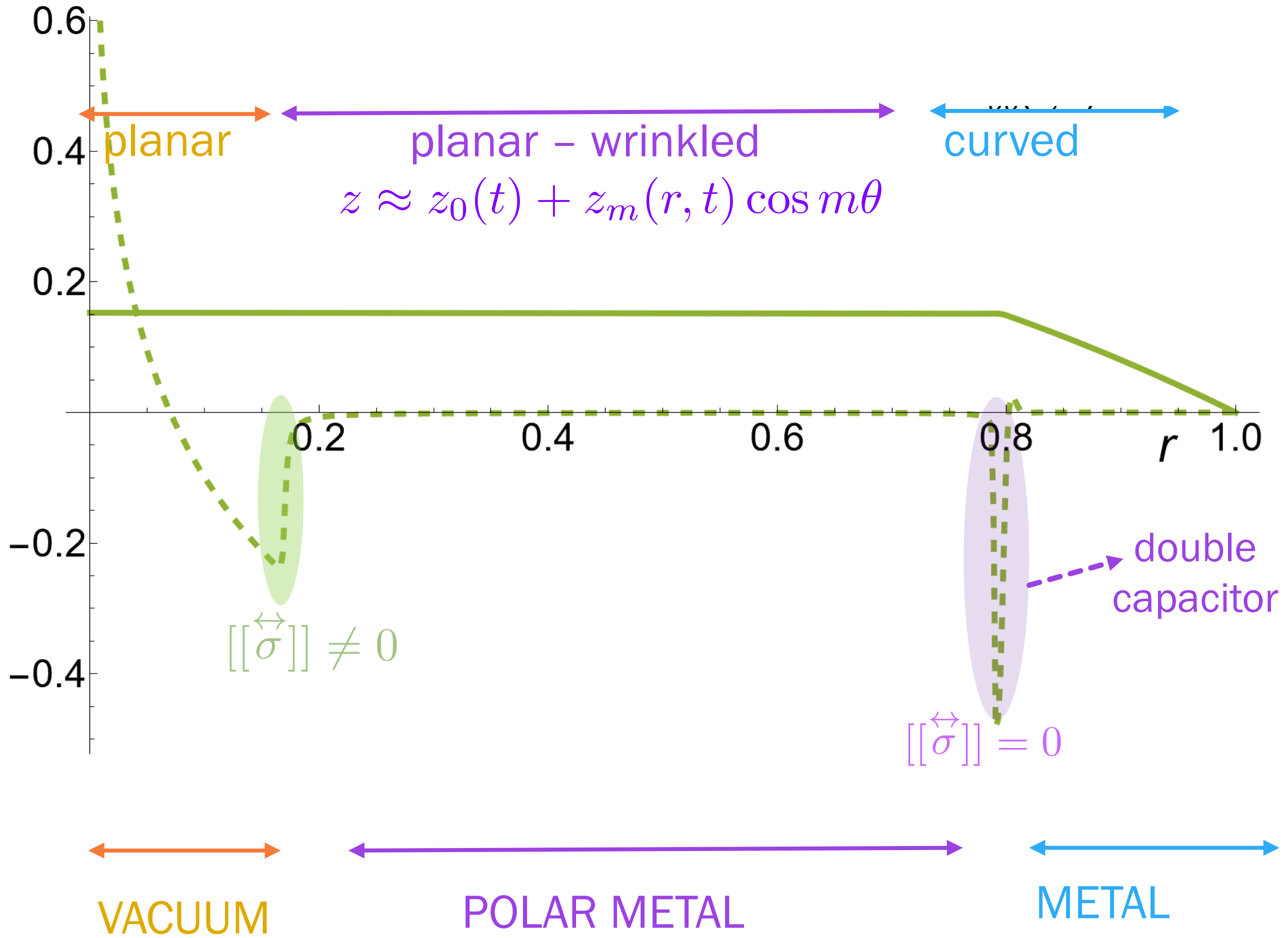


WRINKLED SURFACE DYNAMICS

“POLAR METAL” ANNULUS INVADES “VACUUM” CORE & “CONDUCTING” SEA

$$\vec{E} = \nabla(\mathrm{Tr}\,\overleftrightarrow{\sigma})$$





Simulation of a rapidly collapsing bubble

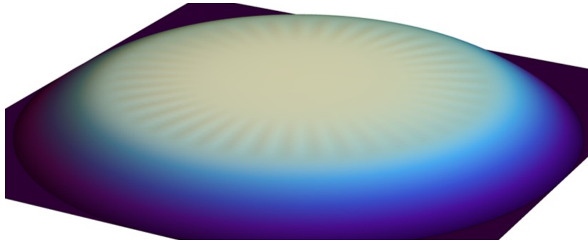
Supplementary video to

How viscous bubbles collapse: topological and symmetry-breaking instabilities driven by curvature-limited dynamics of liquid films

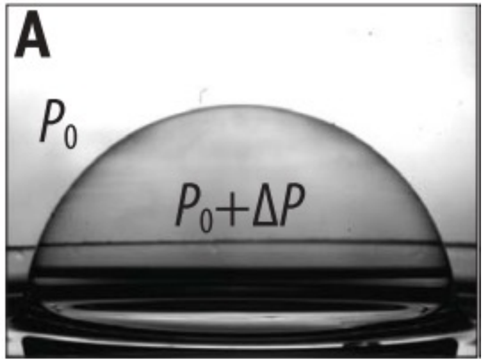
For numerical details see Appendix E

SUMMARY: COLLAPSE OF VISCOUS BUBBLE

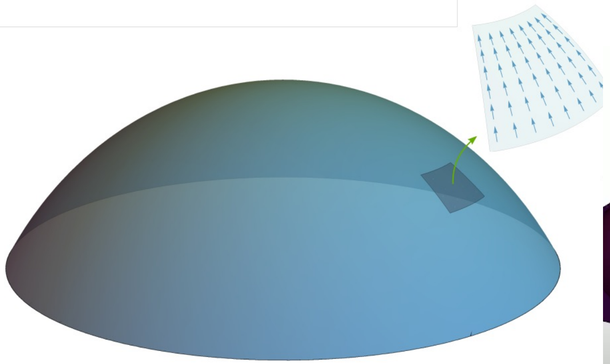
- Rapid depressurization → "Phantom bubble" – steady, spherical, stress-free state
- Topological instability: dynamical nucleation of a "diclination-front" pair
- Axisymmetric flattening process: curvature flows out, damping only on front
- Symmetry breaking instability: wrinkles suppress hoop compression



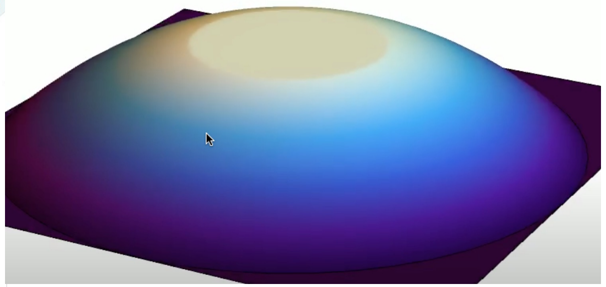
$\tau_{ins} < t < \tau_{vc}$



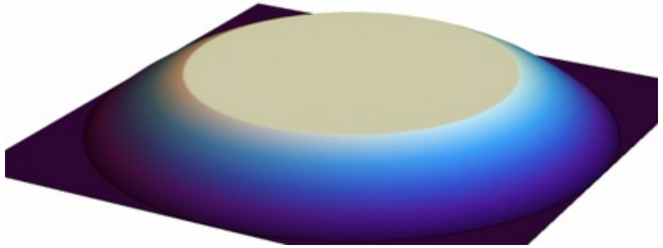
$t \leq 0$



$0 < t \lesssim T$



$t \sim T$



$T < t < \tau_{ins}$

OUTLINE

- Introduction
- *Hydrodynamics* of viscous films *vs.* *Electrostatics* in conducting media:

  momentum conservation ←→ *dynamo-geometric* ***charge*** & *curvature* ***current***

- Hydrodynamics of viscous films *versus* elastic deformations of solids

SOME QUESTIONS PUSHED UNDER THE RUG …

I. Does film's thickening suppress or amplify in-plane compression ? Amplify !

II. What is the role of the meniscus? Crucial !

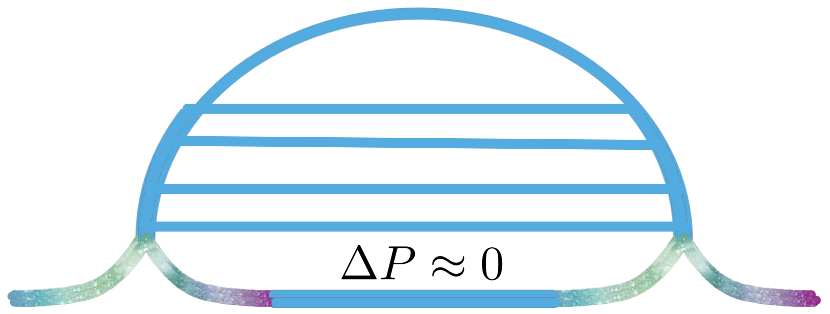
Flatten or Shrink ?

<u>Meniscus (edge pinned)</u>

inward flow →

viscous stress suppress surface tension

$$\overleftrightarrow{\sigma} \propto 2\gamma \overleftrightarrow{I} + \eta h \cdot [\nabla \vec{v} + \partial_t \overleftrightarrow{g}]$$

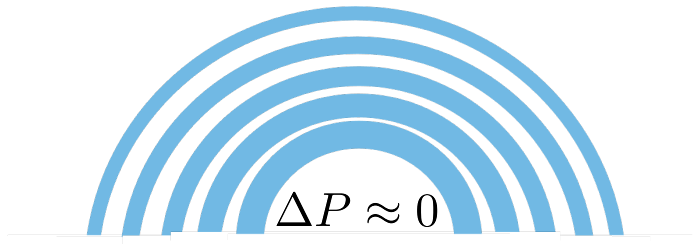


<u>Edge free</u>

Film's thickening →

viscous stress suppress surface tension

$$\overleftrightarrow{\sigma} \propto 2\gamma \overleftrightarrow{I} + \eta h \cdot [\nabla \vec{v} + \partial_t \overleftrightarrow{g}]$$



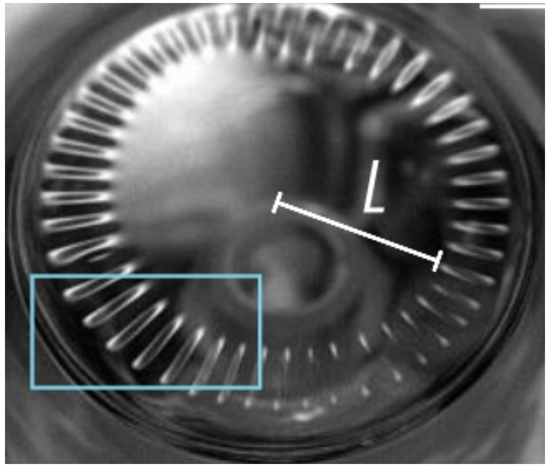
SOME QUESTIONS PUSHED UNDER THE RUG ...

III. What determines the number of wrinkles

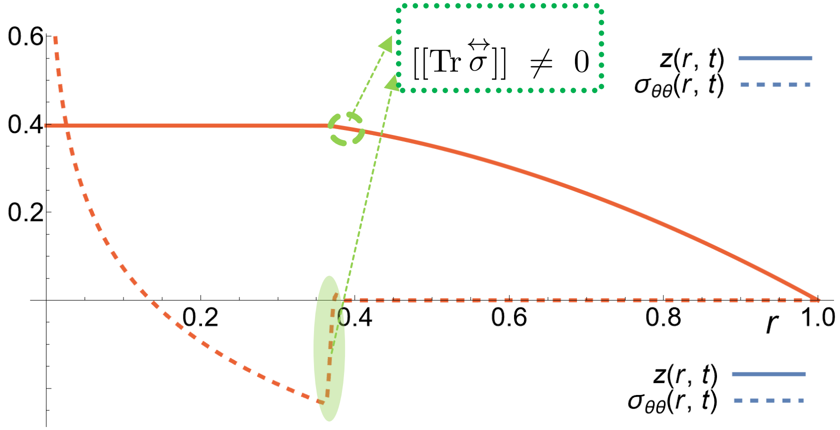
NOT linear stability analysis !

data suggests $n \sim \sqrt{R_0/h}$

far from threshold analysis (ala elasticity) ??



IV. What about "Stokes-Rayleigh analogy" (viscous dynamics ←→ elastic deformations) ?



Signature: asymptotic stress discontinuity

$$\sigma \propto \eta \nabla v \quad \xleftrightarrow{\eta \partial_t u \leftrightarrow E u} \quad \sigma \sim E h \nabla u$$

$$\nabla^4 \Psi \propto \eta \partial_t \det \overleftrightarrow{R}^{-1} \quad \xleftrightarrow{\eta \partial_t u \leftrightarrow E u} \quad \nabla^4 \Psi \propto E \det \overleftrightarrow{R}^{-1}$$

only if $\det \overleftrightarrow{R}^{-1} = 0$

MOMENTUM-CONSERVING VISCOUS FLOW IN 2D: BEYOND CLASSICAL HYDRODYNAMICS

$$\overset{\leftrightarrow}{\sigma} \propto 2\gamma \overset{\leftrightarrow}{I} + \eta h \cdot [\nabla \vec{v} + \partial_t \overset{\leftrightarrow}{g}]$$

in-plane stress (force/length)

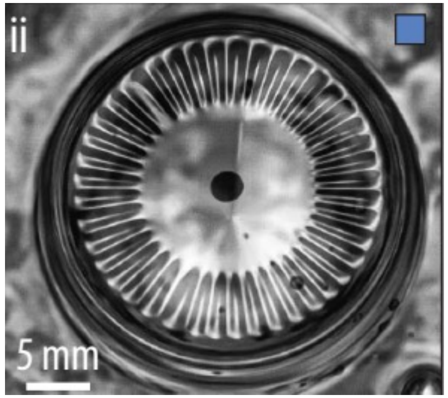
surface tension (homogenous, isotropic)

tangential velocity

metric

$$\nabla \cdot \overset{\leftrightarrow}{\sigma} = 0$$

no normal force



2D Stokes hydrodynamics in curved topography

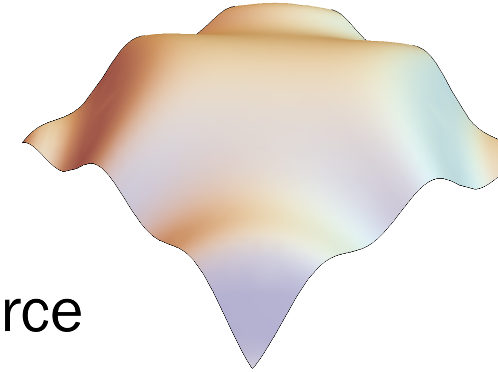
Nonlinear dynamics Imparted by **surface geometry** rather than by fluid inertia

$$\overset{\leftrightarrow}{\sigma} \propto \overset{\leftrightarrow}{\sigma}_{elas} + \eta h \cdot [\nabla \vec{v} + \partial_t \overset{\leftrightarrow}{g}]$$

state-dependent stress

$$\nabla \cdot \overset{\leftrightarrow}{\sigma} = 0$$

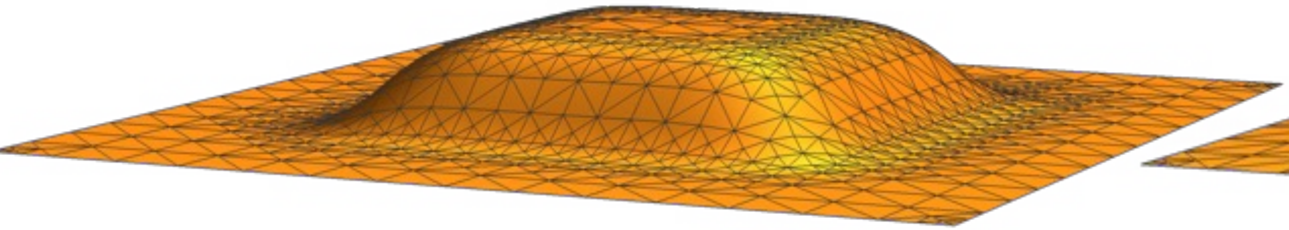
arbitrary normal force

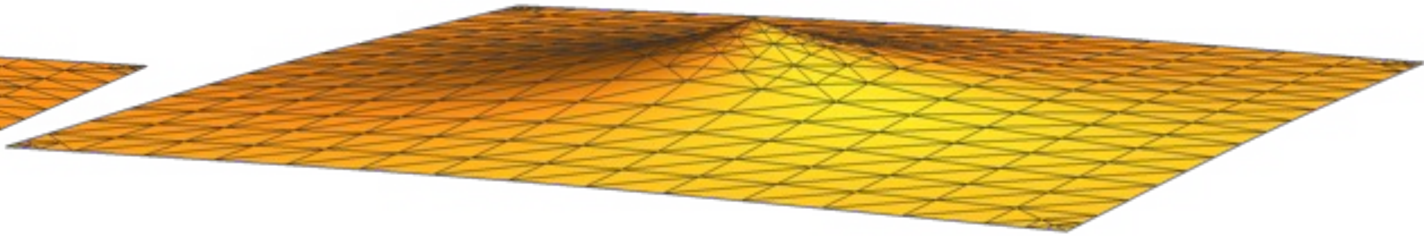


"… if you want to innovate, don't look for a great idea, look for a good problem" (G. Satell)

MOMENTUM-CONSERVING VISCOUS FLOW IN 2D: BEYOND CLASSICAL HYDRODYNAMICS

Example: Grahene drumhead (electrons are strongly correlated, forming viscous 2D fluid)





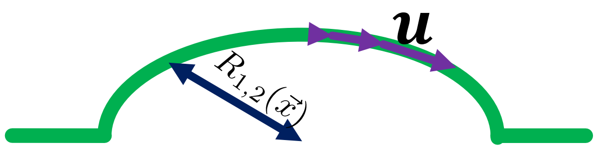
I. at $t<0$, a suspended portion of Graphene flake is deflected by constant force (*e.g.* AFM tip, pressure ..)

II. at $t=0$, the external force is suddenly removed

III. Assume inertia is negligible and **energy** is dissipated solely in heat generated by electric current

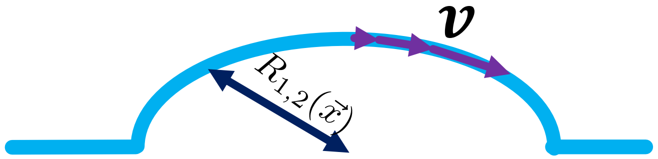
How will the curved shape flatten ?

2D STOKES HYDRODYNAMICS OF GRAPHENE DRUMHEAD



Tangential force balance

(ala 2nd FvK equation)



tangential momentum conserved **separately** in

crystal & **electron liquid**

$$\nabla \cdot \overleftrightarrow{\sigma}^{(c)} = 0 \qquad \nabla \cdot \overleftrightarrow{\sigma}^{(\ell)} = 0$$

$$\nabla^2 \Psi^{(c)} = \mathrm{Tr}\ \overleftrightarrow{\sigma}^{(c)} \qquad \nabla^2 \Psi^{(\ell)} = \mathrm{Tr}\ \overleftrightarrow{\sigma}^{(\ell)}$$

$$\nabla^4 \Psi^{(c)} \propto \underbrace{-\det \overleftrightarrow{R}^{-1}}_{\text{geometric "charge"}}$$

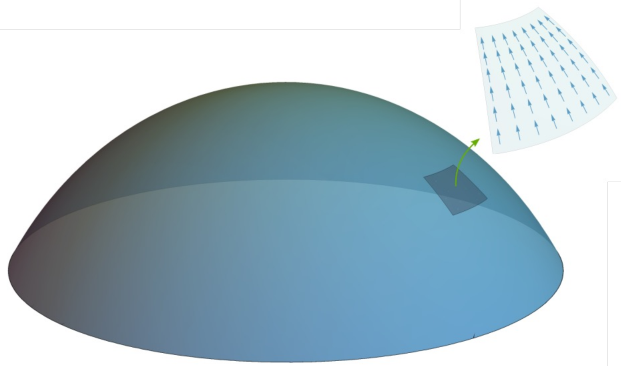
$$\nabla^4 \Psi^{(\ell)} \propto \underbrace{-\partial_t \det \overleftrightarrow{R}^{-1}}_{\text{dynamo-geometric "charge"}}$$

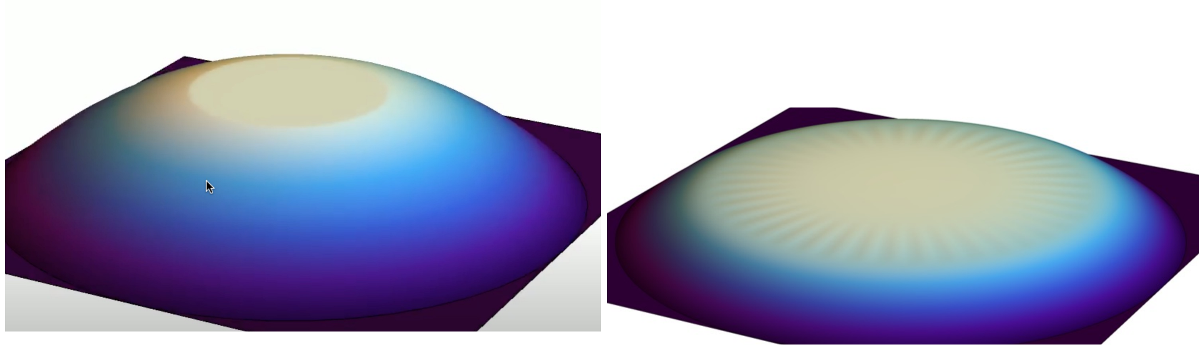
Normal force balance

(ala 1st FvK equation)

$$\left(\overleftrightarrow{\sigma}^{(c)} + \overleftrightarrow{\sigma}^{(\ell)} \right) \cdot \overleftrightarrow{R}^{-1} \approx 0$$

SUMMARY





<u>Curvature driven hydrodynamics in viscous films:</u>

Rapidly depressurized bubble -- a peephole into
a mostly unexplored branch of
”laminar” yet geometrically-nonlinear
fluid mechanics in 2D

$$

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