Speciality of Malcev Algebras

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1 Lie algebras and the PBW theorem

The Poincaré-Birkhoff-Witt (PBW) theorem (Jacobson [2]) implies that any Lie algebra is isomorphic to a subalgebra of the commutator algebra of some associative algebra. This result is established by constructing an associative universal enveloping algebra $U(L)$ for an arbitrary Lie algebra L, together with an injective Lie algebra homomorphism from L to the commutator algebra $U(L)^-$.

2 The speciality problem for Malcev algebras

The first step beyond Lie algebras leads to Malcev algebras. A Malcev algebra is a vector space M with a bilinear product satisfying anticommutativity and the identity

$$[[w,y],[x,z]] = [[[w,x],y],z] + [[[x,y],z],w] + [[[y,z],w],x] + [[[z,w],x],y].$$

The commutator in any alternative algebra satisfies these identities, and so every Lie algebra is a Malcev algebra. The speciality problem for Malcev algebras asks if any Malcev algebra is isomorphic to a subalgebra of the commutator algebra of some alternative algebra. This problem has been open for 50 years since it was first posed in Malcev's paper on analytic loops [4] (where these algebras were called "Moufang-Lie algebras"; they were given their present name by Sagle [6]).

3 Enveloping algebras for Malcev algebras

A solution to a different formulation of the speciality problem for Malcev algebras has recently been provided by Pérez-Izquierdo and Shestakov [5]. They generalize the PBW theorem to Malcev algebras in the following sense: for every Malcev algebra M they construct a universal nonassociative enveloping algebra $U(M)$ and an injective Malcev algebra homomorphism from M to the commutator algebra $U(M)^-$ such that the image of M lies in the generalized alternative nucleus of $U(M)$. The algebra $U(M)$ is in general not alternative nor even power-associative, but it inherits many of the good properties of universal enveloping algebras of Lie algebras, such as the universal mapping property, a PBW-type basis, and a (nonassociative) Hopf algebra structure. Furthermore, if M is a Lie algebra, then $U(M)$ is isomorphic to the familiar (associative) universal enveloping algebra of M.

4 The results we obtained at BIRS

The three of us met in Saskatoon on Friday, April 29, 2005 and drove together to Banff, arriving at BIRS in time for dinner on Saturday, April 30, 2005. On the way from Saskatoon to Banff, we agreed to start by reading and discussing the paper by Pérez-Izquierdo and Shestakov [5]. After doing that, we decided to start with a specific non-Lie Malcev algebra and use the techniques of [5] to compute explicitly the structure constants of the enveloping algebra. In the early paper by Sagle [6] there is an example of a 4-dimensional solvable non-Lie Malcev algebra M (Example 3.1, page 433). ¿From the results of Filippov [1] and Kuzmin [3] it follows that in dimension ≤ 4, this is the only (up to isomorphism) non-Lie Malcev algebra, and that it is solvable and special. We decided that our goals for our stay at BIRS would be:

1. To explicitly construct the enveloping algebra $U(M)$ with PBW-type basis and structure constants.
2. To study the polynomial identities satisfied by the nonassociative algebra $U(M)$.
3. To determine the quotient $A(M)$ of $U(M)$ by the alternator ideal, thereby obtaining an alternative enveloping algebra for M.
4. To determine a finite-dimensional quotient of $A(M)$ containing M in its commutator algebra.

To achieve these goals, we computed (using Maple and Pascal) how to express an arbitrary product of basis monomials of $U(M)$ as a linear combination of basis monomials. To do this we required various reduction algorithms to perform arguments by induction; the essential ideas behind these algorithms appear in the proof of Proposition 2.2 of Pérez-Izquierdo and Shestakov [5]. The techniques we developed at BIRS will allow us to continue this research in the following directions:

1. To solve the same problems for the 5-dimensional non-Lie Malcev algebras (Kuzmin [3]).
2. To do the same for the 7-dimensional simple non-Lie Malcev algebra (Sagle [6], Example 3.2, pages 433–435), and use this to obtain a new construction of the octonions.
3. To do the same for the free Malcev algebra, and use this to search for Malcev s-identities (identities which are satisfied by special Malcev algebras but not by all Malcev algebras).

Our time at BIRS was very productive; we expect to get at least one publication (possibly two or three) from the methods we developed during our "Research in Teams" program.

References

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