Report on the workshop

PROGRESS IN ALGEBRAIC GEOMETRY INSPIRED BY PHYSICS

Organizers

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This is a report of the workshop *Progress in algebraic geometry inspired by physics*, held at the Banff International Research Station, October 8–13, 2005.

This meeting was a great success and a stimulating beginning to the 2005–06 academic year. Some 37 participants attended from top institutions in Canada, the USA, Europe, Korea, Hong Kong, and Japan. As anticipated in its proposal, the workshop covered many of the topics where theoretical physics has most greatly influenced algebraic geometry in recent years.

Gromov-Witten theory, for example, which originated as a quantum field theory governing the propagation of loops or strings on a Ricci-flat spacetime, has become a mathematical theory of the enumerative geometry of algebraic curves on projective varieties. It was discussed in many of the lectures, such as those of Conan Leung and Jim Bryan.

A related topic from physics was discussed in two related lectures by Lothar Göttsche and Hiraku Nakajima: the Nekrasov partition function. This partition function can be regarded, thoroughly physically, as a partition function in an $N = 2$ supersymmetric quantum field theory, but also has mathematical interpretations both in terms of Gromov-Witten invariants and in terms of their analogues and forerunners, the Donaldson invariants of real 4-dimensional manifolds.

Mirror symmetry provides another example of an explicitly physical topic discussed at the meeting. Mirror symmetry began as a duality between quantum field theories, and was reinterpreted in physics as the "T-duality" of Strominger-Yau-Zaslow, in which string theory on a torus of large radius is dual to that on a torus of small radius. Mirror symmetry has received many mathematical interpretations:

- in terms of duality of polytopes by Victor Batyrev and Lev Borisov,
- the "homological mirror symmetry" of Maxim Kontsevich involving derived categories of sheaves and related to the Fourier-Mukai transform, and
- the torus fibrations inspired by Strominger, S.-T. Yau, and Zaslow.

These were represented in the conference by the lectures of Batyrev, Hori and Mark Gross, respectively.

The principal topic of Kentaro Hori's lecture was, however, different, and perhaps more surprising to most participants at the meeting. As one of the few card-carrying physicists present, Hori was able to inform the mathematicians that physics is able to shed light on matrix factorizations of polynomials — certainly a new and intriguing direction that we are likely to hear more of in the future.

But there were also topics of a more purely mathematical nature. Izzet Coskun and Bumsig Kim gave talks on moduli spaces of curves, for example, a more "classical" topic as it goes back in some sense to the nineteenth century. In its modern incarnation, interest dates to the late 1960's, long before the resurgence of physicists' interest in algebraic geometry. Yet it is also clearly a subject that has been revivified and reanimated by the indirect influence of physics. Coskun's talk made this clear: stable curves can be better understood using stable maps, which were only introduced by Kontsevich thanks to the motivation of physics.

Another "classical" topic which kept cropping up was that of K3 surfaces, which were discussed in (at least) the lectures of Leung, Gross, and Bertram. There was no explicit reference to physics in any of these talks, but the indirect influence was clear: these K3 surfaces are (rather elaborate) toy models of Calabi-Yau threefolds, proposed by string theorists to constitute the missing dimensions of space-time.

Some other recurring themes, though less classical, were also purely mathematical. Derived categories of coherent sheaves made an appearance several times, in the lectures of Aaron Bertram, Kentaro Hori, and Alistair Craw, for example. These certainly play a role in physics, as is evident in the work of Michael Douglas, and this was a motivation for Bertram's construction, but the elementary transformations that he described in holomorphic symplectic geometry had a purely mathematical elegance. Craw explained how the study of derived categories could be led in another direction — towards combinatorics — by applying them to the theory of toric varieties.

Another contemporary mathematical theory that was often invoked at the meeting was that of orbifolds or Deligne-Mumford stacks. These are now understood to have a quantum cohomology theory analogous to that of smooth varieties (work due to a number of researchers), and their Gromov-Witten theory, K-theory, and Hochshild cohomology were discussed by Charles Cadman, Takashi Kimura, and Andrei Caldararu respectively.

The concept of topological quantum field theory or TQFT should not be overlooked either. This is not really part of physics; it is more a mathematical formalism, put forward in around 1990 by Michael Atiyah and Graeme Segal, inspired by such physicists as Edward Witten and Robbert Dijkgraaf. But it simplifies and systematizes many calculations in algebraic geometry inspired by physics, any time we want to calculate some invariant on a moduli space by degenerating or cutting up the space on which it is based into smaller constituents (for example, by cutting up a Riemann surface into pairs of pants, interpretable as thrice-punctured spheres). It was discussed, for the enumerative geometry of spaces of admissible covers, in an attractive lecture by Renzo Cavalieri, and alluded to in the talks by Leung and Bryan as well.

There was much informal discussion of all of these topics, and more, at the meeting. The number of formal lectures was intentionally kept small — only sixteen — to provide ample time for informal discussions. However, each of the sixteen speakers was given a full 75 minutes to speak, which ensured an in-depth treatment in each lecture. The topics discussed in the lectures are briefly summarized below.

Conan Leung first reviewed the conjectural Yau-Zaslow formula, which expresses the generating function on the number of curves in a K3 surface as a quasi-modular form.

Then he explained his recent joint work with Junho Lee on the proof of this formula for the index 2 case, generalizing previous work with Jim Bryan for the index 1 case.

The technique employed was the gluing formula for Gromov-Witten invariants.

Lothar Göttsche spoke on his recent work on instanton counting, Donaldson invariants, and line bundles on moduli spaces. (This is joint work with Hiraku Nakajima and Kota Yoshioka.) They computed the Donaldson invariants of a rational surface in terms of the aforementioned Nekrasov partition function, which can be viewed as a generating function for the Donaldson invariants of the affine plane. For a line bundle L on the rational surface X, they computed the holomorphic Euler characteristic

$$\chi(M_X^H(c_1, c_2), \mathcal{O}(\mu(L)))$$

of associated line bundles on the moduli space of H-stable rank 2 bundles on X. Using the Nekrasov conjecture, this yielded explicit generating functions for the Donaldson invariants and the holomorphic Euler characteristics in terms of modular forms and elliptic functions.

Reporting on joint work with Bernd Siebert, **Mark Gross** described a "nonlinear Mumford construction," by which he menat the following. Mumford's construction produces explicit degenerations of abelian varieties, starting with data of a polyhedral decomposition of a real torus and a (multi-valued) convex piecewise linear function on the torus. This can be generalized by replacing the torus with a more general integral affine manifold with singularities. From these data, one can easily produce the central fiber of the degeneration, so the challenge is to smooth this fiber.

Gross showed how Kontsevich and Soibelman's approach translates naturally into this setting, producing explicit smoothings of K3 surfaces. Tropical rational curves emerged naturally out of his construction.

Aaron Bertram spoke about new moduli associated to a K3 surface, studied in joint work with Daniele Arcara. For a K3 surface S whose Picard group is generated by a divisor class C of self-intersection $2g-2$, he considered the "old" moduli space M of stable coherent sheaves on S with invariants $ch_0 = 0$, $ch_1 = H$, $ch_2 = g - 1$ agreeing with those of the push-forward of a sheaf on C of rank 1 and degree $2g - 2$. This is a smooth holomorphic-symplectic manifold.

The object of Bertram's talk was to exhibit a sequence of moduli spaces

$$M \leftrightarrow M' \leftrightarrow M'' \leftrightarrow \cdots$$

that are linked by Mukai flops over projective bundles over products of Hilbert schemes of points on S. These new moduli spaces are not (at least in any manifest way) moduli spaces of coherent sheaves on S, but rather are moduli space of stable objects in the derived category of coherent sheaves on S under a family of stability conditions motivated by physics. Bertram argued that this sequence of flops was the natural generalization of Thaddeus flips to K3 surfaces.

Kentaro Hori reported on his work on matrix factorizations and complexes of vector bundles. Physics shows the equivalence of certain aspects of matrix factorizations of, say, a degree 5 polynomial in 5 variables

$$G(x_1, \ldots, x_5),$$

and complexes of coherent sheaves of the quintic hypersurface $G(x_1, \ldots, x_5) = 0$ in complex projective 4-space. Recently D. Orlov proved the equivalence of the category of matrix factorizations of G and the bounded derived category of coherent sheaves on the quintic.

In his talk, Hori described these equivalences and argued that they are the "right ones" for physics. He suggested that a proper understanding of the physics may have many applications, for example, to stability or to homological mirror symmetry.

Victor Batyrev also spoke about mirror symmetry for Calabi-Yau threefolds, but discussed a subtle feature not previously studied: their integral cohomology. For Calabi-Yau varieties X and Y of dimension d that are mirror to each other, mirror symmetry predicts that the Hodge numbers of X and Y are related by the equality

$$h^{p,q}(X) = h^{d-p,q}(Y).$$

Batyrev's main interest was to understand the relationship between the torsion in their integral cohomology rings. For $d = 3$, he observed that the torsion in H^2 and H^3 must be exchanged by mirror symmetry. His verification of this statement for Calabi-Yau complete intersections in toric varieties reduced to an explicit computation of the fundamental group and the Brauer group.

Izzet Coskun gave a lecture about the cones of ample and effective divisors on Kontsevich moduli spaces. The cones of ample and effective divisors are among the most important invariants associated to any variety. But the study of these cones for moduli spaces is especially important. For example, in a celebrated series of papers in the 1980's, Harris, Mumford, and Eisenbud were able to prove that the moduli space of stable curves is of general type in genus greater than 23 by studying these cones.

In recent work with Joe Harris and Jason Starr, Coskun reduced the computation of the ample cone of the Kontsevich space of (genus zero) stable maps to projective space to a standard conjecture about curves. They also determined the stable effective cone of the Kontsevich moduli spaces. He described these results in his talk and discussed applications to the theory of rational connectivity and the divisor theory of the moduli spaces of pointed stable curves. For example, similar techniques have allowed him to determine the effective cone of the moduli space of pointed (genus zero) stable curves, modulo permutations.

Hiraku Nakajima discussed his joint work with Kota Yoshioka on instanton counting. This refers to the computation of Nekrasov's deformed partition functions of $N = 2$ supersymmetric Yang-Mills theories by integrating in the equivariant cohomology or Grothendieck groups of instanton moduli spaces over four-dimensional Euclidean space, which are quiver varieties associated with the Jordan quiver. These partition functions are analogues of the Donaldson invariants, and equal to the Gromov-Wiotten invariants of certain noncompact Calabi-Yau threefolds. Nakajima reviewed the recent results on these functions.

Alastair Craw reported on work about quivers and exceptional collections for projective toric manifolds. He described how certain collections of line bundles on a projective toric manifold can be used to reconstruct that manifold as a moduli space of quiver representations. To put it another way, he introduced new quiver gauge theory constructions of projective toric manifolds. His condition on the line bundles was remarkably weak, and in particular holds for nice "full strong exceptional collections" (if they exist) that describe the derived category of coherent sheaves. Indeed, Craw's program leads to new examples of such collections. (This was joint work with Greg Smith.)

Harry Tamvakis spoke about the Gromov-Witten invariants of isotropic Grassmannians. He has studied them in joint work with Anders Buch and Andrew Kresch. For a homogeneous space which is the quotient of a classical Lie group by a maximal parabolic subgroup, Tamvakis explained a series of results which show that the three-point genus-zero Gromov-Witten invariants can be equated with, and hence derived from, classical triple intersection numbers on related homogeneous spaces. He applied this principle to prove structure theorems for the small quantum cohomology of these homogeneous spaces, which give new results in the case of a Grassmannian parametrizing non-maximal isotropic subspaces of a vector space equipped with a symplectic or orthogonal form. Buch was also a participant in the workshop, and explained many technical aspects of this work informally in the evenings.

In his lecture on "Hurwitz-Hodge integrals and the crepant resolution conjecture," **Jim Bryan** stated the following. A well-known principle from physics asserts that string theory on an orbifold is equivalent to string theory on any crepant resolution of its coarse moduli space. In mathematics, this can be stated as saying that the Gromov-Witten potentials for the orbifold and the crepant resolution contain equivalent information: that is, one can be transformed to another by an appropriate change of variables. Bryan illustrated this in some examples, showing how it leads to interesting new formulas for integrals of Hodge classes over Hurwitz schemes. The lecture touched on important work joint with Rahul Pandharipande, Andrei Okounkov, Tom Graber, Dagan Karp, and others.

Bumsig Kim spoke about the moduli space of rational plane curves with a unique irreducible singular point. He showed that this moduli space can be decomposed as a union of irreducible smooth rational varieties of varying dimensions. He showed how to compute the degree of the largest component with fixed tangent line at the singular point. He was reporting on joint work with Dosang Joe and Hyungju Park.

Andrei Caldararu gave a stimulating lecture entitled "Towards computing the Hochschild cohomology ring of orbifolds," in which he attempted to explain the ingredients that should go into proving the generalization of Kontsevich's Theorem for complex manifolds to orbifolds. More explicitly, he went over the proof of Kontsevich's Theorem and pointed out what changes have to be made when dealing with orbifolds. For example, the inertial orbifold appears in a natural way in the course of the argument.

Takashi Kimura described the latest results from his long-standing collaboration with Tyler Jarvis. They apply to the setting of a global quotient, that is, a smooth projective variety equipped with the action of a finite group G. To these data, they have associated a G-equivariant Frobenius algebra, which they call the "stringy K-theory," whose G-coinvariants yield the orbifold K-theory of the quotient. They then introduced a stringy Chern character, which is a ring isomorphism from stringy K-theory to its cohomological counterpart. It contains "corrections" to the ordinary Chern character. The proof of the isomorphism follows from a new, simple reformulation of the relevant obstruction bundle, which does not involve stable maps. Hence their work significantly simplifies earlier work in simpler situations.

Renzo Cavalieri spoke about his doctoral work which gave the intersection numbers on moduli spaces of admisible covers the structure of a topological quantum field theory. More precisely, he explained how to construct a two-level weighted topological quantum field theory whose structure coefficents are equivariant intersection numbers on moduli spaces of admissible covers. Such a structure is parallel (and related, albeit somewhat mysteriously) to the local Gromov-Witten theory of curves of Jim Bryan and Rahul Pandharipande.

Cavalieri described the explicit computation of the theory using techniques of localization on moduli spaces of admissible covers of a parametrized projective line. The Frobenius algebras he obtained were one parameter deformations of the class algebra of the symmetric group. In certain special cases he could produce explicit closed formulas for such deformations in terms of the representation theory of the symmetric group.

Charles Cadman also described the work of his doctoral thesis, which uses high technology from the theory of stable maps to Deligne-Mumford stacks to solve a thoroughly classical problem, namely the enumeration of rational plane curves with tangency conditions to a fixed cubic. His key idea was to consider what he calls the "stack of nth roots" associated to a scheme X with a Cartier divisor D: that is, the stack whose objects are morphisms to X together with sections of an nth root of the pullback of the line bundle $\mathcal{O}(D)$, whose nth powers correspond to the natural section of $\mathcal{O}(D)$. This is a Deligne-Mumford stack whose coarse moduli space is X, and (for smooth X and D) stable maps to this stack correspond to maps to X with tangencies of order n along D. Recursions solving the enumerative problem can then be obtained, following Kontsevich, by applying the Witten-Dijkgraaf-Verlinde-Verlinde equations in the quantum cohomology of the stack of nth roots.

Participants.

- Abramovich, Dan, Boston University
- Batyrev, Victor, University of Tubingen
- Bertram, Aaron, University of Utah
- Bryan, Jim, University of British Columbia
- Buch, Anders, Aarhus University
- Cadman, Charles, University of Michigan
- Caldararu, Andrei, University of Wisconsin
- Cavalieri, Renzo, University of Michigan
- Chen, Linda, Ohio State University
- Ciocan-Fontanine, Ionut, University of Minnesota
- Coskun, Izzet, Massachusetts Institute of Technology
- Craw, Alastair, Stony Brook University
- Gholampour, Amin, University of British Columbia
- Gottsche, Lothar, International Centre for Theoretical Physics

- Gross, Mark, University of California at San Diego
- Hori, Kentaro, University of Toronto
- Jarvis, Tyler, Brigham Young University
- Kim, Bumsig, Korea Institute for Advanced Study
- Kimura, Takashi, Boston University
- Lee, Yuan-Pin, University of Utah
- Leung, Nai Chung (Conan), The Chinese University of Hong Kong
- Li, Jun, Stanford University
- Mare, Augustin-Liviu, University of Regina
- Mizerski, Maciej, University of British Columbia
- Mustata, Andrei, University of Illinois, Urbana-Champaign
- Nakajima, Hiraku, Kyoto University
- Payne, Sam, University of Michigan
- Purbhoo, Kevin, University of British Columbia
- Roth, Michael, Queen's University
- Shapiro, Jacob, University of British Columbia
- Song, Yinan, University of British Columbia
- Tamvakis, Harry, University of Maryland
- Thaddeus, Michael, Columbia University
- Tseng, Hsian-hua, University of British Columbia
- Vakil, Ravi, Stanford University
- Watts, Jordan, University of Calgary