Algebraic groups, quadratic forms and related topics

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1 A brief historical introduction

The origins of the theory of algebraic groups can be traced back to the work of the great French mathematician E. Picard in the mid-19th century. Picard assigned a "Galois group" to an ordinary differential equation of the form

$$\frac{d^n f}{dz^n} + p_1(z)\frac{d^{n-1}f}{dz^{n-1}} + \ldots + p_n(z)f(z) = 0\,,$$

where $p_1, \ldots, p_n$ are polynomials. This group naturally acts on the n-dimensional complex vector space V of holomorphic (in the entire complex plane) solutions to this equation and is, in modern language, an algebraic subgroup of $\mathrm{GL}(V)$.

This construction was developed into a theory (now known under the name of "differential Galois theory") by J. F. Ritt and E. R. Kolchin in the 1930s and 40s. Their work was a precursor to the modern theory of algebraic groups, founded by A. Borel, C. Chevalley and T. A. Springer, starting in the 1950s. From the modern point of view algebraic groups are algebraic varieties, with group operations given by algebraic morphisms. Linear algebraic groups can be embedded in GL_n for some n, but such an embedding is no longer a part of their intrinsic structure. Borel, Chevalley and Springer used algebraic geometry to establish basic structural results in the theory of algebraic groups, such as conjugacy of maximal tori and Borel subgroups, and the classification of simple linear algebraic groups over an algebraically closed field. (The latter used the classification of simple Lie algebras, developed earlier by Lie, Cartan, Killing and Weyl.) A more detailed historical account of these developments can be found in [19].

The main focus of the workshop was on linear algebraic groups over fields that are not necessarily algebraically closed. In this context the theory of linear algebraic groups turned out to be closely related to several areas in algebra which previously had an independent existence. Among these areas are Galois theory, the theory of central simple algebras (including Brauer groups and Brauer-Severi varieties), the algebraic theory of quadratic forms, and non-associative algebra. Informally speaking, these connections may be viewed as another manifestation of the idea, championed by F. Klein at the turn of the 20th century. Klein believed many mathematical objects (in particular, in geometry) are best understood and described in terms of their symmetry groups. A crucial role in implementing this idea in the algebraic context (where the objects to be studied are central simple algebras, quadratic forms, octonion algebras, etc.) is played by the theory of Galois cohomology pioneered by J.-P. Serre and J. Tate in the 1950s and 60s.

This approach has been particularly successful within the algebraic theory of quadratic forms. In the context of number theory the study of quadratic forms goes back to Gauss (and probably earlier). The algebraic theory of quadratic forms began with a seminal paper of Witt in 1937, in which what are now called ”Witt's Theorem” and the ”Witt ring” first appeared. But it was not until a remarkable series of papers by Pfister in 1965 - 1967 that the theory was transformed into a significant field in its own right. In these papers, Pfister generalized the well-known two, four, and eight square identities of Euler and Cayley, determined the minimum number of squares representing -1 in an arbitrary field, and developed the finer ring structure of the Witt ring of quadratic forms. This phase of the subject is well documented in the books of Lam [12] and Scharlau [17]. The connection with the theory of algebraic groups was introduced into he subject by T. A. Springer who, in 1959, recasted some of the classical invariants of quadratic forms in terms of the Galois cohomology of the orthogonal group.

2 Recent Developments and Open Problems

In the past 25 years there has been rapid progress in the theory of quadratic forms (and more generally, in the theory of algebraic groups) due to the introduction of powerful new methods from algebraic geometry and algebraic topology. This new phase began in 1981 with the first use of sophisticated techniques from algebraic geometry and K-theory by A. Merkurjev and A. Suslin who established a deep relationship between Milnor's K-groups and Brauer groups. The Merkurjev-Suslin theorem was a starting point of the theory of motivic cohomology constructed by V. Voevodsky. Voevodsky developed a homotopy theory in algebraic geometry similar to that in algebraic topology. He defined a (stable) motivic homotopy category and used it to define new cohomology theories such as motivic cohomology, K-theory and algebraic cobordism. Voevodsky's use of these techniques resulted in the solution of the Milnor conjecture (for which he was awarded a Fields Medal in 2002) and more recently of the Bloch-Kato conjecture (a detailed proof of the latter is yet to appear in print). For a discussion of the history of the Milnor conjecture and some applications, see [15].

These developments have, in turn, led to a virtual revolution in the theory of quadratic forms. Using motivic methods and Steenrod operations (defined by Voevodsky in motivic cohomology and independently by P. Brosnan on Chow groups), Merkurjev, Karpenko, Izhboldin, Rost and Vishik, and others have made dramatic progress on a number of long-standing open problems in the field. In particular, the possible values of the u-invariant of a field have been shown to include all positive even numbers (by A. Merkurjev, disproving a conjecture of Kaplansky), 9 by O. Izhboldin, and every number of the form $2^n + 1$ by A. Vishik. (Vishik's result is new; it was first announced at the workshop.) Another break-through was achieved by Karpenko, who described the possible dimensions of anisotropic forms in the nth power of the fundamental ideal I^n in the Witt ring, extending the classical theorem of Arason and Pfister.

An unrelated important development in the theory of central simple algebras is the recent proof, by A. J. de Jong, of the long standing period-index conjecture; see [8]. This conjecture asserts that the index of a central simple algebra defined over the function field of a complex surface coincides with its exponent. Previously this was only known in the case where the index of had the form $2^n \cdot 3^m$ (this earlier result is due to M. Artin and J. Tate). In a subsequent paper de Jong and J. Starr found a new striking solution of the period-index problem by constructing rational points on families of Grassmannians. Yet another geometric approach for index-period problem was developed by M. Lieblich. Lieblich's approach is based on constructing compactified moduli stacks of Azumaya algebras and studying their properties. These methods and their refinements are likely to play an important role in future research on currently open problems in the theory of algebraic groups; in particular, on Serre's Conjecture II, Albert's conjecture on cyclicity of central simple algebras of prime degree and Bogomolov's conjecture on the Galois group of a maximal pro-normal closure.

Many fundamental questions in algebra and number theory are related to the problem of classifying G-torsors and in particular of computing the Galois cohomology set $H^1(k, G)$ of an algebraic group defined over an arbitrary field k. In general the Galois cohomology set $H^1(k, G)$ does not have a group structure. For this reason it is often convenient to have a well-defined functorial map from this set to an abelian group. Such maps, called cohomological invariants have been introduced and studied by J-P. Serre, M. Rost and A. Merkurjev. Among them, the Rost invariant plays a particularly important role. This invariant has been used by researchers in the field for over a decade but the details of its definition and basic properties have not appeared in print until the recent publication of the book [10] by S. Garibaldi, A. Merkurjev and J-P. Serre.

This book is expected to give further impetus to this line of research.

The presentations at the workshop were loosely grouped into the following general categories:

- Galois theory
- K-theory
- Algebraic stacks
- Homogeneous spaces
- Arithmetic groups
- Brauer groups
- Quadratic forms in characteristic $\neq 2$
- Quadratic forms in characteristic 2

We will now briefly report on the contents of these presentations.

3 Galois theory

Lecture by Florian Pop. Let K be an arbitrary field containing an algebraically closed subfield. Let p be a prime number, $p \neq \operatorname{char} F$. Set $G_{K,p}$ to be a Sylow p subgroup of the absolute Galois group of K. Bogomolov's freeness conjecture asserts that the commutator group $[G_{K,p}, G_{K,p}]$ of $G_{K,p}$ is a free pro-p group; or equivalently, it has cohomological dimension one.

This conjecture was motivated by considerations about the cohomology ring $H^*(K)$ of K, with coefficients in, say, μ_p, and in particular by the Merkurjev-Suslin theorem. The evidence before Pop's work relied on generalizations of Tsen's theorem which asserts that if k is algebraically closed and the transcendence degree K/k is 1 then the cohomological dimension of the absolute Galois group is 1. Also if K satisfies Bogomolov's conjecture, then so do its fields of formal power series $K((t))$ in t over K.

This type of question was also investigated by Chernousov–Gille–Reichstein [9], who ask whether the maximal abelian extension K^{ab} of a field as above has cohomological dimension one; more concretely, whether the maximal abelian extension of the rational field in two variables $\mathbf{C}(t,u)^{ab}$ over the complex numbers has cohomological dimension one. If so, then this would have applications to tackling Serre's Conjecture II.

One could say that the power series $K((t))$ are "local" objects over K, thus one should rather speak here about an "obvious evidence" for the above conjectures. The work of Pop aims at giving less obvious evidence for the above two conjectures. In fact, the examples presented by Pop given evidence for an even stronger conjecture, namely that in the cases of interest (i.e., if one considers function fields $K|k$ over some base fields k which contain all the roots of unity) K^{ab} has a *free profinite absolute Galois group*.

The evidence given by Pop is the following:

1) Suppose that k is an algebraic extension of a local field such that k contains all roots of unity. If $K|k$ is a function field in one variable over k, then the absolute Galois group of K^{ab} is profinite free.

2) A more global version of (1): Let p be a prime number. Set $K = \bar{\mathbf{F}}_p(t,u)$, where $\bar{\mathbf{F}}_p$ is an algebraic closure of a field of order p and t,u are algebraically independent variables over $\mathbf{F}_p$. Let k_0 be a rational function field of one variable over $\mathbf{F}_p$ in K and let $\tilde{k}_0$ be a maximal algebraic extension of k_0 in K_{sep} which is unramified over k_0 outside the infinite place. (K_{sep} is a separable closure of K.) Finally set K^{ab} to be a maximal abelian extension of K. Then the compositum $K^{ab}\tilde{k}_0$ has a free profinite Galois group.

This work is a promising step towards settling Bogomolov's conjecture for fields like $\bar{\mathbf{F}}_p(t,u), \mathbf{C}(t,u), \ldots$ The use of the field $\tilde{k}_0$ in Pop's construction is rather interesting. The advantage of using the field extension

$\tilde{k}_0|k_0$ is that it has a well understood arithmetic description. Suppose that $k_0 = \mathbf{F}_p(X)$. Then the roots of Artin-Schreier equations $y^p - y = f(X)$ (where p does not divide $\deg f(X)$) are in $\tilde{k}_0$. Moreover by Abhyankar's conjecture (see [11] or [16]) it is known that all finite groups which are generated by p-subgroups (finite quasi p-groups) are quotients of $\mathrm{Gal}(\tilde{k}_0/k_0)$. More precise information about the solution of Galois embedding problems inside $\tilde{k}_0/k_0$ may be obtained from [16]. Hence $\tilde{k}_0$ is a rather large extension of k_0.

Lecture by John Swallow. Absolute Galois groups of fields are mysterious. Therefore one would like to identify quotients of Galois groups which are non-trivial, and yet possible to completely classify.

A pro-p group A is called a T-group if there exists some maximal closed abelian subgroup B of A (This means in particular that $[A : B] = p$) such that the exponent of B divides p.

Let Γ be a pro-p group and let Δ be a closed subgroup of index p in Γ. Set also $\Phi(\Delta)$ to be a Frattini subgroup of Δ. (This means $\Phi(\Delta) = \Delta^p[\Delta, \Delta]$, the closed subgroup of Δ generated by pth-powers and commutators.) Then we define $T(\Gamma/\Delta) := \Gamma/\Phi(\Delta)$ to be the T-group associated with the pair Γ, Δ. If Γ is an absolute group of a field F and Δ fixes a cyclic extension E/F, then we say that $T := T(\Gamma/\Delta) = T(E/F)$ is the T-group associated with the extension E/F. (In [BeLMS], all such groups are classified.) In fact, one does not need to require that the absolute Galois group of F is a pro-p group; we restrict our attention to this case to simplify the exposition.

Each $T(E/F)$ as above is a T-group. In order to classify all $T(E/F)$ among T-groups one defines certain invariants of T-groups. First recall that the central series $T_{(i)}$ of a group T is defined recursively as follows

$$T_{(1)} = T, T_{(i+1)} = [T, T_{(i)}], i = 1, 2, \ldots.$$

Further, $Z(T)$ is the center of T and $Z(T)[p]$ is the subgroup of elements of $Z(T)$ of order dividing p.

Swallow defined invariants $t_1, t_2, \ldots, t_p$ and u of T by

$$\begin{aligned} t_1 &= \dim_{\mathbf{F}_p} H^1\left(\tfrac{Z(T)[p]}{Z(T)\cap T_{(2)}}, \mathbf{F}_p\right), \\ t_i &= \dim_{\mathbf{F}_p} H^1\left(\tfrac{Z(T)\cap T_{(i)}}{Z(T)\cap T_{(i+1)}}, \mathbf{F}_p\right), 2 \le i \le p, \\ u &= \max\{i : 1 \le i \le p, T^p \subset T_{(i)}\}. \end{aligned}$$

and gave a complete description of which values of $t_1, \ldots, t_p$ can occur for T-groups. For an odd prime p he also explained for which values of $t_1, \ldots, t_p$ there exists a field extension E/F as above such that $T \cong T(E/F)$.

Thus if p is an odd prime, the possible quotients $T(E/F)$ of the absolute groups Γ_F are substantially restricted. These restrictions imply further restrictions on the presentation of Γ_F via generators and relations; for details see [5] and [6]. In contrast, if $p = 2$, there is no restriction on $T(E/F)$, and all pro-2 T-groups occur for suitable quadratic extensions E/F.

Swallow also described $\mathbf{F}_p[\mathrm{Gal}(E/F)]$ module structure of $H^n(\Gamma_E, \mathbf{F}_p)$, where F/F is a cyclic extension of degree p, F contains a primitive pth-root of 1, and $\Gamma_E \subset \Gamma_F$ are absolute Galois groups of E and F respectively. For details see [13]

In recent joint work with F. Chemotti and J. Mináč, Swallow described the $\mathbf{F}_p[\mathrm{Gal}(E/F)]$-module structure of $H^1(\Gamma_E, \mathbf{F}_2)$ in the case where $\mathrm{Gal}(E/F)$ is $C_2 \times C_2$. An interesting byproduct of this description is that although the Klein-4 group has infinitely many indecomposable modules over $\mathbf{F}_2$, only finitely many of these modules can occur as a summand of $H^1(\Gamma_E, \mathbf{F}_2)$. This fact points out the possibility of obtaining the full structure of Galois modules for other Galois groups, even in cases where the classification of indecomposable modules is a hopeless task.

Lecture by Eva Bayer-Fluckiger. Let F be a field of characteristic different from two and G be a finite group. A G-form is a pair (M, φ) with M an $F[G]$ module of finite F-dimension and φ a quadratic form such that $\varphi(gx, gy) = \varphi(x, y)$ for all $x, y \in M$ and for all $g \in G$. The problem is to determine when are two G-forms $\varphi := (M, \varphi)$ and $\psi := (M, \psi)$, i.e., are G-isomorphic. This problem is a natural generalization of the classical problem of determining when (M, φ) admits a self-dual basis, i.e., when $(M, \varphi) \cong (F[G], q)$. Here $q(\sigma, \tau) = \delta_{\sigma\tau}$ for each $\sigma, \tau \in G$ and $\delta_{\sigma\tau} = 1$ or 0 depending upon whether $\sigma = \tau$ or $\sigma \neq \tau$. The problem of the existence of a self-dual basis is especially interesting when the module $F[G]$ is a Galois field extension L of F, $G = \mathrm{Gal}(L/F)$ and form q is a trace form $q_L : L \times L \to F, (x, y) \to \mathrm{Tr}_{L/F}(xy)$. In

[1] it was proved that any Galois algebra has a self-dual basis. The situation is more complicated when $\mid G \mid$ is even; the paper [4] treated the cases where the 2-Sylow subgroup G_2 of the Galois group G is elementary abelian (i.e., $G_2 \simeq C_2 \times \ldots \times C_2$) or a quaternion group of order 8 ($G_2 \simeq Q_8$).

Let $W(F)$ be the Witt ring of non-degenerate quadratic forms over F and $I(F)$ its fundamental ideal of even dimensional forms and $I^n(F)$ its nth power. Suppose that $\mathrm{cd}_2 F$, the cohomological 2-dimension of F, is finite and equal to d. This means that the absolute Galois group Γ_F of F has cohomological dimension d. Let L and L' be two G-Galois algebras and let φ and ψ be their corresponding trace forms. Then in [7] it was proved that $\rho \otimes \varphi \cong_G \rho \otimes \psi$ for any $\rho \in I^d(F)$. Now let L be any Galois algebra over F with Galois group G. Let Γ be an absolute group of F. Then L can be viewed as an element of $H^1(\Gamma, G)$, where the action of Γ on G is trivial. In particular there is a corresponding 1-cocycle $\varphi : \Gamma \to G$, which is just a continuous homomorphism associated to L. Consider any homomorphism $x \in H^1(G, \mathbf{F}_2)$. Set $x_L = x_0\varphi \in H^1(\Gamma, \mathbf{F}_2)$. Keeping our assumption that L and L' are two Galois extensions of F having Galois group G, assume now that $\rho \in I^{d-1}(F)$. Then Bayer-Fluckiger showed that $\rho \otimes \varphi \cong_G \rho \otimes \psi$ if and only if $e_{d-1}(\rho) \cup x_L = e_{d-1}(\rho) \cup x_{L'}$ for all $x \in H^1(G, \mathbf{Z}/2\mathbf{Z})$. Here e_i are the isomorphisms given by the Milnor conjecture $I^i(F) \to H^i(\Gamma_F, \mathbf{Z}/2\mathbf{Z})$. An analogous result holds for (finite) ordered systems of quadratic forms (or hermitian forms) $\Sigma := (\varphi_1, \ldots, \varphi_m)$, with the obvious notion of isomorphism. If Σ and Σ' are two such ordered systems of quadratic forms (respectively hermitian forms) of size m such that they become isomorphic over the separable closure of F then $\rho\Sigma \cong \rho\Sigma'$ for all $\rho \in I^d(F)$ (respectively, $\rho \in I^{d-1}(F)$).

At the end of the lecture Bayer-Fluckiger showed that similar results hold if the field F is replaced by (D, σ), where D is an F-division algebra with involution σ. One can also look at ordered systems of such hermitian forms; for details see [2, 3]

4 K-theory

Lecture by Stefan Gille. Let X be noetherian scheme and $X^{(i)}$ the set of points in X of codimension i. If $x \in X^{(i)}$ let $F(x)$ be the residue field. We have a Gersten complex

$$(*) \quad 0 \to K_n'(X) \to \oplus_{X^{(0)}} K_n'(F(x)) \to \oplus_{X^{(1)}} K_n'(F(x)) \to \cdots$$

in coherent K-theory. The sequence $(*)$ is exact (Gersten Conjecture) for $X = \text{Spec } R$ if R is a regular semi-local ring by work of Quillen and Panin. One wants a similar result for Hermitian Witt groups. Let $\mathcal{M}_c(X)$ be the category of coherent $\mathcal{O}_X$-modules. We have a filtration by Serre subcategories $\mathcal{M}_c(X) = \mathcal{M}^0 \supset \mathcal{M}^1 \supset \cdots$ with $\mathcal{M}^i := \{\mathcal{F} \in \mathcal{M}_c(X) \mid \text{codim supp } \mathcal{F} \geq i\}$. If $\dim X$ is finite, we get a spectral sequence $E_1^{p,q} := K_{p-q}(\mathcal{M}^p/\mathcal{M}^{p+1}) \Rightarrow$ cohomological K-theory of X and $K_{p-q}(\mathcal{M}^p/\mathcal{M}^{p+1}) = \oplus_{X^{(p)}} K_{-p-q}(F(x))$ by Gabriel's thesis and devissage. If A is an Azumaya algebra over X, we can look at $\mathcal{M}_c(A)$ the category of coherent (left) A-modules and filter it by $\mathcal{M}_A^p := \mathcal{M}_c(A) \cap \mathcal{M}^p$. We get another spectral sequence and can ask if the Gersten conjecture is true for it. Replacing X by A and $F(x)$ by $A \otimes F(x)$ in $(*)$, we get a complex introduced by Colliot Thélène and Ojanguran and showed to be exact by them and Panin-Suslin. Gille studies this problem if A has an involution which consists of an automorphism σ of X of order two and an $\mathcal{O}_X$-linear map $\tau : A \to \sigma_* A$ satisfying $\sigma_*(\tau) \circ \tau = 1_A$ and $\tau(ab) = \tau(b)\tau(a)$. The map τ is of the first kind of $\sigma = 1$ and the second kind otherwise. Assume that τ is of the first kind. Gille shows that if (A, τ) is an Azumaya algebra over a regular scheme X of finite dimension with τ of the first kind then there exist two complexes, the *hermitian* and *skew hermitian Gersten-Witt* complexes

$$(*) \quad 0 \to W^{\pm}(A, \tau) \to \oplus_{X^{(0)}} W^{\pm}(A \otimes F(x), \tau \otimes F(x)) \to \cdots$$

and this complex is exact if X is the spectrum of a semi-local ring of a smooth variety. Such a result could not be true if τ is of the second kind as, in general, it would not induce automorphisms of the residue fields. Gille then constructed these Gersten-Witt groups. To show exactness, one follows Quillen's proof but modifying Quillen's last argument on the additivity of functors to Gille's result that given a Gorenstein ring R of finite Krull dimension, an Asumaya algebra A over R with an involution τ of the first kind, and $t \in R$ an element satisfying $\pi : R \to R/R/Rt$ has a flat splitting then the transfer $\pi_* : W^i(A/tA, \tau/t\tau) \to W^{i+1}(A, \tau)$ is zero.

Lecture by Alexander Nenashev. Balmer-Witt theory does not have Chern classes as it is not oriented and the Projective Bundle Theorem fails. Nenashev showed how to construct a twisted Thom isomorphism and deformation to the normal cone in the theory and used it to show the existence of a pushforward f_* : $W^n(Y, f^*L \otimes \omega_{Y/X}) \to W^{n+c}(X, L)$ for any projective morphism $f : Y \to X$ of pure codimension c where L is a line bundle on X and $\omega_{Y/X}$ is the relative dualizing sheaf. The difficult point is to show if $j : Z \to Y$ and $i : Y \to X$ are closed imbeddings then the pushforwards $(ij)_* = j_* i_*$ which uses the theory of "double" deformation spaces.

Lecture by Marco Schlichting. Balmer-Witt groups of a regular scheme X have long exact Mayer-Vietoris sequences. In general, this is no longer true if the scheme is singular. Schlichting lectured on a way to deal with this by defining new Witt groups called *stabilized Witt groups* generalizing certain L-groups defined by Ranichi and Witt rings with involution defined by Karoubi. Although this theory is not known to hold for triangulated categories with involution, it does hold for categories of rings with involution, exact categories with involution, dg categories, and exact categories with isomorphisms weak equivalences. In particular, in these cases, one can generalize the notion of suspensions and cones. Thes stabilized Witt groups have periodicity 4 and satisfy Mayer-Vietoris and homotopy invariance. They coincide with Witt-Balmer Witt rings if $K_n X = 0$ for all negative n, e.g., if X is regular. The case when the characteristic of the underlying field is zero was also discussed and the relationship with blowups, reflecting work done jointly with G. Cortinas, C. Haesemeyer, and C. Weibel.

5 Algebraic stacks

Lecture by Patrick Brosnan. Let $\mathcal{F}$: fields$/F \to$ sets be a functor. Merkurjev, generalizing the idea of Buhler-Reichstein defined the essential dimension of a set a to be ed $a := \min\{\text{tr deg}_F K \mid L/K/F \text{ with } a \in \text{im}(\mathcal{F}(K) \to \mathcal{F}(L))\}$ and the essential dimension of $\mathcal{F}$ to be ed $\mathcal{F} := \{\text{ed } a \mid a \in \mathcal{F}(L),\ L/F\}$. If G is a group let ed $G :=$ ed $H^1(-, G)$. Generalizing the definition of essential dimension to include stacks, Brosnan discussed his joint work with Z. Reichstein. An Artin stack can be viewed as a functor from rings to categories (usually groupoids) satisfying various properties. An interesting example of χ is the moduli stack of smooth curves of genus g; there are many other interesting examples, related, e.g., to various other families algebro-geometric objects, such as curves, hypersurfaces, abelian varieties, etc., possibly with additional structures, such as marked points. If χ is a stack then edχ is defined as the essential dimension of the functor $L \longrightarrow \{\text{isomorphism classes of objects in } \chi_L\}$. Assuming that an Artin stack has a filtration of closed stacks $\chi = \chi_n \supset \chi_{n-1} \supset \cdots \supset \chi_0 = \emptyset$ with $\chi_i \setminus \chi_{i-1} = [Y_i/G_i]$, the stack associated to the G_i-torsors of scheme Y_i where G_i is a linear algebraic group, then they show ed χ is finite. Brosnan also discussed his theorem that the essential dimension of a complex abelian variety A (i.e., of the Galois cohomology functor $H^1(-, A)$) is $2\dim(A)$.

Lecture by Angelo Vistoli. Vistoli lectured on the use of stacks to investigate the theory of hyperelliptic curves. (Cf. the summary of Brosnan's lecture for definitions.) If X is a scheme and G a group acting on X, let $[X/G]$ be the stack associated to G-torsors of X. For example, the stack of moduli spaces of genus g is $[X/PGL_N]$. Define the homology of a stack $\mathcal{F} \to \text{schemes}/F$ by $H^*(\mathcal{F}, \mathbf{Z}) := H^*_G(X, \mathbf{Z})$. Then $\text{Pic}([X/G]) = \text{Pic}_G(X)$, where $\text{Pic}_G(X)$ is the G-equivariant Picard group. The stack of elliptic curves is $[U/\mathbf{G}_m]$ where $U := \{(a, b) \in \mathbf{A}^2 \mid -4a^3 - 27b^2 \neq 0\}$ for elliptic curves (given in Weierstrass form): $y^2 = x^3 + ax + b$. This difficult theorem shows that the stack of elliptic curves is a quotient stack. The problem is to generalize this to find X_g so that the stack of curves of genus g is $[X_g/GL_g]$. For $g = 2$, it an be shown that X_2 is a subspace of $\mathbf{A}^7$. This generalizes to the stack of hyperelliptic curves of genus g, a closed substack of the stack of curves of genus g if $g \geq 2$. Together with A. Arsie, this stack was identified as $[X/G]$ with $X = \{f \mid f \text{ a homogeneous form of degree } 2g+2 \text{ with distinct zeros}\}$ an open subset of $\mathbf{A}^{2g+3}$ and $G = GL_2$ if g is even and $G = \mathbf{G}_m \times PGL_2$ if g is odd (with specified action). Moreover the Picard group of this stack is $\mathbf{Z}/(2g+1))$ if g is even and $\mathbf{Z}/(4(2g+1)))$ if g is odd. The case of trigonal curves was also discussed.

6 Homogeneous spaces

Lecture by Prakash Belkale. Let G be a simply connected simple algebraic group and P a maximal parabolic subgroup. Belkale lectured on his joint study with S. Kumar on the ring structure of $H^*(G/P, \mathbf{C})$ in terms of structure constants for multiplication of the Schubert basis. By introducing a new twisted product on this basis, they are able to apply to give additional information to the eigenvalue problems and its relation to the Horn Conjecture and the Klyachko, Knutson-Tao theorem on the sum of eigenvalues of hermitian matrices.

Lecture by Kirill Zainoulline. Let G be a linear algebraic group over F and X a projective homogeneous G-variety. One wishes to decompose the Chow motive $\mathcal{M}(X)$ of X into a sum of motives of varieties Y having "trivial splitting patterns". This has been done for some cases, e.g., if G is split, if X has a rational point (by V. Chernousov, S. Gille and A. Merkurjev), or if G is isotropic (by P. Brosnan). So assume that G is anisotropic. If X is an Pfister form then Rost showed that $\mathcal{M} = \oplus R(i)$ with $R(i)$ indecomposable, $R_{\bar{F}} = \mathbf{Z} \oplus \mathbf{Z}(2^{n-1} - 1)$. But $R(i)$ is not the motive of a variety. N. Karpenko showed that the motive of the Severi-Brauer variety of a division algebra is indecomposable A. Vishik decomposed the motive of an anisotropic quadric. Zainoulline discussed other cases. A projective smooth variety over F is called *generically split* if $\mathcal{M}(X_{F(X)}) \cong \oplus_* \mathbf{Z}(*)$ and L/F is called a *splitting field* for X if X_L is generically split. Fix a prime p. Let $\bar{A} = \mathrm{CH}(X_L)/p$ where L is a splitting field of X and $\bar{A}_{rat} := \mathrm{im}(\mathrm{CH}(X)/p \to \mathrm{CH}(X_L)/p)$ (cf. the generically discrete invariant of Vishik). If p is prime and there exists a $\rho \in \bar{A}^r$ satisfying $\bar{A}^s = \bar{A}^s_{rat}$ for all $s < r$, $\bar{A}^r = \langle \rho, \bar{A}^r_{rat} \rangle$ and there exists finite subset $\mathcal{B}$ of $\bar{A}_{rat}$ such that $\mathcal{B} \times \{\rho^i\}_{i=0,p-1}$ is a basis for $\bar{A}$ then $\mathcal{M}(X) \otimes \mathbf{Z}/p = \oplus_* R(*)$ with R indecomposable if and only if R has no 0-cycles of degree 1. If X and Y both satisfy the conditions of this result for the same r, X splits over $F(Y)$, and Y splits over $F(X)$ then $R_X \cong R_Y$. This applies to the case of G split with $G = \xi G$, $\xi \in H^1(\Gamma_F, G)$ (an inner form) with Γ_F the absolute Galois group of F and $X = {}_\xi(G/P)$, P a parabolic subgroup, This applies when $X = SB(\mathbf{M}(D))$, where D is an F-division algebra of degree p, an n-fold Pfister form with $p = 2$, ${}_\xi(F_4/P_1)$ with $p = 2$ or 3, and ${}_\xi(E_8/P_8)$ with $p = 5$.

7 Arithmetic groups

Lecture by Philippe Gille. Let Γ_F be the absolute Galois group of a number field F. If v is a place of F, we will denote the completion of F at v by F_v and the algebraic closure of F_v by $\overline{F_v}$. We will also denote a finite set of primes by S, the ring of integers in F by A, the ring of S-integers by A_S, and the ring of integerst in $\overline{F_v}$ by $\overline{A_v}$. The Borel-Serre Theorem states that for a linear algebraic group G over F, the map $W'_S(F,G) := \ker(H^1(\Gamma_F, G) \to \prod_{v \notin S} H^1(\Gamma_{F_v}, G))$ is proper, i.e., has finite fibers. Gille discussed his joint work with L. Moret-Bailly on the integral version of this theorem.

Let X be variety over F having an action of a linear algebraic group G on it and $Z_0 \subset X$ a flat closed A_S-subscheme. Let $W'_s(x_0) := G(F)\backslash\{x \in X(F) \mid x \in G(F_v)x_0$ for all $v \notin S\}$. This is a finite set. Suppose that G/A_S is a flat affine group scheme and X/A_S is a flat scheme with an algebraic action $G \times_{A_S} X \to X$ given by $g \cdot x \mapsto \rho(g) \cdot x$. Then $G(A_S)\backslash\mathrm{loc}(Z_0)$ is finite, where $\mathrm{loc}(Z_0) := \{Z \subset X \mid Z$ a flat closed A_S-subscheme with $\rho(g_v) : Z \times_{A_S} \overline{A_v} \xrightarrow{\sim} Z \times_{A_v} \overline{A_v}$ for some $g_v \in G(\overline{A_v})$ for all $v \notin S\}$. An example of this is $G = GL_m/\mathbf{Z}$ acting on G by conjugation. Suppose this is the case. Fix $g_0 \in G$. Then there are only finitely many $g \in GL_n\mathbf{Z}$ satisfying $g = g_p g_0 g_p^{-1}$ for $g_p \in GL_m\mathbf{Z}_p$ for all p. The theorem follows from a more general one, viz., if G/A_S is an affine group scheme (but not necessarily flat) then the cohomology set $H^1_{fppg}(A_S, G)$ is finite where $fppf$ is the faithfully flat of finite presentation topology. To prove this one makes various reductions. First one reduces to a flat group scheme over A_S. This can be done because over a number field as the normalization of G is still a group scheme. One shows that the result holds for a flat affine group scheme. Reducing to the case that G is also connected, the result for such G is proven.

Lecture by Uzi Vishne. Vishne discussed his joint work with M. Katz and M. Schaps on traces in congruence subgroups $\Gamma(I)$ of finite index in an arithmetic lattice Γ. Let K be a totally real number field lying in $\mathbf{R}$ via one of the real embeddings so $K \otimes \mathbf{R} = \mathbf{R} \times \mathbf{R}^{d-1}$. Let $\mathcal{O}_K$ be the ring of integers in K and D/K a quaternion algebra with $D \otimes \mathbf{R} = \mathbf{M}_2(\mathbf{R})$ but $D \otimes_\sigma \mathbf{R}$ a division algebra at the $(d-1)$ non-inclusion real embeddings. Let Q be an order in D. The lattice Γ is taken to be Q^1, the elements of norm one. Let

$X := \Gamma\backslash\mathcal{H}$ where $\mathcal{H}$ is the upper half plane and $X_I := \Gamma(I)\backslash\mathcal{H}$ where I is an ideal in $\mathcal{O}_K$. Then $X_I \to X$ is a cover of Riemannian manifolds. Let $g(X)$ be the genus of X. The length of the shortest non-trivial closed loop in $\pi_1(X)$ is called the *girth* of X. Vishne and his collaborators showed that for any metric Riemannian surface Y of genus g, one has $(\text{girth}(Y))^2/\text{area}(Y) \leq (\log g)^2/\pi g$ and for $Z = X$, or $X(I)$ above that $(\text{girth}(Z))^2/\text{area}(Z) \geq 4(\log g(Z))^2/9\pi g(Z)$. So $(\text{girth}(X_I) \geq (2 \cdot 2/1 \cdot 3)(\log(g(X_I) - c)$ for some constant $c = c(\Gamma)$. All the integers in the coefficient are constants that can be explained except for the second 2. For example, the first 2 is the trace of 1. More generally, if $\pm 1 \neq x \in \Gamma(I) := \ker(Q^1 \to (Q/IQ)^1$ then $|\text{tr}\, x| \geq (N(I)^2/2^d N(2\mathcal{O}_K + \gamma I)) - 2$ where $Q \subset (1/\gamma)Q_0$ with Q_0 the standard order $\mathcal{O}_K[i,j]$ in Q and γ minimal. Computation shows that there exists a constant $\lambda_{D,Q}$ satisfying $[\Gamma : \Gamma(I)] \leq \lambda_{D,Q} N(I)$. This is used to show $\text{girth}(X_I) \geq 4/3(\log(g(X_I)) - \log 2^{3d-5}\text{vol}(X)\lambda_{D,Q}/\pi$. For a Hurwitz surface, i.e., a compact Riemann surface X the order of whose automorphism group achieves the maximum possible size $84(g-1)$, this gives $\text{girth}(X) \geq (4/3)\log(X)$.

8 Brauer Groups

Lecture by Daniel Krashen. Krashen discussed joint work with M. Lieblich. Let F be a perfect field, D a central F-division algebra, and C a curve over F of genus 1. Krashen discussed the problem of determining the index of $D_{F(C)}$. They show that the index ind $D_{K(C)} := \min\{[E : F(C)] \mid D_E \text{ splits }\}$ is in fact equal to $\min\{[L : F] \mid D_{L(C)} \text{ splits}\}$. This solves the problem if F is a local field, viz., ind $D_{K(C)} = \min\{[L; F] \mid$ ind $D/\text{gcd}(\text{ind}\, D, [L : F])$ divides ind $C_L\}$ where ind $(C) := \min\{[E : F] \mid C(E) \neq \emptyset\}$. Krashen then discussed the theory of twisted sheaves and its relation to the index. Let X be a nice scheme over F, i.e., integral, noetherian, ... Let $\alpha \in H^2(X, \mathbf{G}_m)$ (the cohomological Brauer group). An α-*twisted sheaf* on X is a collection of $\mathcal{O}_{U_i}$-modules M_i where $\{U_i\}$ is an (étale) open cover of X with (glueing) isomorphisms $\varphi_{ij} : M_i|_{U_i \cap U_j} \to M_j|_{U_i \cap U_j}$ satisfying $\varphi_{ij}\tilde{\alpha} = \varphi_{jk}\varphi_{ij}$ with $\tilde{\alpha}$ a (Cech) cocycle in the class of α. (This can be shown to be independent of choices.) There exists an α-twisted locally free sheaf of rank r on X if and only if there exists an Azumaya algebra A on X of degree r such that the class of A is α, i.e., α lies in the Brauer group of X. Going to the generic point, this implies that there exists an α-twisted coherent sheaf of rank r on X if and only if ind $\alpha_{F(X)} \mid r$. Next Krashen discussed how this relates to the problem of determining when an element $l \in \text{Pic}\, C$ comes from $L \in \text{Pic}\, C(F)$, i.e., $l = [L]$; equivalently l arises from a line bundle $\mathcal{L}$ on $C_{\bar{F}}$, where $\bar{F}$ is the algebraic closure of F. Choosing isomorphisms $\varphi_{\sigma,\tau} : {}^{\sigma}\mathcal{L} \to {}^{\tau}\mathcal{L}$ for σ, τ in the absolute Galois group of F may not be compatible glueing data but does give a 2-cocycle $\alpha_{\sigma,\tau,\gamma}$ in $\mathbf{G}_m$ hence leads to an α_C-twisted line bundle defined over F hence in the Brauer group of F. If C is an elliptic curve and V an α_C-locally free sheaf of rank n then ind $\alpha_{F(C)} = n$ implies there exists E/F of degree n such that α_{C_E} splits.

9 Quadratic forms in characteristic $\neq 2$

Lecture by Alexandr Vishik. The u-invariant of a field F is defined to be the maximal dimension of anisotropic quadratic forms defined over F. For fields of characteristic different from two, it known that u cannot be $3, 5$, or 7. Merkurjev showed that any even integer could be the u-invariant of a field and Izhboldin showed the value of 9 was achievable. Vishik lectured on his construction of fields having u-invariant $2^n + 1$ for any $n \geq 3$. Let $G(Q,i)$ be the Grassmannian of i-dimensional projective planes in a smooth D-dimensional quadric Q over F for $0 \leq i \leq d := [D/2]$. The *generic discrete invariant* $GDI(Q)$ is defined to be the image of $\text{Ch}^*(G(Q,i) \to \text{Ch}^*(G(Q,i)/\bar{F})$ where $\text{Ch}^*(X)$ is the Chow group of X mod 2 and $\bar{F}$ is the algebraic closure of F. If $Fl(Q,0,i)$ is the flag variety of Q, there exists a correspondence $f : Q \longrightarrow G(Q,i)$. Let $z_j(i-d) = f_*(l_{D-i-j})$ for $D-d-i \leq j \leq D-i$ where $l_0, l_1, \ldots, l_d$ in $\text{CH}_i(Q/\bar{F})$, $\leq i \leq d$, are the classes of projective subspaces of $Q_{\bar{F}}$ of dimension i (choose one if d is even). The ith *elementary discrete invariant* $EDI(Q,i)$ of Q is set $\{j \mid z_j(i-j)$ is defined over F mod 2. To each Q one can draw a $d \times d$ square with the lattice point (x,y) colored if $z_x(y-d)$ is defined over F. Vishik proved the if the characteristic of F is zero and $D = 2^r - 1$ with $r \geq 3$ and the square for Q has only the (d,d) point possibly colored then for all quadrics Q' of dimension $> D$ the invariant $EDI(Q'_{F(Q)})$ will have the same property. In particular, $Q'_{F(Q)}$ is anisotropic. Using this result one can construct a field having u-invariant

$2^r + 1$ for any $r \geq 3$ in the usual way. The proof of the theorem utilizes a more general result that Vishik proved, viz., if the characteristic of F is zero and $y \in \mathrm{Ch}^m(Y/\bar{k})$ with Y a smooth quasi-projective variety over F and Q as above then for any $m \leq [(1+D)/2]$, the element y is defined over F if and only if $y|_{F(Q)}$ is defined. The proofs use symmetric operations in cobordism theory. Because of interest in the result, Vishik gave a second lecture with more details of the proofs.

10 Quadratic forms in characteristic 2

Lecture by Ricardo Baeza. Let F be a field of characteristic two. Let $W(F)$ denote the Witt ring of non-singular symmetric bilinear forms and $I(F)$ the fundamental ideal of even dimensional forms. Let $I^n(F)$ be the nth power of $I(F)$. (Cf. the summary of Hoffman's lecture for definitions and notation.) Let $W_q(F)$ be the Witt group of (even dimensional) non-singular quadratic forms over F; it is a $W(F)$-module. If $a, b \in F$ let $[a, b]$ be the binary quadratic form $ax^2 + xy + by^2$. Every non-singular quadratic form is an orthogonal sum of such binary forms. The submodule $I_q^{n+1}(F) := I^n(F)W_q(F)$ is generated by $(n+1)$-fold quadratic Pfister forms $\varphi \otimes [1, a]$ with φ a bilinear n-fold Pfister form. J. Arason and R. Elman found a presentation for $I^n(K)$ when the field K was of characteristic different from two. Baeza with J. Arason found analogous presentations for $I^n(F)$ and $I_q^{n+1}(F)$ for all n. For $I^n(F)$ the generators are isometry classes $[\mathbf{b}]$ of bilinear n-fold Pfister forms $\mathbf{b}$ with generating relations given by

1. $[\mathbf{b}] = 0$ if $\mathbf{b}$ is metabolic.

2. $[\langle 1, a\rangle \otimes \mathbf{c}] + [\langle 1, b\rangle \otimes \mathbf{c}] = [\langle 1, a+b\rangle \otimes \mathbf{c}] + [\langle 1, ab(a+b)\rangle \otimes \mathbf{c}]$ with $\mathbf{c}$ an $(n-1)$-fold Pfister form and $a + b \neq 0$.

3. $[\langle 1, ab\rangle \otimes \langle 1, c\rangle \otimes \mathbf{d}] - [\langle 1, a\rangle \otimes \langle 1, c\rangle \otimes \mathbf{d}] = [\langle 1, ac\rangle \otimes \langle 1, b\rangle \otimes \mathbf{d}] - [\langle 1, a\rangle \otimes \langle 1, b\rangle \otimes \mathbf{d}]$ with $\mathbf{d}$ an $(n-2)$-fold Pfister form.

where the second relation is only needed if $n = 1$ and for $I_q^n(F)$ the generators are isometry classes of quadratic n-fold Pfister forms $[\varphi]$ with generating relations given by

1. $[\mathbf{c} \otimes [1, d_1 + d_2]] - [\mathbf{c} \otimes [1, d_1]] + [\mathbf{c} \otimes [1, d_2]]$ with $d_1, d_2 \in F$ and $\mathbf{c}$ a (bilinear) $(n-1)$-fold Pfister form.

2. $[\langle 1, a\rangle \otimes \varphi] + [\langle 1, b\rangle \otimes \varphi] = [\langle 1, a+b\rangle \otimes \varphi] + [\langle 1, ab(a+b)\rangle \otimes \varphi]$ with φ a quadratic $(n-1)$-fold Pfister form and $a + b \neq 0$.

where the second relation is only needed for $n = 1$ The proof uses the ideas to prove this result if the field is of characteristic different from two together with a result about forms $[[a_1, \ldots, a_n]]$ defined to be $\otimes_{i=1}^n \langle 1, a_i\rangle \otimes [1, a_1 \cdots a_{n+1}]$ if $a_1, \ldots a_n \in F^\times$ otherwise to be zero. These generate $I^{n+1}(F)$ with generating relations

1. $[[a_1, \ldots, a_{n+1}]] = 0$ if some $a_i = 1$.

2. $[[a_1, \ldots, r^2 a_i, \ldots a_j, \ldots, a_{n+1}]] = [[a1, \ldots, a_i, \ldots r^2 a_j, \ldots, a_{n+1}]]$.

3. $[[a_1, \ldots, a_{n+1}]] = 0$ if some $a_1, \ldots, a_{n+1} \in \wp(F)$.

Lecture by Detlev Hoffman. Let F be a field of characteristic two and $\mathbf{b}$ be a non-degenerate symmetric bilinear form over F. The form $\mathbf{b}$ decomposes as an orthogonal sum of an anisotropic part, unique up to isometry, and a metabolic part and each metabolic form is a sum of binary metabolic forms isometric to $\begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}$. The form $\mathbf{b}$ is diagonalizable if it represents a non-zero element. In particular, the similarity classes of non-degenerate symmetric bilinear form the Witt ring $W(F)$. The even dimensional forms constitute the fundamental ideal $I(F)$ of this ring. We have the usual filtration by the powers $I^n(F)$ of $I(F)$ and $I^n(F)$ are generated by n-fold Pfister forms $\otimes_{i=1}^n \langle 1, a_i\rangle$ for some non-degenerate diagonal binary forms $\langle 1, a_i\rangle$. Let $\overline{I}^n(F) := I^n(F)/I^{n+1}(F)$. The Arason-Pfister Hauptsatz holds, i.e., the non-metabolic forms in $I^n(F)$ have dimension at least 2^n. To each $\mathbf{b}$, we can associate the corresponding quadratic form

$\varphi_{\mathbf{b}}, v \mapsto \mathbf{b}(v,v)$. This form is totally singular, i.e., its polar form is trivial. Let $F(\mathbf{b}) := F(\varphi_{\mathbf{b}})$ be the function field of the projective quadric determined by $\varphi_{\mathbf{b}}$. Laghribi showed that $\mathbf{b}_{F(\mathbf{b})}$ is metabolic if and only if $\mathbf{b}$ is a scalar multiple of a Pfister form just as in the case that the field is of characteristic not two. Moreover, we can construct a splitting tower by inductively defining $F_0 = F$ and $F_i = F(\mathbf{b}_i)$ where $\mathbf{b}_i$ to be the anisotropic part of $\mathbf{b}_{F(\mathbf{b}_{i-1})}$. If h is the smallest integer such that $\dim \mathbf{b}_h \leq 1$ then $\mathbf{b}_{h-1}$ is a scalar multiple of an n-fold Pfister form for some n called the degree of $\mathbf{b}$. Let $J_n(F) := \{\mathbf{b}_i \mid \deg \mathbf{b} \geq n\}$ (with the zero form having infinite degree). Then Laghribi showed $J_n(F) = I^n(F)$. If φ is a quadratic form over F then it is an orthogonal sum of a non-degenerate (non-singular) part φ_{ns} and a totally singular part. If φ is a quadratic form over F let $\overline{I}^n(F(\varphi)/F) := \ker(I^n(F) \to \overline{I}^n(F(\varphi))$. Hoffman showed that the following (which proves the second Laghribi result): Let φ be a quadratic form over F. If the non-degenerate part of φ is of dimension at least two then $\overline{I}^n(F(\varphi)/F) = 0$ for all $n \geq 0$ and if $\varphi := \langle 1, a_1, \ldots a_l\rangle$, so totally singular, and $2^m = [F^2(a_1, \ldots a_l) : F^2]$ then $\overline{I}^n(F(\varphi)/F) = 0$ for $m > n$ and $\overline{I}^n(F(\varphi)/F)$ is generated by the forms $\psi \otimes (\otimes_{i=1}^m \langle 1, b_i\rangle) + I^{n+1}(F)$ with $\psi \in I^{n-m}(F)$ and $b_1, \ldots, b_m$ satisfying $F^2(b_1, \ldots b_m) = F^2(a_1, \ldots a_l)$. This uses the analogue of the Milnor conjecture for quadratic forms in characteristic not two proven by Kato using differential forms.

Lecture by A. Laghribi. Let F be a field of characteristic two. We use the notation and definitions in the talks by R. Baeza and D. Hoffmann. If K/F is a field extension, let $i_K : W(F) \to W(K)$ and $j_K : W_q(F) \to W_q(K)$ be the maps induced by the inclusion $F \subset K$. In the case of fields of characteristic not two, kernels of these maps for various field extensions were studied by R. Elman, A. Wadsworth, T.-Y-. Lam, J.-P. Tignol, and R. Fitzgerald. In characteristic two, the multiquadratic case was studied by D. Hoffmann and Laghribi. Let p be an irreducible monic polynomial in the polynomial ring $F[T] := F[t_1, \ldots, t_n]$ (monic relative to a fixed lexicographic ordering) and $F(p)$ the quotient field of $F[T]/(p)$. M. Knebusch proved the Norm Theorem: If $\mathbf{b}$ is an anisotropic symmetric bilinear form then $\mathbf{b}_{F(p)}$ is metabolic if and only if $\mathbf{b}_{F[T]} \cong p\mathbf{b}_{F[T]}$ (without a characteristic assumption) using the theory of specializations and induction, where the case $n = 1$ is handled by the Milnor exact sequence for $W(F(t))$. Aravire-Jacob used the analogue of this sequence for $W_q(F)$ if F is perfect and another if F is not perfect to prove the analogue of the Norm Theorem for non-singular quadratic forms with hyperbolic replacing metabolic. Let φ be a quadratic form then $\varphi \cong \varphi_{ns} \perp \varphi_{ts}$ with φ_{ns} non-singular and φ_{ts} totally singular. (The form φ_{ns} is not unique but φ_{ts} is.). Call a form *semi-singular* if neither summand is trivial. We can study three cases: the form is non-singular, totally singular, or semi-singular. The Norm Theorem for totally singular forms was proven by Hoffmann-Laghribi. This leaves the case of semi-singular quadratic forms. We can also write $\varphi \cong \varphi_{\mathbf{H}} \perp \varphi_0 \perp \varphi_{an}$ where $\varphi_{\mathbf{H}}$ is hyperbolic, φ_0 is the trivial form of some dimension, and φ_{an} is the anisotropic part. Let $i_W(\varphi) = (1/2)\dim \varphi_{\mathbf{H}}$, the *Witt index* of φ and $j_d(\varphi) := \dim \varphi_0$, the *defect index* of φ. Call $i_t(\varphi) = i_W((\varphi) + j_d(\varphi)$ the *total index* of φ. The form φ is called *quasi-hyperbolic* if $\dim \varphi$ is even and $i_t(\varphi) \geq \dim \varphi/2$. The Norm Theorem holds for semi-singular quadratic forms. Its proof depends on this notion of quasi-hyperbolicity replacing hyperbolicity (as it does in the totally singular case). Laghribi-Mammone prove the following Norm Theorem: If φ is anisotropic semi-singular then $\varphi_L = p\varphi_L$ implies that p is inseparable and $\varphi_{F(p)}$ is quasi-hyperbolic and if p is a totally singular quadratic form representing 1 then the converse is true. They also prove a Subform Theorem: If φ is even dimensional and anisotropic and p is a quadratic form such that $\varphi_{F(p)}$ is quasi-hyperbolic then p is totally singular and for all values a of φ_{ns}, non-zero values b of φ_{ts}, and non-zero values c of p , there exists a non-singular form ψ such that $\varphi \cong \psi \perp \varphi_{ts}$ with abp a subform of ψ and acp a subform of $ab\varphi_{ts}$. The proof also uses there theorem that if $p - t_1^{2^m} + d$ (with $m \geq 1$) and $i_w(\varphi_{F(p)}) = \dim \varphi_{ns}/2$ then there exists a non-singular ψ over F such that $\psi_{F(p)}$ is hyperbolic and $\varphi \cong \psi \perp varphi_{ts}$. This also leads to the generalization of when a quadratic form splits over its function field.

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