BIRS 07w5033: Interactions of geometry and topology in low dimensions

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1 Introduction

This workshop focussed on interactions between symplectic geometry, gauge theory, contact topology, and applications to low-dimensional manifolds. While each of these areas has been very active for many years in an independent fashion, the theory of low-dimensional manifolds has greatly benefited from interactions with the other subjects represented at the event. Our workshop can be seen as a follow-up to the BIRS-MSRI Hot Topics event of November 2003 on Floer homology. Four years after this event, it was very productive to gather gauge theorists, contact topologists, symplectic geometers and topologists all together to share their insights and to foster collaborative investigation and research.

One of the (pleasant) difficulties in organizing such an event is to carefully select a group of 40 world experts with a good balance between research areas, experience versus budding mathematical activity and try to represent as accurately as possible the most important current trends. In the end, this was achieved with great success. In particular, while some of the most established and senior researchers were present (Kirby, Stern, Akbulut, Boyer, Kirk, etc), it was a great inspiration to witness the emergence of young mathematicians through their important scientific work, but also through a lively presence at the conference, asking many questions, making good observations and stimulating discussions, as exemplified by the likes of Hedden, Perutz, Ng, Grigsby among others.

2 Overview of the Field

In recent years collaborations between contact and symplectic geometers, gauge theorists, and low-dimensional topologists have been highly fruitful, leading to solutions to long-standing conjectures in topology, illuminating the world of contact and symplectic manifolds, and providing new perspectives on fundamental questions in low-dimensional topology.

For some time, gauge theory has provided geometric topologists with powerful techniques yielding spectacular results on the classification problem for 4-dimensional manifolds, including the early 4-manifold invariants of Donaldson, the Seiberg-Witten invariants, and the more recent invariants of Ozsvath-Szabo. In each case, when applied to 4-manifolds with boundary, the invariants take values in the Floer homology groups of the bounding 3-manifold. These Floer homology groups are important 3-manifold invariants in their own right and have been applied to a variety of problems, such as knot theory and the structure of the homology cobordism group in dimension three. Frequently, these invariants are easier to compute in the presence of extra structure. For example, all the above mentioned invariants are non-zero for symplectic 4-manifolds. These non-vanishing results are instrumental in the interactions between contact and symplectic geometry and low-dimensional topology.

In 2001 P. Ozsvath and Z. Szabo introduced the Heegaard Floer homology groups. Their theory was motivated by Seiberg-Witten theory, but is defined in a completely different way: it is a variant of lagrangian Floer homology that uses much of the 3-manifold topology in its definition (in particular a Heegaard decomposition). In many instances, this new theory has been able to reprove fundamental results obtained in gauge theory over the last few decades and has also provided many new applications. One of its advantages is that there are powerful exact sequences enabling one to obtain hard topological information. For example, it was shown by Y. Ni that it detects if a knot in S^3 fibres, see also [12]. This theory is conjecturally equivalent to Seiberg-Witten Floer homology by an analogue of the Atiyah-Floer conjecture, and a very interesting research program of Yi Jen Lee proposes an approach for proving this conjecture. Heegaard Floer homology is also (conjecturally) closely related to the contact homology groups introduced by Y. Eliashberg, H. Hofer and others and it is possible that a tool derived from this set-up, namely the embedded contact homology of Hutchings will provide a relation between Heegaard-Floer and Seiberg-Witten theories. A good explanation of these correspondences would help determine exactly what exactly Heegaard Floer groups measure.

Emmanuel Giroux revolutionized contact geometry by proving an equivalence between contact structures on 3-manifolds up to isotopy and open book decompositions up to stabilization. Open book decompositions are a classical topological concept and have been studied for some time. This result is analogous to Simon Donaldson's proof that symplectic 4-manifolds always admit Lefschetz pencils, and the analogy is strengthened by Bob Gompf's proof that all Lefschetz pencils admit symplectic structures. Both these correspondences relate geometric concepts to topological ones and have been the foundation for many of the applications of symplectic geometry to questions in low-dimensional topology. For example, this correspondence leads to two notable results, namely Giroux and Noah Goodman's positive resolution of Harer's conjecture that all fibered knots in S^3 are related by Hopf plumbings, and Ozsvath and Szabo's proof of Gordon's conjecture that the unknot is the only knot on which p surgery yields the lens space $-L(p,1)$. The correspondence also has implications in the other directions as well. It is the basis of the non-vanishing of the Ozsvath Szabo invariant of symplectic 4-manifolds mentioned above. It is also the key tool in Eliashberg and John Etnyre's proof that any symplectic filling of a contact manifold can be embedded in a closed symplectic manifold. This result in turn is an integral part of Peter Kronheimer and Tom Mrowka's proof of the Property P conjecture that a nontrivial surgery on a nontrivial knot in S^3 has nontrivial fundamental group or that $\mathbb{R}\mathrm{P}^3$ cannot be obtained by Dehn surgery along a non-trivial knot in S^3.

3 Recent Developments and Open Problems

There were many active areas represented at the workshop, but if pressed to determine one or two where the breakthroughs have been most important recently, one would have to look in the direction of combinatorial approaches to Heegaard-Floer homology and the impact of symplectic field theory ideas in the contact world through the embedded contact homology of Hutchings, with expected applications to 3-manifold topology.

The problem of combinatorially constructing Heegaard-Floer groups without resorting to counting pseudo-holomorphic curves had taken a very promising turn a few months before the workshop when knot Floer homology was given a purely combinatorial interpretation. This was an underlying theme in several of the talks at the workshop, in particular Ng's talk focused on applications of these ideas to the classification of transverse knots in contact manifolds (see [13]), while Plamenevskaya explained how the Heegaard-Floer contact invariant can be defined in the combinatorial context of Manolescu, Ozsvath and Sarkar, see [14].

Further, as the knot Heegaard Floer homology categorifies the Alexander polynomial in much the same way that Khovanov cohomology categorifies the Jones polynomial, an interesting open problem is to relate the Heegaard Floer and Khovanov cohomologies. A few of the talks at the workshop focused on this aspect, and the most recent result along these lines is the existence of a spectral sequence interpolating between the two theories.

A few months before the conference, Taubes took everyone by surprise in the area of symplectic topology by proving in [15] the 3-dimensional case of the Weinstein conjecture (for any compact oriented 3-manifold M and α a contact 1-form on M, the vector field that generates the kernel of the 2-form $d\alpha$ has at least one closed integral curve). The proof in [15] involved a come-back of the Seiberg-Witten equations, but it also gave very good reasons to focus on further properties of the embedded contact homology (ECH). In turn, Taubes' proof can be seen as a step towards showing that ECH is isomorphic to Seiberg-Witten-Floer homology.

Another example of the interactions between the subjects represented at the workshop is the study of slice numbers of knots. Many talks at the event (Owens, Grigsby, Jabuka, Hedden, Chantraine) involved work on the slice number of classes of knots. The tools for studying this are today quite varied - Heegaard-Floer homology, classical gauge theory, contact topology methods - and there was much discussion at the conference about these problems, including recent work of Lisca (January 2007) on the very classical slice-ribbon conjecture (every slice knot in S^3 is a ribbon knot) which settled the issue in the case of 2-bridge knots.

4 Presentation Highlights

The meeting featured 22 one-hour talks on various aspects of 3-dimensional and 4-dimensional manifold theory. The central themes were:

- **4-manifolds, smooth and symplectic structures.** *(Talks 1, 2, 5, 12, 19, 20)*
- **3-manifolds, contact structures and invariants.** *(Talks 3, 6, 7, 8, 9, 10, 14, 15, 17, 18, 22)*
- **Heegaard-Floer homology and analogues.** *(Talks 6, 7, 10, 11, 13, 14, 15, 16, 18, 21)*
- **Knots and invariants (especially the knot concordance problem).** *(Talks 4, 6, 11, 15, 16, 21)*

Below is a detailed list of speakers, titles, and brief descriptions of their talks.

1. **Scott Baldridge** (Louisiana State) *Small symplectic building blocks and the geography problem*
   Constructions were given of many simply connected and non-simply connected symplectic and smooth manifolds, including a minimal symplectic manifold homeomorphic to $\mathbb{CP}^2\#3(\overline{\mathbb{CP}}^2)$ containing a symplectic genus 2 surface with simply connected complement.

2. **Inanc Baykur** (Michigan State) *Folded-Kähler structures on 4-manifolds*
   This talk focused on the existence of a generalization of symplectic structures on arbitrary closed smooth oriented 4-manifolds, called "folded-Kähler structures". This result comes with a decomposition theorem, which states that every closed smooth oriented 4-manifold can be decomposed into two compact Stein manifolds (one with reversed orientation), such that the induced contact structures agree on a separating convex hypersurface. There is also a natural topological counterpart for these structures: "folded Lefschetz fibrations".

3. **Steven Boyer** (UQAM) *On families of virtually fibred Montesinos link exteriors*
   William Thurston conjectured over twenty years ago that every compact hyperbolic 3-manifold whose boundary is a possibly empty union of tori is virtually fibred, that is, has a finite cover which fibres over the circle. If true, it provides a significant amount of global information about the topology of such manifolds. To date, there has been remarkably little evidence to support the conjecture. For instance, there is only one published non-trivial example of a closed virtually fibred hyperbolic rational homology 3-sphere. (Non-trivial in this context means that the manifold neither fibres nor semi-fibres.)

   This talk outlined a proof of this conjecture for the exteriors of many Montesinos links and constructed an infinite family of closed virtually fibred hyperbolic rational homology 3-spheres. As a result, one obtains a finite index bi-orderable subgroup in the fundamental groups of the exteriors of many Montesinos links.

4. **Baptiste Chantraine** (UQAM) *Cobordisms of Legendrian knots*
   One generalization of cobordism theory of knots in the Legendrian category is asking that such cobordisms are realised by Lagrangian surfaces in the symplectization. From a study of relative Gromov-

Lee theorems and the behaviour of the classical invariants, one sees that this relation is rigid (unlike in the topological case where only the Thom-Pontryagin construction is needed). The first step in studying this relation (and the relation of Lagrangian concordance) is to show that Legendrian-isotopic knots are Lagrangian concordant. This talk gave the basic definitions and theorems which are the starting point of the theory and provided non-trivial examples of Lagrangian cobordisms implying that this relation is non-symmetric. As an application, a contact-topology proof of the local Thom conjecture was obtained.

5. **Stefan Friedl** (UQAM) *Symplectic 4-manifolds with a free circle action*
Let W be a symplectic 4-manifold with free circle action. It was shown that if the fundamental group of the orbit space satisfied certain separability properties, then the orbit space fibers over S^1 . Using the Lubotzky alternative, the same result can be proved if the canonical class is trivial.

6. **Julia Grigsby** (Columbia) *Knot concordance and Heegaard Floer homology invariants in branched covers*
The smooth concordance order of a knot K is defined as the smallest positive integer n for which the connected sum of n copies of K bounds a smoothly-embedded disk in the four ball. This talk described two new invariants which yield an obstruction to a knot having finite smooth concordance order. These invariants are defined by examining analogues of "classical" Heegaard Floer homology invariants in the double-branched cover of K. Using a simple combinatorial description of these invariants in the case where K is a two-bridge knot, we are able to conclude that all two-bridge knots of 12 or fewer crossings for which the concordance order was previously unknown have infinite concordance order.

7. **Matthew Hedden** (MIT) *Lens space surgeries, contact structures, the braid group, and algebraic curves*
This talk gave a proof of the following statement: If Dehn surgery on a knot K yields a lens space, then K arises as the transverse intersection of an algebraic curve in $\mathbb{C}^2$ with the three-sphere. Furthermore, the genus of the piece of the curve inside the four-ball is equal to the Seifert genus of K. Knots arising in this way are more general than the well-understood links of singularities, as their singular sets may be more complicated. The result follows from a theorem stating that an invariant defined using Ozsvath-Szabo theory detects when fibered knots arise from algebraic curves with a genus constraint as above. This theorem, in turn, follows from connections between Ozsvath-Szabo theory and Giroux's work on three-dimensional contact geometry, and work of Rudolph relating the knot theory of algebraic curves to the braid group.

8. **Benjamin Himpel** (Bonn) *A splitting formula for spectral flow and the SU(3) Casson invariant for spliced sums*
The $SU(3)$ Casson Invariant does not behave well under spliced sums, however, for splicings of complements of torus knots, there is a conjectured formula relating the $SU(3)$ Casson invariant to the $SU(2)$ Casson invariant of the two knots. A proof of this conjecture was presented, and the main tool for this is a splitting formula for the $su(n)$ spectral flow of the twisted odd signature for 3-manifolds cut along a torus coupled to a path of $SU(n)$ connections.

9. **Ko Honda** (Southern California) *Invariants of exact Lagrangian cobordisms*
Constructions of exact Lagrangian cobordisms between two Legendrian knots in the symplectization of the standard contact $\mathbb{R}^3$ were given, and invariants arising from Legendrian contact homology and Khovanov homology were discussed.

10. **Michael Hutchings** (UC Berkeley) *Embedded contact homology*
The goal of this talk was to interpret Seiberg-Witten and Ozsvath-Szabo Floer homology via contact geometry in 3-dimensions and J-holomorphic curves in 4-dimensions. The candidate theory is embedded contact homology (ECH), which is a kind of Floer theory defined for a contact 3-manifold Y , whose differential counts certain embedded pseudoholomorphic curves in the symplectization $\mathbb{R} \times Y$. This talk defined the theory and discussed some associated open problems and conjectures.

11. **Stanislav Jabuka** (Nevada, Reno) *Knot concordance and Heegaard Floer homology*
After describing some of the open questions for knot concordance, some of them dating back to the seminal work of J. Milnor and R. Fox in the mid 1960's, some new results obtained using Heegaard

Floer homology were presented. An obstruction for a knot to be of order $n > 0$ in the concordance group was defined and examined for small crossing knots (as of this writing there are still 8-crossing knots with unknown concordance order), this work obviously being related to the problem of existence of n-torsion knots for > 2 in the concordance group. Also presented was a Heegaard Floer proof of a theorem first obtained by Fintushel and Stern according to which the only 3-stranded pretzel knot $K(p, q, r)$ (with p, q, r odd) with trivial Alexander polynomial and which is of finite concordance order, is the unknot. Generalizations of this result to pretzel knots with nontrivial Alexander polynomial were given.

12. **Hee Jung Kim** (McMaster) *Topological triviality of smoothly knotted surfaces in 4-manifolds*
Some generalizations and variations of the Fintushel-Stern rim surgery are known to produce smoothly knotted surfaces. If the fundamental groups of their complements are cyclic, then these surfaces are topologically unknotted. Using a twist-spinning construction from high-dimensional knot theory, examples of knotted surfaces whose complements have cyclic fundamental groups were constructed.

13. **Thomas Mark** (Virginia) *On perturbed Heegaard Floer invariants*
This talk constructed a version of Heegaard Floer homology with coefficients in certain Novikov rings depending on a choice of perturbation, namely a 2-dimensional real cohomology class. For nontrivial perturbations, the reducible part of the Floer homology vanishes, in close analogy with Seiberg-Witten theory. This leads to analogues of the Ozsvath-Szabo invariants for closed 4-manifolds with $b^+ > 0$ and well-behaved relative invariants for 4-manifolds with boundary.

14. **Gordana Matic** (Georgia) *Open books and contact class in Heegaard Floer Homology*
An alternate, simple description of the Ozsvath-Szabo contact class in Heegaard Floer homology for a closed contact 3-manifold was given, and some consequences of this approach were discussed. This variant of the contact class was extended to define a new contact class for a contact manifold with convex boundary which lives in Juhasz's sutured Floer homology.

15. **Lenhard Ng** (Duke) *Transverse knots and Heegaard Floer homology*
The recently discovered combinatorial form for knot Floer homology has an unexpected application in contact geometry. In fact, one can use knot Floer homology to produce an effective invariant of transverse knots, and this talk gave several new examples of transversely non-simple knot types.

16. **Brendan Owens** (Louisiana State) *Slicing numbers of knots*
The slicing number $u_s(K)$ of a knot K in S^3 is the least number of crossing changes to convert K to a slice knot. This gives an upper bound for the slice genus $g_s(K)$. Livingston defined an invariant $U_s(K)$ which takes into account signs of crossings, with $g_s \leq U_s \leq u_s$. Heegaard Floer theory and also Donaldson's theorem can be used to give information on these numbers, and this talk described an infinite family of knots K_n with slice genus n and $U_s > n$.

17. **Tim Perutz** (Cambridge) *A symplectic Gysin sequence and the Floer homology of connected sums*
This talk described a project to study the structure of symplectic models for gauge-theoretic TQFTs on singularly-fibred 3-and 4-manifolds. The Gysin sequence for the cohomology of a sphere-bundle has a symplectic Floer-theoretic counterpart, which in turn is precisely analogous to a sequence describing Floer homology for connected sums of 3-manifolds.

18. **Olga Plamenevskaya** (SUNY Stony Brook) *A combinatorial description of the Heegaard Floer contact invariant*
Manolescu, Ozsvath, and Sarkar recently proved that certain Heegaard Floer homologies admit a purely combinatorial description. In particular, Sarkar and Wang developed an algorithm to modify a given Heegaard diagram for a 3-manifold so that the holomorphic disks can be combinatorially understood. Using the geometric description of the contact invariant due to Honda, Kazez, and Matic, a version of this algorithm was described in the context of open books to show that the Heegaard Floer contact invariant is combinatorial.

19. **Nikolai Saveliev** (Miami) *Dirac operators on manifolds with periodic ends*
This talk focused on relationships between classical invariants of low-dimensional topology and certain

invariants arising in 4-dimensional gauge theory, and in particular on Dirac operators on non-compact spin manifolds with periodic ends of dimension at least four. A necessary and sufficient condition for such an operator to be Fredholm for a generic endperiodic metric was given, and these end-periodic Dirac operators were used to prove that an invariant introduced by Cappell and Shaneson in the 1970's provides an obstruction to the existence of metrics of positive scalar curvature on some non-orientable 4-manifolds. As an application, it was shown that some exotic 4-manifolds do not admit a metric of positive scalar curvature (in some cases, even if their orientation double covers do).

20. **Ron Stern** (UC Irvine) *Reverse engineering and smooth structures on simply-connected smooth manifolds*
This talk introduced a procedure called "reverse engineering" which can be used to construct infinite families of smooth 4-manifolds in a given homeomorphism type. This is a very general technique that recovers many of the known techniques for producing smooth structures on a given simply-connected 4-manifold.

    Reverse engineering is a three step process for constructing infinite families of distinct smooth structures on simply connected 4-manifolds. One starts with a model manifold which has nontrivial Seiberg-Witten invariant and the same euler number and signature as the simply connected manifold that one is trying to construct, but with $b_1 > 0$. The second step is to find b_1 essential tori that carry generators of H_1 and to surger each of these tori in order to kill H_1 and, in favorable circumstances, to kill π_1. The third step is to compute Seiberg-Witten invariants. After each surgery one needs to be careful to preserve the fact that the Seiberg-Witten invariant is nonzero. One can construct exotic structures on many simply-connected smooth manifolds starting with an irregular complex surface.

21. **Hao Wu** (Massachusetts) *The Khovanov-Rozansky cohomology and Bennequin inequalities*
Bennequin type inequalities were established using various versions of the Khovanov- Rozansky cohomology. A new proof of a Bennequin type inequality was established and new Bennequin type inequalities for knots using Gornik's version of the Khovanov-Rozansky cohomology were given. These generalize results by Shumakovitch, Plamenevskaya and Kawamura using the Rasmussen invariant.

22. **Chris Wendl** (MIT) *Intersection theory and compactness for holomorphic curves in low dimensions*
This talk described a recent result strengthening the standard compactness theorem for a geometrically natural class of embedded holomorphic curves in contact 3-manifolds: it turns out the intersection theory of punctured holomorphic curves can be used to rule out multiple covers in the limit, so that transversality is never a problem. This has applications to the theory of finite energy foliations (generalizations of planar open book decompositions), and also suggests an approach for defining distinctly low dimensional versions of Contact Homology and SFT. Also described were some related results in symplectic 4-manifolds and nontrivial symplectic cobordisms. These are part of a larger program to justify the statement that "nice holomorphic curves degenerate nicely".

5 Scientific Progress Made

The workshop brought together leading experts from several different areas, and this sparked much scientific interaction. Despite the large number of proposed talks, the organizers were committed to the idea that such a workshop based in interactions between subjects should have more than enough scheduled time for scientific discussions. This was accomplished thanks to a careful scheduling of long breaks at the lunch period and not overloading the talk timetable so that informal discussions were many during the evenings. In some instances this also provided valuable time for teams of researchers who rarely get to meet to work together.

The meeting also established several collaborations that have already born fruit. Two examples readily come to mind. One evening while discussing in the BIRS lounge, the post-doctoral researcher Paolo Ghiggini and Ph.D. student Jeremy van Horn-Morris elaborated with Ko Honda a way to prove that Giroux torsion kills the contact invariant of Ozsvath-Szabo (see [4] for a write up of their results) and since then, Ghiggini and Honda have been working on extensions of this work to the case of twisted coefficients.

The workshop was also the perfect stage for intense discussions around the classification problem for smooth structures and the geography problem for symplectic 4-manifolds. In the weeks leading to the conference, many independent approaches to these problems were put forward in the form of preprints, with competing claims made by various authors. A large group of the researchers involved in these developements attended the workshop: Scott Baldridge, Inanc Baykur, Paul Kirk, Doug Park, and Ron Stern. It was at this BIRS meeting that many issues about rigorous proofs were raised and discussions between the various teams sorted out the details and even brought many of the key players together in collaboration, as evidenced by the recent papers [1, 2].

We conclude with the following bibliography which is far from being extensive, its goal being mostly to provide some background articles, all of them still in preprint form, to some of the latest topics discussed at the workshop and articles that have emerged since then. Consulting these articles and, in turn, their own references will give a much broader perspective on the various areas represented at the event.

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